Gradient descent assimilation for the point vortex model

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Outline

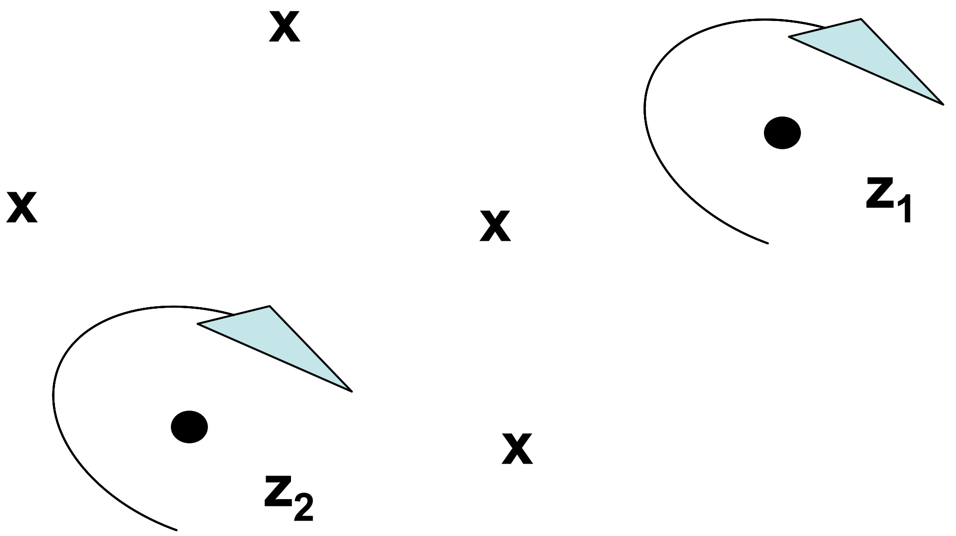
- Lagrangian data
- The point vortex model
- Gradient descent methodology
- Assimilation with full observations
- Partial observations
- Summary

Lagrangian data assimilation

- Much of the data for geophysical systems are Lagrangian
- Lagrangian observations refer to a sequence of position measurements along the flow trajectory
  - Data not in terms of model variables
- Solution is to augment the state space by including coordinates of Lagrangian tracers

Point vortex model

- We want to assimilate noisy observations of a passive tracer into the model of flow for point vortices



Point vortex model

- **Vortex dynamics governed by local fluid velocity; superposition of vortex velocities**

$$\frac{dz_m}{dt} = \frac{i}{2\pi} \sum_{l=1, m \neq l}^{N_v} \frac{\Gamma_l}{z_m^* - z_l^*}$$

where N_v = **No. vortices**

Γ_l = **Circulation strength**

- **Sequence of tracer positions, governed by**

$$\frac{d\xi_n}{dt} = \frac{i}{2\pi} \sum_{n,l=1}^{N_v} \frac{\Gamma_l}{\xi_n^* - z_l^*}$$

Point vortex model

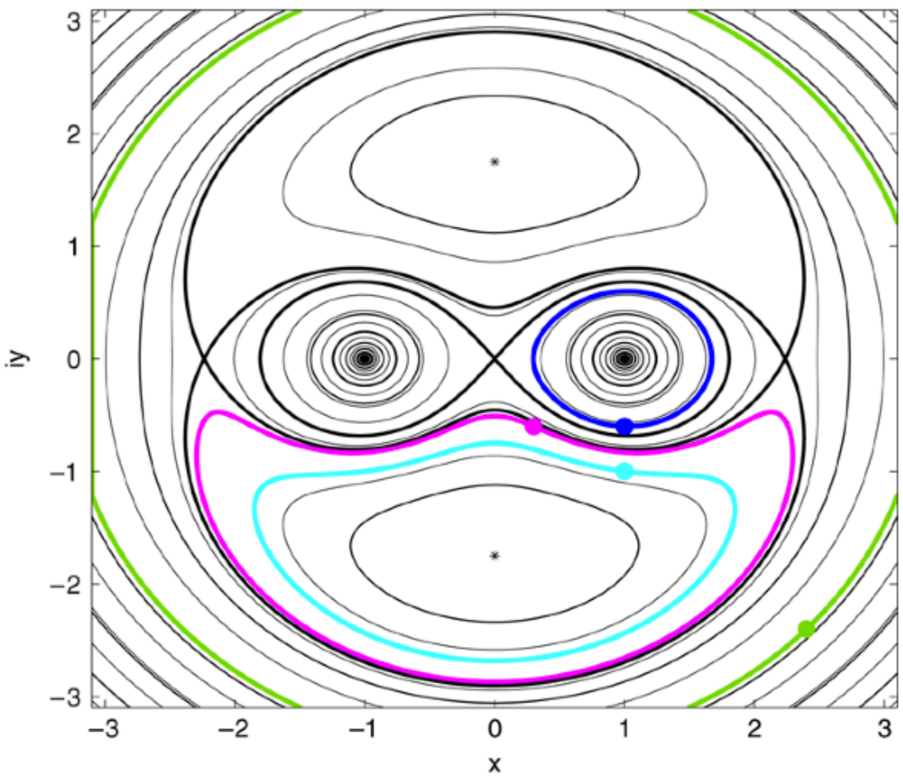
- Model of point vortex flow for 2 vortices, 1 tracer
- Assimilate tracer trajectories and solve equations of motion given a set of initial conditions
- Forecast vortex positions in the case of partial observations of the system
    - Comparison with EKF, Particle filter

