Models of Fluid - Solid Coupling and Exact Controllability

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Coupling between Navier-Stokes Equations and Lamé System

$$\begin{aligned}
\Omega_F(t) &\subset \mathbb{R}^3 \\
\Omega_S(t) &= \Omega \setminus \overline{\Omega_F(t)} \\
v_t + (v \cdot \nabla_x)v - \mathrm{div}_x \sigma(v,p) &= 0 \text{ in } \Omega_F(t), 0 < t < T \\
\mathrm{div}_x v &= 0 \text{ in } \Omega_F(t),\ 0 < t < T \\
v(x,0) &= v_0(x) \text{ in } \Omega_F(0) \\
\sigma(v,p) &= \nu(\nabla_x v + \nabla_x v^T) - pI
\end{aligned}$$

Lagrangian trajectories

$$\begin{aligned}
\frac{\partial X}{\partial t}(y,t) &= v(X(y,t),t) \\
X(y,0) &= y, \quad y \in \Omega_F(0)
\end{aligned}$$

Boundary Conditions

$$\begin{aligned} v &= 0 \text{ on } \Gamma_e \times (0,T) \\ v(X(y,t),t) &= \frac{\partial w}{\partial t}(y,t) \text{ on } \Gamma_S(0) \times (0,T) \end{aligned}$$

Lamé System

$$w_{tt} - \mathrm{div}_y \sigma(w) = 0 \text{ in } \Omega_S(0) \times (0,T)$$

$$w(y,0) = w_0(y), \frac{\partial w}{\partial t}(y,0) = w_1(y) \text{ in } \Omega_S(0)$$

$$\sigma(w) = \lambda \text{ trace } \varepsilon(w) + 2\mu\varepsilon(w)$$

$$\varepsilon(w) = \frac{1}{2}(\nabla_y w + \nabla_y w^T)$$

Boundary Conditions

$$\sigma(w)n = (\sigma(v,p) \circ X) \text{ Cof } (\nabla_y X)n \quad \text{ on } \Gamma_S(0) \times (0,T)$$

Features.

Evolution problem, non-stationary problem posed on a non-cylindrical domain in space-time.

Free Interface problem.

Mixed Nature (Hyperbolic-Parabolic)

Hyperbolic metric and Parabolic metric

Hyperbolic scaling and Parabolic scaling

Characteristic speeds do not match at the interface

Natuaral spaces do not match at the interface.

Idea. Solid deformations may have only finitely many degrees of freedom.
Hyperbolicity is negligible.
Parabolic dominates.
Solid is treated as a perturbation of the fluid system.

Motion of Rigid Bodies

x_B - position of centre of mass

ω - angular velocity

Q - Orientation of solid (orthogonal matrix)

$$
\begin{aligned}
S(t) &= Q(t)S(0) + x_B(t) \\
x &= Q(t)y + x_B(t)
\end{aligned}
$$

Velocity of Solid

$$
\begin{aligned}
v(x,t) &= \omega(t) \wedge (x - x_B(t)) + x_B'(t) \text{ (Interface condition)} \\
Q'(t)y &= \omega(t) \wedge Q(t)y
\end{aligned}
$$

Newton's Laws:

$$
\begin{aligned}
m_B''(t) &= -\int_{\partial S(t)} \sigma(v,p)n \\
I\omega'(t) &= -\int_{\partial S(t)} (x - x_B(t)) \wedge \sigma(v,p)n
\end{aligned}
$$

STOKES MODEL

$$
\begin{aligned}
y' - \Delta y + \nabla p &= u 1_U \quad \text{in} \quad Q = \Omega \times (0,T) \\
div y &= 0 \quad \text{in} \quad Q \\
y &= s' \quad \text{on} \quad \Sigma_i = \Gamma_i \times (0,T) \\
y &= 0 \quad \text{on} \quad \Sigma_e = \Gamma_e \times (0,T) \\
y(0) &= y_0 \quad \text{in} \quad \Omega
\end{aligned}
$$

$$
s'' + s = -\int_{\Gamma i} \sigma(y,p) n \quad \text{in} \quad (0,T)
$$

$$
s(0) = s_0, s'(0) = s_1
$$

$$
\sigma = \nu(\nabla y + \nabla y^T) - pI
$$

Heat -Solid Model

$$
\begin{aligned}
y' - \Delta y &= u 1_U \quad \text{in} \quad Q \\
y &= 0 \quad \text{on} \Sigma_e \\
y &= s' \cdot n \quad \text{on} \Sigma_i \\
y(0) &= y^0 \quad \text{in} \quad \Omega
\end{aligned}
$$

$$
s'' + s = -\int_{\Gamma_i} (\partial_n y) n \quad \text{in} \quad (0, T)
$$

$$
s(0) = s_0, s'(0) = s_1
$$

Helmholtz Model

$$\begin{aligned} y'' - \Delta y &= u 1_U \quad \text{in} \quad Q = \Omega \times (0,T) \\ y &= 0 \quad \text{on} \quad \Sigma_e = \Gamma_e \times (0,T) \\ \partial_n y &= s' \cdot n \quad \text{on} \quad \Sigma_i = \Gamma_i \times (0,T) \end{aligned}$$

$$y(0) = y_0, y'(0) = y_1 \quad \text{in} \Omega$$

$$s'' + s = -\int_{\Gamma i} y' n \quad \text{in} \quad (0,T)$$

$$s(0) = s_0, s'(0) = s_1$$

Resume

$$\text{NSE} + \text{Lame} \longrightarrow \text{NSE} + \text{Rigid Body}$$

Eliminates infinite degrees of freedom for solid deformation. Infact, no solid deformation at all.

