Knots, graphs and Khovanov homology I

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IISER Mohali, India
Dec 2013

Outline

P.G. Tait

Modern knot theory began in late 1800's when Tait, Little and others tried to make a periodic table of elements by tabulating knot diagrams by crossing number:

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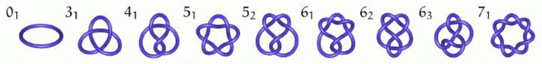

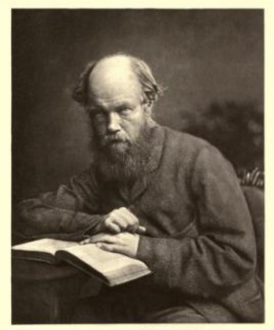

Tait graphs

Any link diagram D with a checkerboard coloring corresponds to a plane graph with signed edges G_D called the **Tait graph**.

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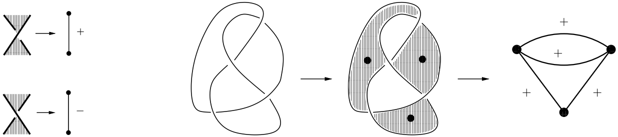

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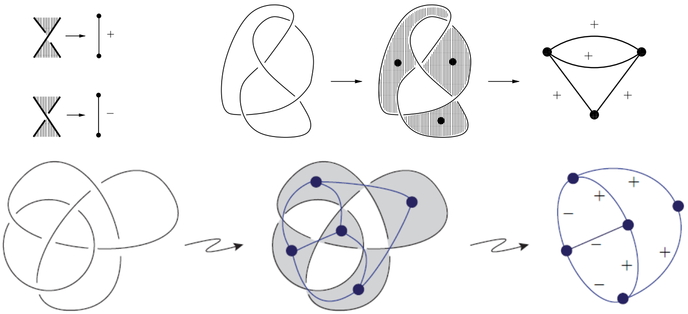

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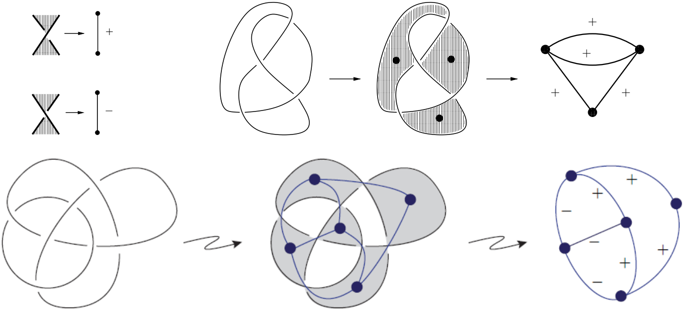
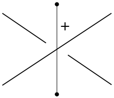


Exercise Any link diagram can be checkerboard colored.

Conversely, we can recover the diagram from any signed plane graph by taking its medial graph, and making crossings according to the sign on each edge:



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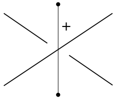

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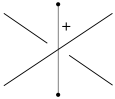

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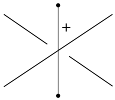

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Tait emphasized the importance of alternating diagrams:

Conjecture (Tait)   A reduced alternating diagram has minimal crossing number among all diagrams for that link.

A proof had to wait about 100 years until the Jones polynomial (1984), which led to several new ideas that were used to prove Tait's conjecture.

Jones polynomial

In 1984, V. Jones discovered $V_L(t) \in \mathbb{Z}[t, t^{-1}]$ by studying representations of the braid group,

$V_L(t)$ satisfies a skein relation:

$t^{-1}\, V$  $(t) - t\, V$  $(t) = \left(\sqrt{t} - 1/\sqrt{t}\right) V$  (t)

First polynomial link invariant to distinguish the right and left trefoils:

V  $(t) = t + t^3 - t^4, \quad V$  $(t) = -t^{-4} + t^{-3} + t^{-1}$

Open Problem: If $V_K(t) = 1$, is $K = \bigcirc$?