Point vortex model

- Stream function for point vortex model

- Flow transformed to Lagrangian coordinates

- Tracer paths for different initial conditions can follow different types of flow



E. T. Spiller et al., Physica 237 (2008)

Existing methods

- **Failure rate for existing methods with different initial conditions**

	$0.3 - 0.6i$	$1 - 0.6i$	$1 - i$	$2.4 - 2.4i$
standard PF	7.8%	3.0%	4.6%	12.0%
standard BPF w/doubling	6.4%	2.6%	2.6%	9.6%
cloud expanding BPF	0.4%	0.2%	0.2%	4.0%
directed doubling BPF	0.4%	0.0%	0.0%	4.8%
perturbed observation BPF	1.6%	1.6%	2.0%	7.0%
extended Kalman filter	4.6%	78.2%	0.6%	2.0%

E. T. Spiller et al., Physica 237 (2008)

- **In some methods assimilation fails in the region of saddle points**

Gradient descent assimilation

- **Let $x_t \in \mathbb{R}^m$ be the trajectory of the model for a series of states at times *t=1…n*, given by**

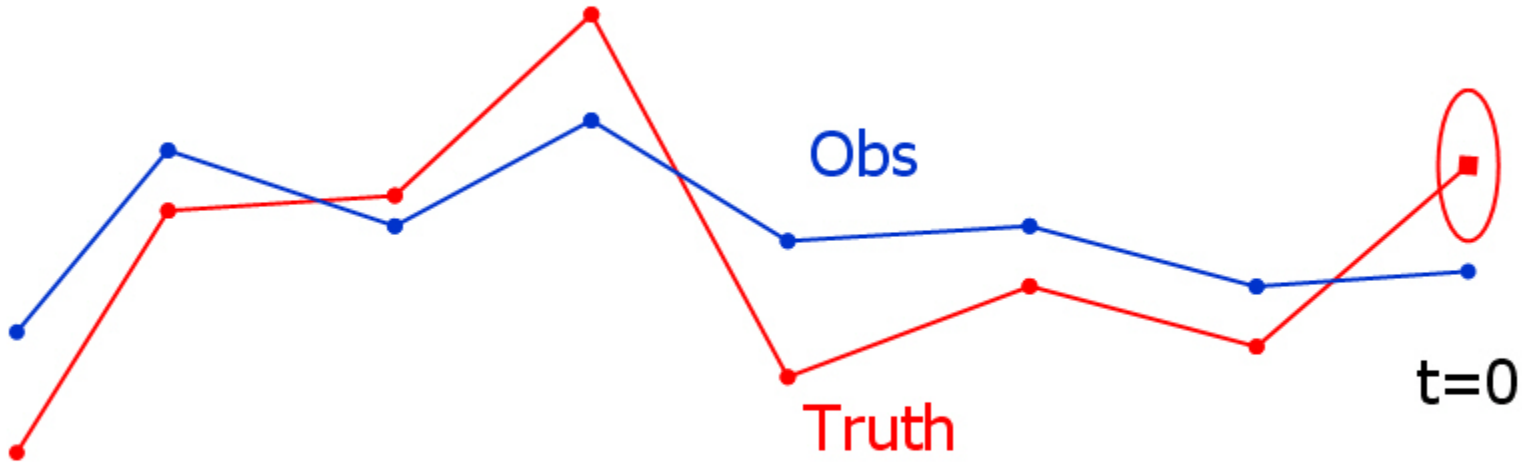
$$x_{t+1} = F(x_t)$$

- **We have a sequence of *n* noisy (IID distributed) observations, s_t, of the *m* dimensional system**