NSE + Rigid body $\longrightarrow$ Stokes Model.

Allows some finite degrees of freedom for solid deformation; contains viscosity effects.

Stokes Model $\longrightarrow$ Heat-Solid Model.

Elimintes pressure effects,

contains viscosity effects

contains some some solid deformation.

Heat -Solid Model $\longrightarrow$ Helmholtz Model.

Eliminates viscous effects

Adds weak compressibility effects

contains some solid deformation.

Linearization principle holds for control problems

Nonlinear equation $\longrightarrow$ Linear equation

Viscous effects are treated via Carleman observability estimates.

Finite Energy Solutions (Helmholtz Model)

Energy Space

$$Z = V \times L^2(\Omega) \times I\!R^2 \times I\!R^2$$

$$V = \left\{ v \in H^1(\Omega); v = 0 \text{ on } \Gamma_e \right\}$$

Skew adjoint operator,

Energy conservation

Dynamics is well defined.

Stone's Theorem .

Exact observability

Consider free dynamics (no control)

$$
\begin{aligned}
\phi'' - \Delta\phi &= 0 \quad \text{in } Q \\
\phi &= 0 \quad \text{on } \Sigma_e \\
\partial_n \phi &= w' n \quad \text{on } \Sigma_i \\
w'' + w &= -\int_{\Gamma_i} \phi' n \quad \text{in } (0,T)
\end{aligned}
$$

Initial conditions

$$
\phi(0) = \phi_0,\ \phi'(0) = \phi, w(0) = w_0, w'(0) = w_1
$$

Observation is ϕ' on U during $0 < t < T$

$$\|(\phi_0, \phi_1, w_0, w_1)\|^2_{Egy} \leq c \int_0^T \int_U |\phi'|^2$$

Exact observability estimate

Infinite dimensional Kalman Rank Condition.

Consequence of observability estimate

Unique continuous principle (UCP)

observation $\longrightarrow$ solution is injective

observation $= 0 \Rightarrow$ solution $= 0$.

This is not obvious.

Its validitry depends on (U, T)

Example. $U = \Omega$.

UCP is not enough; we require observability estimate for controllability.

Exact Controllability

Theorem. (Observability⇒Controllability)

Assume $T > 0$.

Assume Exact observability estimate for (U, T).

Then for any $(y_0, y_1, s_0, s_1) \in Z$, there is a control $u \in L^2(U \times (0, T))$ such that

$$(y(T), y'(T), s(T), s'(T)) = (0, 0, 0, 0).$$

Besides, u can be obtained as follows.

Define $J : Z \longrightarrow \mathbb{R}$ by

$$J(\phi_0, \phi_1, w_0, w_1) = \frac{1}{2}\int_0^T\int_U |\phi'|^2 + \int_\Omega \nabla\phi_0 \cdot \nabla y_0 + \int_\Omega \phi_1 y_1$$

$$+ w_0 \cdot s_0 + w_1 \cdot s_1$$

Then J has a unique minimizer $(\tilde{\phi_0}, \tilde{\phi_1}, \tilde{w_0}, \tilde{w_1})$. Let $(\tilde{\phi}, \tilde{w})$ be the corresponding solution . Take $u = \tilde{\phi} 1_U$. Moreover this has minimal norm in $L^2(U \times (0, T))$ among all possible controls.

Remark.

Exact observability estimate ensures the coercivity of the functional J and thereby guarantees the stability of minimizing sequences.

Proving Observability estimate (I)

- Multiplier Method
- Multiplier $\rho\phi, \rho s$ with $\rho(t) = t^2(T-t)^2$

$$\|(\phi_0, \phi_1, w_0, w_1)\|_Z \leq c \left\{ \int_0^T \int_\Omega |\phi'|^2 + \int_0^T |w|^2 \right\}$$

Next step. Compactness arguments

$$\int_0^T |w|^2 \leq c \int_0^T \int_\Omega |\phi'|^2$$

This method works if $U = \Omega, T > 0$ arbitrary.

Control acts on entire fluid domain.

Observation is made on entire fluid domain.

Proving observability (II)

Frequence variable method

Localized Control for Helmholtz Model

$$y'' - \Delta y = u 1_U \quad \text{in} \quad Q$$

$$y = 0 \quad \text{on} \quad \Sigma_e, \qquad \partial_n y = s' \cdot n \quad \text{on} \quad \Sigma_i$$

$$s'' + s = -\int_{\Gamma i} y' n \quad \text{in} \quad (0, T)$$

Initial Conditions.