Kauffman bracket polynomial

Simplest combinatorial approach to the Jones polynomial:

Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ defined recursively by

1. $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$
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Example:

$$\begin{aligned} \langle \infty \rangle &= A\langle \bigcirc\bigcirc \rangle + A^{-1}\langle \bigcirc\!\!\bigcirc \rangle \\ &= A \cdot \delta + A^{-1} \cdot 1 = -A^{-1} - A^{3} + A^{-1} = -A^{3} \end{aligned}$$

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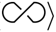
Example:

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$$= A \cdot \delta + A^{-1} \cdot 1 = -A^{-1} - A^{3} + A^{-1} = -A^{3}$$

If we adjust for this ambiguity, and change variables by $t = A^{-4}$, then

$$\left(-A^{-3}\right)^{w(L)} \langle L \rangle = V_L(t)$$

Kauffman states

Besides the axiomatic definition, Kauffman expressed $\langle L \rangle$ as a sum of all possible *states* of L:

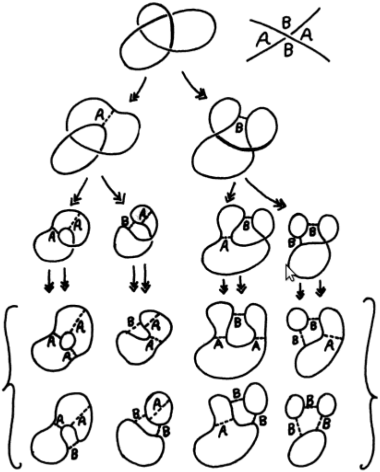
If L has n crossings, all possible A and B splices yield 2^n states s.

Let $a(s)$ and $b(s)$ be the number of A and B splices, resp., to get s.

Let $|s|$ = number of loops in s.

$$\langle L \rangle = \sum_{\text{states } s} A^{a(s)-b(s)} \, (-A^2 - A^{-2})^{|s|-1}$$

Kauffman states



$$\left\{ \begin{array}{c|c|c|c} A^3\delta & A & A & A^{-1}\delta \\ A & A^{-1}\delta & A^{-1}\delta & A^{-3}\delta^2 \end{array} \right\}$$

Spanning tree model of Jones polynomial

A- & B-smoothings of a crossing correspond to contraction and deletion of an edge of the Tait graph We want to use tools from graph theory to study knots.

1. (1954) The Tutte polynomial is defined using contraction and deletion of edges. It is also defined by a spanning tree expansion:

$$T_G(x, y) = \sum_T x^{i(T)}\, y^{j(T)}$$

where $i(T)$ is the number of internally active edges and $j(T)$ is the number of externally active edges of G for a given spanning tree T.

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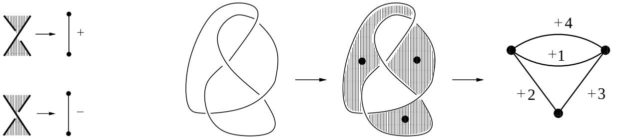
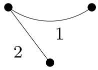
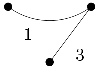
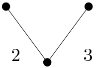
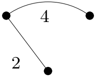
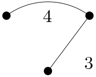
2. (1987) Applying Tutte's results, Thistlethwaite defined a spanning tree expansion for the Jones polynomial of links. If L is alternating, $V_L(t) \doteq T_{G_L}(-t, -1/t)$.