- **In the perfect model scenario *F* is known exactly, so is the noise model**

Gradient descent methodology

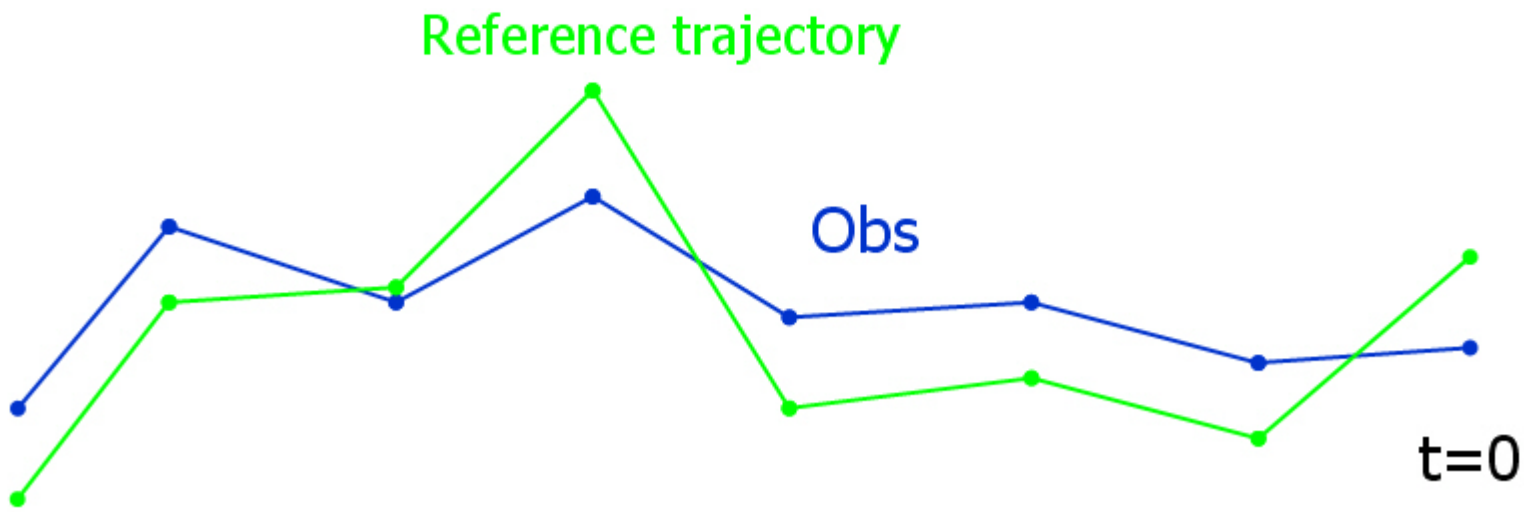
Goal: To generate a trajectory consistent with the model dynamics and observations



Kevin Judd Leonard Smith, Antje Weisheimer, Physica D 190 (2004)

Gradient descent methodology

How to find a reference trajectory?



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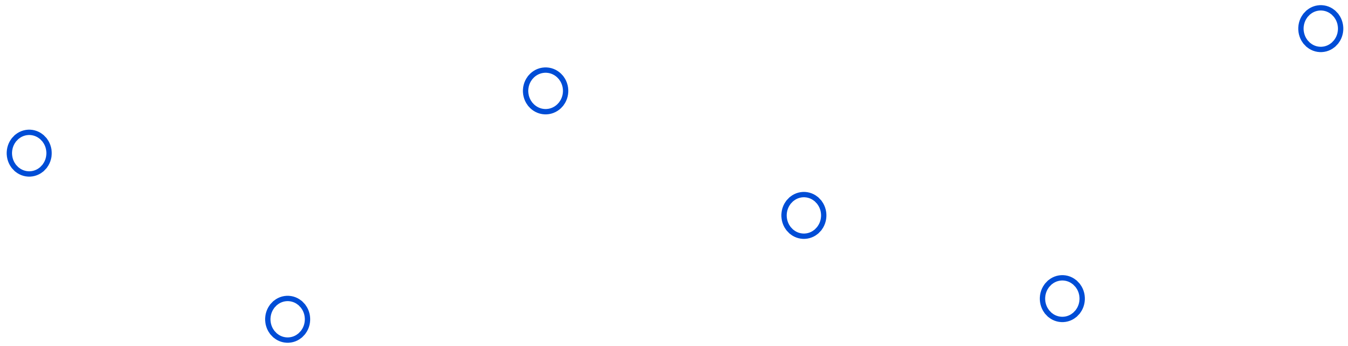
Gradient descent methodology

- **For the sequence of *n* observations each with *m* dimensions, define an extended sequence space of *$n \times m$* dimensions**
    - ***$6n$* for 2-point vortex model**
- **Create initial trajectory, *u*, by 1-step forecast from observations**