Theorem. Assume U is a neighbourhood of $\Gamma_e \cup \Gamma_i$. Then exists time $T > 0$ such that we have exact controllability and observability in the energy space. No information on time of controllability. Control, observations only in a neighbourhood of Γ_e and Γ_i.

HELMHOLTZ Model as I order System

$$z'(t) = Az(t) + Bu(t)$$

$$z(t) = (y(t), y'(t), s(t), s'(t))$$

$$A(f, g, k, l) = (g, \Delta f, l, -k - \int_{\Gamma i} gn)$$

$$D(A) = \{(f, g, k, l); f \in H^2(\Omega), g \in H^1(\Omega)$$
$$f = g = 0 \quad \text{on} \Gamma_e$$
$$\partial_n f = l \cdot n \quad \text{on} \Gamma_i$$
$$k, l \in \mathbb{C}^2\}$$

$Bv = (0, v1_U, 0, 0)$

A is skew - adjoint $A^* = -A$

$B^*(f, g, k, l) = g1_U$

Observability Inequality/Estimate

Passage : Time $t \to$ Frequency ω

Infinite dimensional **Hautus test** (cf. TUCSNAK-WEISS)

Proposition $z'(t) = Az(t) + Bu(t)$

A skew - adjoint, with compact resolvent

$\{\Phi_k\}$ on basis of eigenvectors $\{i\mu_k\}$ eigenvalues. Define

$$E_\lambda = \overline{\langle \Phi_k ; |\mu_k| > \lambda \rangle}$$

(1) High Frequency Condition:

$\exists \delta > 0, \alpha > 0$ such that $\forall \omega \in I\!R, |\omega| > \alpha$

$$||(i\omega I - A)\Phi||^2 + ||B^*\Phi||^2 \geq \delta^2 ||\Phi||^2$$

$$\forall \Phi \in E_\alpha \cap D(A).$$

(2) Low Frequency Condition:

$B^*\Phi \neq 0 \quad \forall$ eigenvector Φ of A

Then we have exact observability.

No information on time T of observability

(High & Low Frequency Conditions)

Verification (2)

$$A\Phi = i\omega\Phi$$

. This is equivalent to

$$\begin{cases} g = i\omega f, \quad \Delta f = i\omega g \\ l = i\omega k, \quad -k - \int_{\Gamma i} gn = i\omega l \\ \text{Boundary Conditions} \end{cases}$$

- zero is not an eigenvalue of A

- If $B^*\Phi = 0$ then $g = 0$ on U.

$\Delta f + \omega^2 f = 0, f = 0$ on U. UCP $\Rightarrow$ $f \equiv 0$.

Hence $\Phi = 0$

Verification (1)

Proof by Contradiction, technical.

Roughly, in high frequency part, we can neglect solid

Helmholtz model $\approx$ uncoupled model

$y'' - \Delta y = 0$ in $Q, y = 0 \quad$ on $\quad \Sigma_e,$

$\partial_n y = 0$ on $\ \Sigma_i$

choice of $U \Rightarrow (1)$

Proving Observability (III)

- Method of lateral propagation of energy for for the wave equation in one space variable.

Theorem. Assume U is a neighbourhood of Γ_e only. (No control near the solid boundary)
(H) Assume solid is a ball /disc.
Then observability estimate holds with an optimal time.

(H) ensures that no energy is trapped near the solid boundary.

Here, we need control/observation only in neighbourhood of Γ_e only. No control is required near the solid.

Introduce Polar coordinatres (r, θ). Decomposition of the problem into spherical harmonics.

Surprise. Only the first harmonic mode is coupled to the solid. Other modes are free. Thus we are led to consider the wave equation in one -space variable coupled with the solid.

$$\psi'' - \frac{1}{r}\partial_r(r\partial_r\psi) + \frac{1}{r}\psi = 0$$

$$r_0 < r < R, 0 < t < T.$$

$$\psi(t, R) = 0$$

$$\partial_r\psi(t, r_0) = \pi w'(t)$$

$$w''(t) + w(t) = r_0\psi'(t, r_0)$$

Initial conditions

Observability estimate

$$\int_{r_0}^{R} |\psi'(0,r)|^2 r dr + \int_{r_0}^{R} |\partial_r \psi(0,r)|^2 r dr + \\ + \int_{r_0}^{R} \frac{1}{r} |\psi(0,r)|^2 dr + |w'(0)|^2 + w(0)|^2 \\ \leq c \int_{0}^{T} \int_{R_0}^{R} |\psi'(t,r)|^2 r dr dt$$

UCP. If $T > 2(R_0 - r_0)$ and

$$\int_{0}^{T} \int_{R_0}^{R} |\psi'(t,r)|^2 r dr = 0$$

then $(\psi, w) \equiv (0,0)$.

Lateral propagation of waves.

Interchange space and time.

Integrate energy density with respect to time at a fixed space point.

Conjecture. When solid region is convex, there si controllability/observabilitay with control/observation made only on Γ_e with optimal time.

References

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Two dimensional problems with rigid bodies: Tucsnak and his collaborators

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