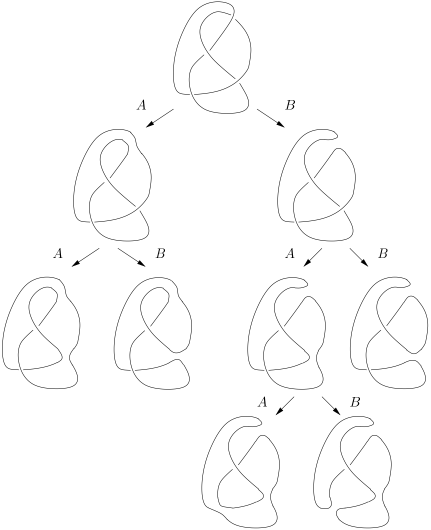
Example



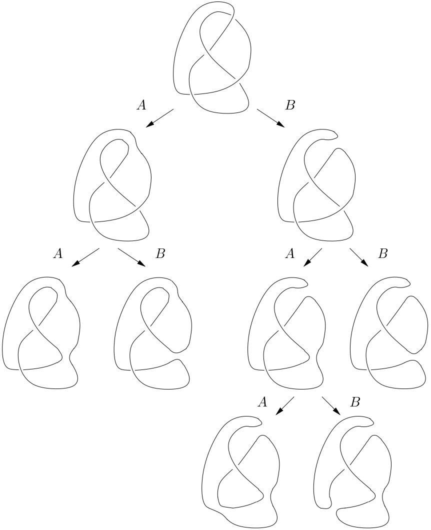
Spanning trees					
Activities	$LLdd$	$LdDd$	ℓDDd	ℓLdD	$\ell\ell DD$
Weights	A^{-8}	$-A^{-4}$	$-A^{4}$	1	A^{8}

$$\langle D\rangle = A^{-8} - A^{-4} + 1 - A^{4} + A^{8}, \quad \text{and writhe } w(D) = 0$$

$$\text{Let } t = A^{-4}: \qquad V_K(t) = t^{-2} - t^{-1} + 1 - t + t^{2}$$



The activity word for T determines the twisted unknot $U(T)$ as a partial splicing of the link diagram.



Each unknot $U(T)$ contributes a monomial to $\langle D \rangle$: Let $\sigma(U) = \#A - \#B$,

$$\langle D \rangle = \sum_U (-A)^{3w(U)} \cdot A^{\sigma(U)} = \sum_T \mu(T)$$

Tait Conjecture

Thistlethwaite proved Tait's Conjecture using the above spanning tree expansion.

Theorem(Thistlethwaite'87) If L is reduced and alternating then

- $V_L(t)$ is an alternating polynomial, and
- the $\text{span}(V_L(t)) = \#$ crossings.

Thus the crossing number of a reduced, alternating diagram is an invariant of the knot !

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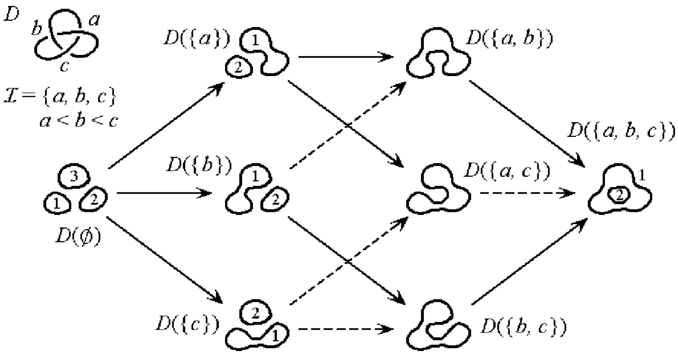
Kauffman, Murasugi and Turaev also proved Tait's conjecture using different methods.

Khovanov Homology



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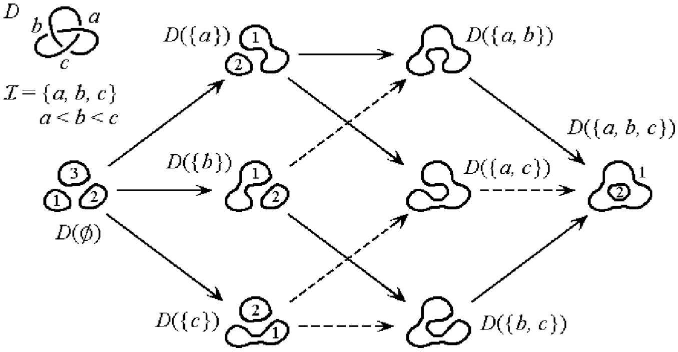

Bigraded complex $\{C^{i,j}(D), \partial\}$, with homology groups $H^{i,j}(D)$ which are invariant of the link and

$$\chi(H^{i,j}) = \sum_{i,j} (-1)^i q^j \mathrm{rank}(H^{i,j}) = (q + q^{-1}) V_L(q^2)$$

Viro's model for Khovanov homology

Enhanced Kauffman state s of D: Assign a $\pm$ sign to every loop.