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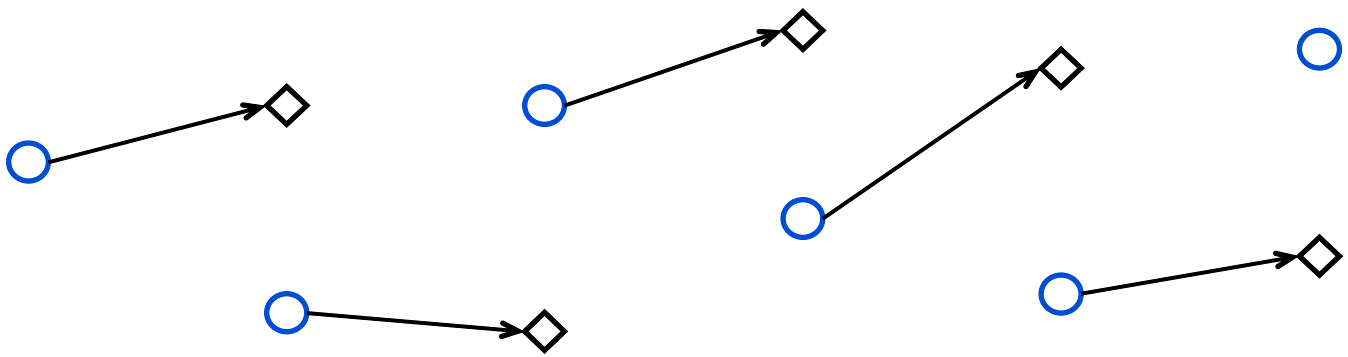

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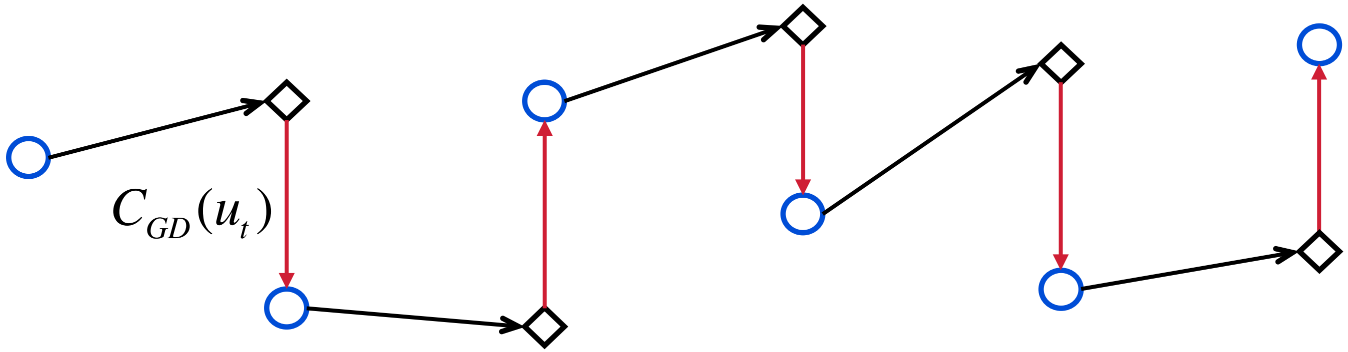
Gradient descent methodology

- **Define the mismatch (error cost function) as:**

$$C_{GD}(u) = \sum_{t=1}^{n} \left| F(u_t) - u_{t+1} \right|^2$$

- **Apply gradient descent algorithm, starting at the observations to minimise the set of mismatches simultaneously**



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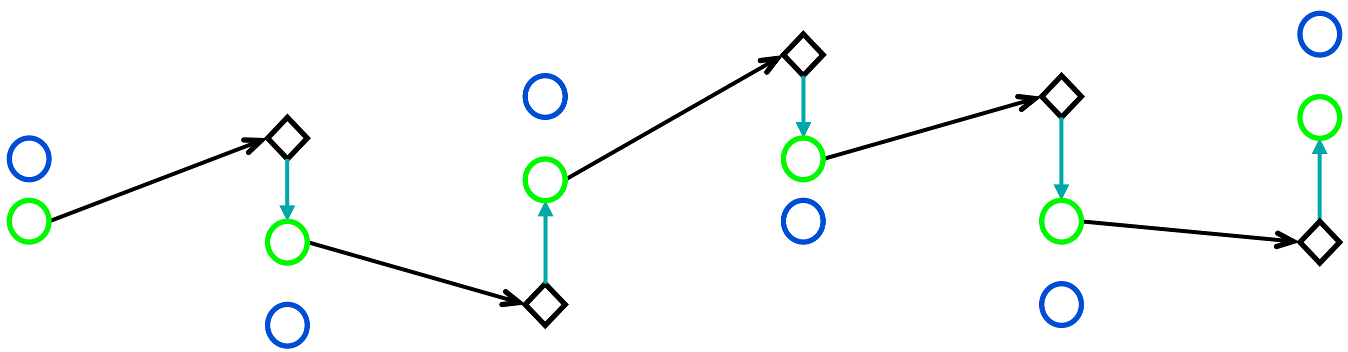

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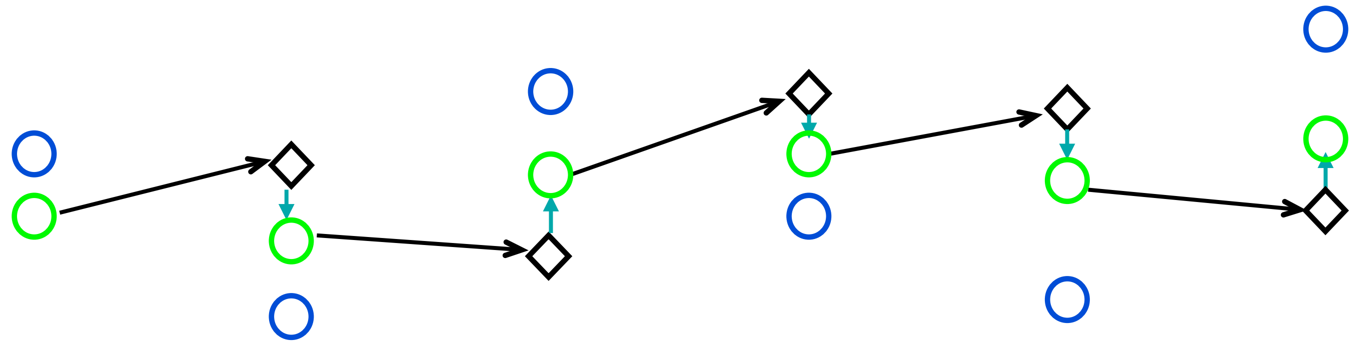

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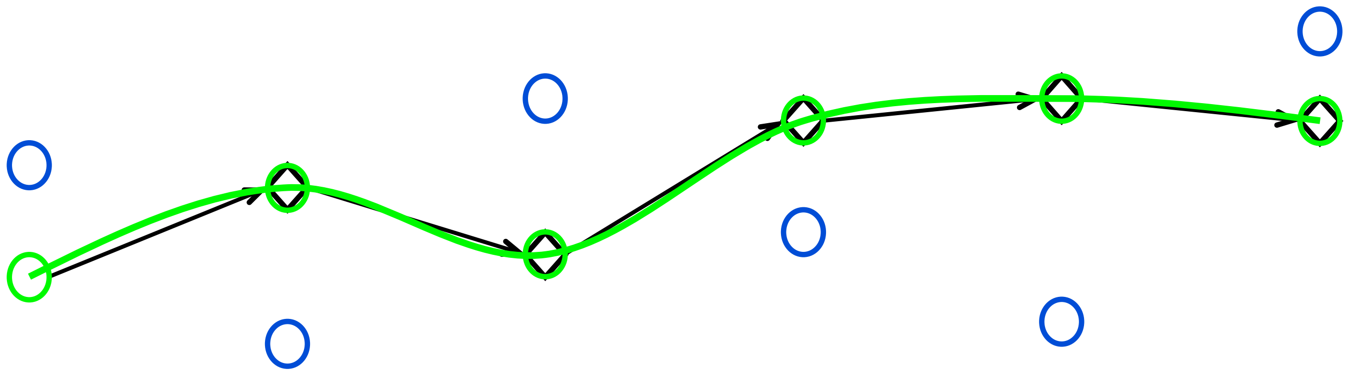