Let $\sigma(s) = \#A - \#B$ and $\tau(s) = \# \bigcirc^{+} - \# \bigcirc^{-}$

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Bigrading: $i(s) = (w(D) - \sigma(s))/2$ and $j(s) = i(s) + w(D) - \tau(s)$

Bigraded complex:
$C^{i,j}(D) = \langle s \mid i(s) = i,\ j(s) = j \rangle \xrightarrow{\partial} C^{i+1,j}(D)$
where $s_1 \xrightarrow{\partial} s_2$ if and only if

- One A–smoothing)(in s_1 changed to B–smoothing $\asymp$ in s_2
  i.e. $\sigma(s_2) = \sigma(s_1) - 2$.
- Same signs on common loops of s_1 and s_2, and
  $\tau(s_2) = \tau(s_1) + 1$.

Reduced Khovanov homology

Choose a base point on D, and define reduced enhanced states to have $+$ sign on the loop which has the basepoint.

Reduced complex and reduced Khovanov homology is defined similarly.

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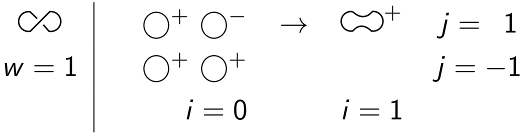
Example using reduced Khovanov homology:



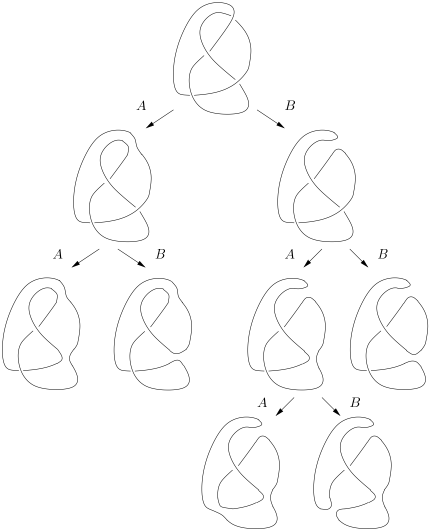
$\widetilde{H}^{i,j} = \mathbb{Z}^{(0,-1)}$

Example: 8_{19}

	0	1	2	3	4	5	6	7	8
20								$\frac{0}{1}$	$\xrightarrow{1} \frac{0}{1}$
18						$\frac{0}{1}$	$\xrightarrow{1} \frac{0}{9}$	$\xrightarrow{8} \frac{0}{12}$	$\xrightarrow{4} \frac{0}{4}$
16					$\frac{0}{5}$	$\xrightarrow{5} \frac{\mathbf{1}}{\mathbf{36}}$	$\xrightarrow{30} \frac{0}{55}$	$\xrightarrow{25} \frac{0}{31}$	$\xrightarrow{6} \frac{0}{6}$
14				$\frac{0}{11}$	$\xrightarrow{11} \frac{0}{73}$	$\xrightarrow{62} \frac{0}{127}$	$\xrightarrow{65} \frac{0}{92}$	$\xrightarrow{27} \frac{0}{31}$	$\xrightarrow{4} \frac{0}{4}$
12			$\frac{0}{13}$	$\xrightarrow{13} \frac{\mathbf{1}}{\mathbf{78}}$	$\xrightarrow{64} \frac{\mathbf{1}}{\mathbf{148}}$	$\xrightarrow{83} \frac{0}{127}$	$\xrightarrow{44} \frac{0}{55}$	$\xrightarrow{11} \frac{0}{12}$	$\xrightarrow{1} \frac{0}{1}$
10	$\frac{0}{1}$	$\xrightarrow{1} \frac{0}{8}$	$\xrightarrow{7} \frac{\mathbf{1}}{\mathbf{41}}$	$\xrightarrow{33} \frac{0}{78}$	$\xrightarrow{45} \frac{0}{73}$	$\xrightarrow{28} \frac{0}{36}$	$\xrightarrow{8} \frac{0}{9}$	$\xrightarrow{1} \frac{0}{1}$	
8	$\frac{0}{2}$	$\xrightarrow{2} \frac{0}{8}$	$\xrightarrow{6} \frac{0}{13}$	$\xrightarrow{7} \frac{0}{11}$	$\xrightarrow{4} \frac{0}{5}$	$\xrightarrow{1} \frac{0}{1}$			
6	$\frac{\mathbf{1}}{\mathbf{1}}$								