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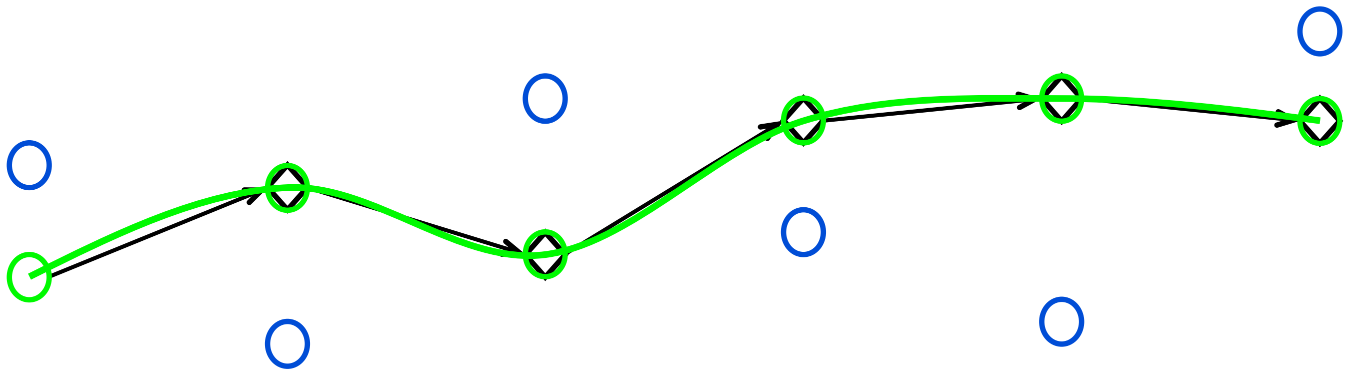
Gradient descent methodology

- **Assimilation converges to a trajectory of the model ‘near’ the ‘true’ trajectory**

- **Observations are used as a starting point, but ignored after initial descent**



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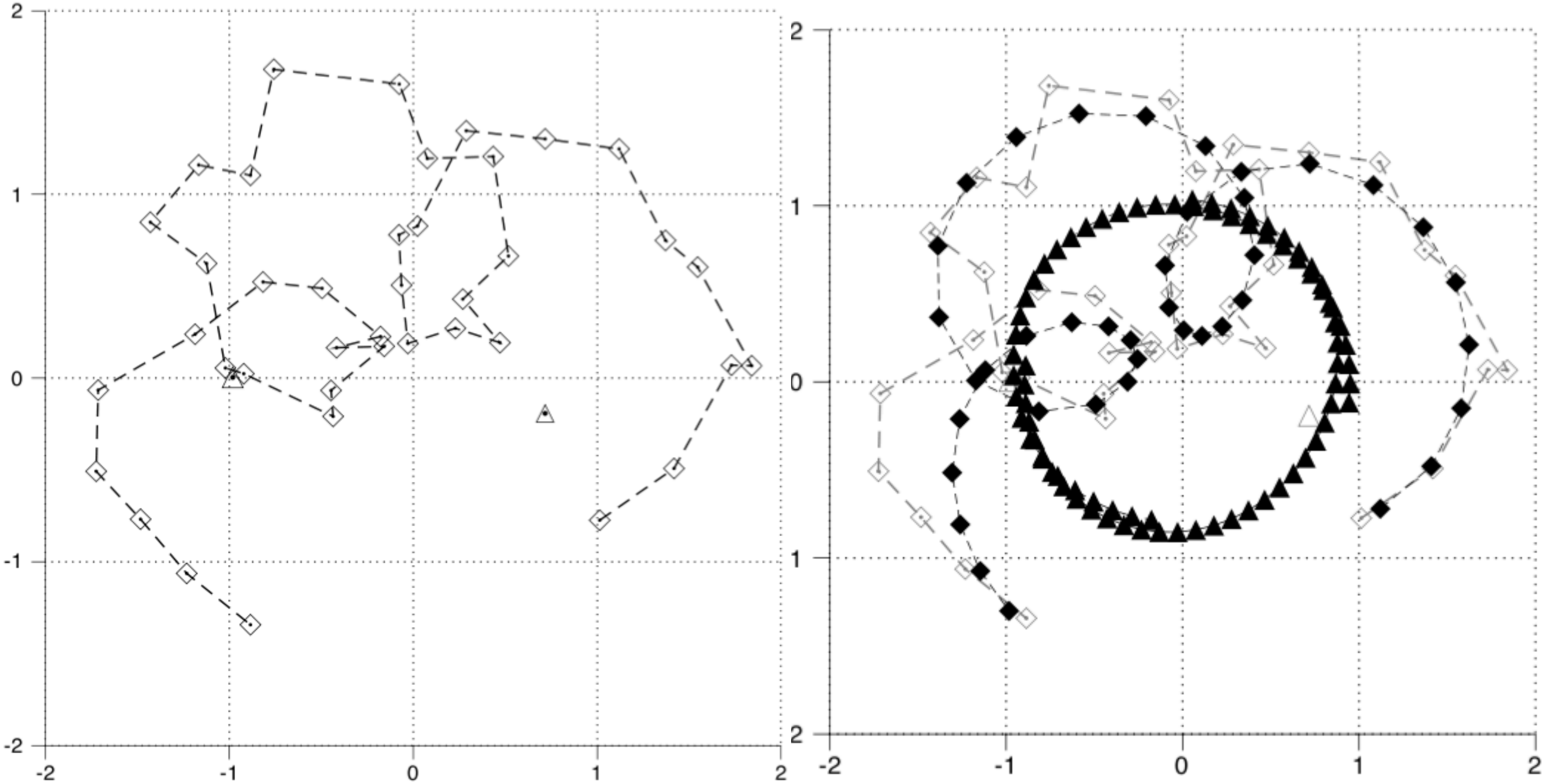
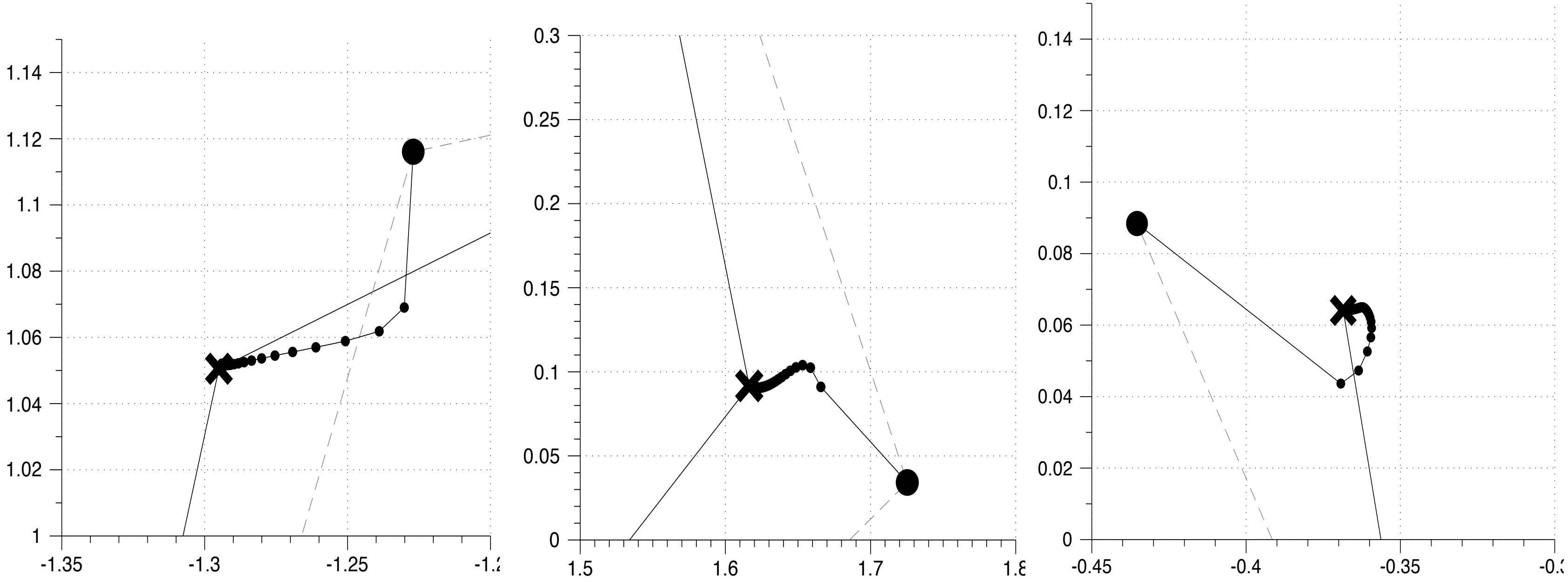
Assimilation with full observations

- **Tracer and vortex positions observed at all times**



Assimilation with full observations

- **Movement of individual drifter positions over the assimilation**



Partial observations

- **Initial locations of vortices are observed as well as tracer positions at all times**