Ranks of the chain groups $\widetilde{\mathcal{C}}^{i,j}$, differentials, and homology $\widetilde{\mathcal{H}}^{i,j}$ of the reduced Khovanov chain complex for 8_{19}

Spanning tree model for Khovanov homology



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Every spanning tree T of Tait graph $G \longleftrightarrow$ twisted unknot $U(T)$

Bigrading on spanning trees: $u(T) = -w(U)$, $v(T) = E_+(T)$, where $w(U) =$ writhe of twisted unknot, $E_+(T) =$ number of positive edges in spanning tree T.

Spanning tree complex: $\mathbb{C}(D) = \bigoplus\limits_{u,v} \mathbb{C}^u_v(D)$, where $\mathbb{C}^u_v(D) = \mathbb{Z}\langle T \subset G |\ u(T) = u,\ v(T) = v\rangle$

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Spanning Trees $\longleftrightarrow$ Twisted unknots $\longleftrightarrow$ Contractible complex $\longleftrightarrow$ Fundamental cycle

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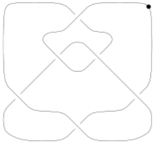
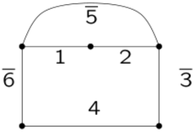
Theorem (C-Kofman) For a knot diagram D, there exists a spanning tree complex $\mathbb{C}(D) = \{\mathbb{C}_v^u(D), \partial\}$ with $\partial : \mathbb{C}_v^u \to \mathbb{C}_{v-1}^{u-1}$ that is a deformation retract of the reduced Khovanov complex,

$$\widetilde{H}^{i,j}(D;\mathbb{Z}) \cong H_v^u(\mathbb{C}(D);\mathbb{Z})$$

with $u = j - i + k_1$ and $v = j/2 - i + k_2$.

Example: 6-crossing trefoil





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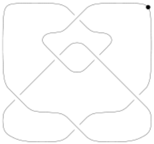

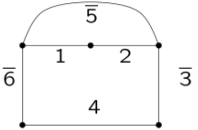

$v \backslash u$	-2	-1	0	1	2
3					$T_1 : LL\bar{d}D\overline{d}\overline{D}$ $T_3 : LL\overline{L}L\overline{d}\overline{d}$
2		$T_8 : \ell D\overline{D}D\overline{d}\overline{D}$ $T_{11} : \ell D\overline{L}L\overline{D}\overline{d}$	$T_9 : \ell D\bar{\ell}D\overline{D}\overline{D}$	$T_2 : LL\overline{L}d\overline{d}\overline{D}$ $T_7 : Ld\overline{D}D\overline{d}\overline{D}$ $T_6 : Ld\overline{L}L\overline{D}\overline{d}$	$T_4 : Ld\bar{\ell}D\overline{D}\overline{D}$
1	$T_{10} : \ell D\overline{L}d\overline{D}\overline{D}$		$T_5 : Ld\overline{L}d\overline{D}\overline{D}$		

Applications

Theorem (C-Kofman, Lee, Shumakovitch) The reduced Khovanov homology of an alternating knot is

- supported on one diagonal (H-thin),
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- supported on one diagonal (H-thin),
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Theorem (C-Kofman, Asaeda-Przytycki, Manturov) The reduced Khovanov homology of a non-split k-almost alternating link is supported on at most $(k+1)$ diagonals.

Related Open Problems

- Find a formula for the boundary of the spanning tree complex.

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- Find a formula for the boundary of the spanning tree complex.
- Study invariants coming from knot homologies e.g. Rasmussen's s invariant, in terms of Tait graphs and spanning trees.
- There is an associated spectral sequence coming from the spanning tree homology, which converges to Khovanov homology. Studying this spectral sequence would lead to more information about Khovanov homology.

Questions

Questions

Thank You

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Thank You

Slides available from :

http://www.math.csi.cuny.edu/abhijit/