- **Assimilation is performed in a two-stage process**

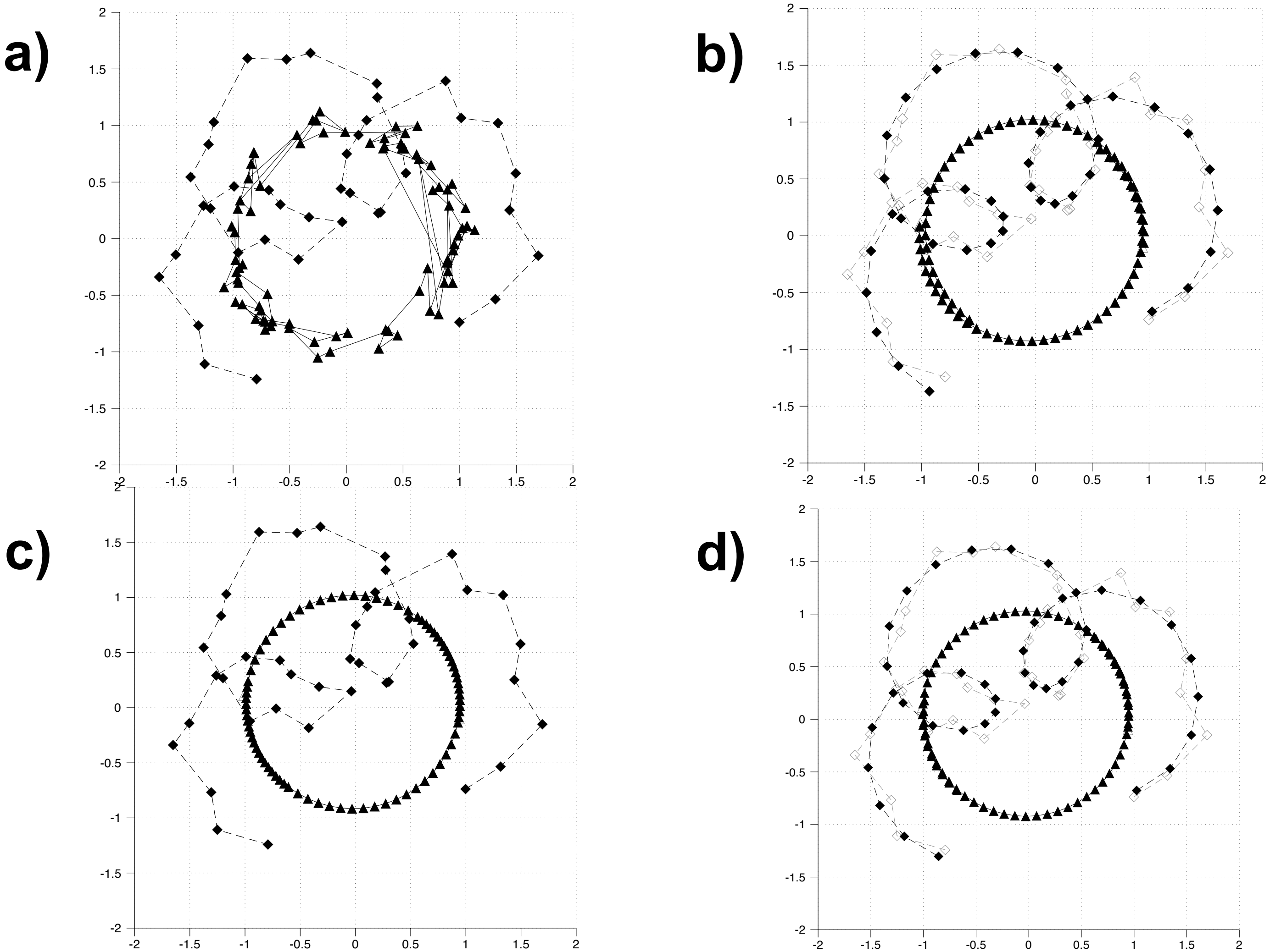
    - **Estimates of the unobserved components of state *x* are calculated by resampling from observational noise on initial vortex positions**

    - **Integrate equations forward to populate full state**

    - **Gradient descent to search for a trajectory**

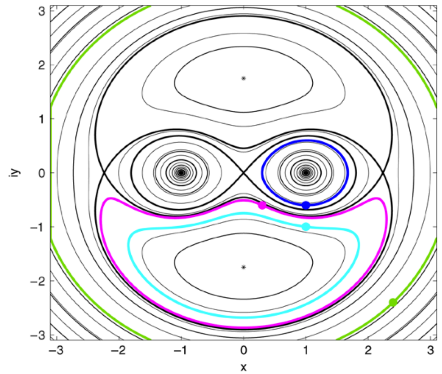
    - **Reset tracer positions to observed values and repeat with new estimate for vortex positions**

Partial observations



Saddle point problem

- **Some methods are known to fail to assimilate trajectories around saddle points**
- **Gradient descent assimilation has been shown overcome this problem**
- **Further investigations will be carried out in the future, including comparisons with available existing methods**



Summary

- Methodology for applying gradient descent assimilation in PMS has been shown to work well

- Work is ongoing to provide robust comparisons with other methods in the point vortex model (Particle filter, EKF)

- Algorithm successfully applied for the case of partial observations

- Gradient descent assimilation has been successfully applied to imperfect model scenarios and is shown to provide informative estimates of model error (see Lenny Smith's talk for more on model error)