Two Body & Three Body Scattering Theory

Formulation of Two Body Scattering

- Time-dependent Schrodinger Equation

$$i\frac{\partial}{\partial t}\Psi_a^{(+)}(t) = H\Psi_a^{(+)}(t)$$

ϕ_a

Its solution is

$$\Psi_a^{(+)}(t) = e^{-iHt}\Psi_a$$

Reference Wave-Packet

ϕ_a . *Its time dependence*

$$\phi_a(t) = e^{-iH_0t}\phi_a$$

$$\lim_{t\to\infty}\left\| e^{-iHt}\Psi_a^{(+)} - e^{-iH_0t}\phi_a \right\| = 0$$

- This physical requirement implies a certain restriction on the range of the potentials;
- Expressed mathematically

$$\Psi_a^{(+)} = s - \lim_{t \to -\infty} e^{iHt} e^{-iH_0 t} \phi_a \equiv \Omega^{(+)} \phi_a$$

- This equation defines the Moller operator

$$\Omega^{(\pm)} = s - \lim_{t \to \mp\infty} e^{iHt} e^{-iH_0 t}$$

- The Moller operator $\Omega^{(-)}$ produces the scattering state $\Psi_a^{(-)}(t)$ which goes over into the free state for $\phi_a(t)$ $t \to \infty$
- Extending the definition for any finite time

$$s - \lim_{t \to \mp} e^{iH(t+\tau)} e^{-iH_0(t+\tau)} = e^{iH\tau} \Omega^{(\pm)} e^{-iH_0\tau}$$

- $= \Omega^{(\pm)}(\tau)$

- By differentiation, we get

$$0 = \frac{d\Omega^{(\pm)}}{d\tau} = e^{iH\tau}(H\,\Omega^{(\pm)} - \Omega^{(\pm)}\,H_0)e^{-iH_0\tau}$$

- Giving us the commutation relation.

$$H\,\Omega^{(\pm)} = \Omega^{(\pm)}\,H_0$$

- Domain of Moller operators is the Hilbert space of square integrable free states.

- Work by Faddeev has shown that the domain can , however, be extended to include states of sharp momentum.
- We are thus allowed to use momentum eigen states
- $|\vec{p}\rangle$ instead of wave-packets ϕ_a and
- Apply the Moller operators $\Omega^{(+)}$ on them to get scattering states $|p\rangle^{+}$ consisting of a plane wave plus an outgoing spherical wave.
- The commutation relation given above shows

- Applying the Hermitian conjugate operator $\Omega^{(\pm)^+}$
- to the bound states of H, i.e., $|\Psi_n\rangle$ they give zero;

$$\left\langle \Omega^{(\pm)^+}\Psi_n \middle| \vec{p} \right\rangle = \left\langle \Psi_n \middle| \Omega^{(\pm)} ! \vec{p} \right\rangle = \left\langle \Psi_n \middle| \vec{p} \right\rangle^{(\pm)}$$

- Expression on the r.h.s is zero because states
- $|\Psi_n\rangle$ *and* $|p\rangle^{(\pm)}$ are eigenstates of H having different energies. Since states $|\vec{p}\rangle$ form a complete set, $\Omega^{(\pm)^+}|\Psi_n\rangle = 0$

Time limit in $\Omega^{(\pm)}$ can be replaced by Euler limit:

$$\Omega^{(\pm)} = s - \lim_{t \to \mp\infty} e^{iHt} e^{-iH_0 t}$$

$$= s - \lim_{t \to \mp\infty} \pm \varepsilon \int_{\mp}^{0} dt\, e^{\pm \varepsilon t} e^{iHt} e^{-iH_0 t}$$

Note

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{0} dt\, \varepsilon\, e^{\varepsilon t} f(t) = f(\infty) \int_{-\infty}^{0} dx\, e^{x} = f(\infty)$$

By introducing Euler limit, time dependence disappears in scattering theory

Resolvent Equation & L-S Equation

- We can not carry out the integration in the integral representation for $\Omega^{(\pm)}$, for the simple reason H_0 *and* H do not commute; the two exponential functions
- $$e^{iHt}\, e^{-iH_0 t} \neq e^{i(H-H_0)t}$$
- Since Moller operators can operate on plane wave states, we write

$$|\vec{p}\rangle^{(\pm)} = \lim_{\varepsilon\to 0} \pm\varepsilon \int_{\mp\infty}^{0} dt\, e^{\pm\varepsilon t} e^{iHt}\, e^{-iEt} |\vec{p}\rangle$$

$$= \lim_{\varepsilon \to 0} \pm \varepsilon \int_{\mp}^{0} dt\, e^{i(H - E \mp i\varepsilon)t} \left| \vec{p} \right\rangle = \lim_{\varepsilon \to 0} \pm i\varepsilon (E \pm i\varepsilon - H)^{-1} \left| \vec{p} \right\rangle$$

- Here we get the definition of a resolvent or Green's function $g(z) = (z - H)^{-1} \qquad (z = E \pm i\varepsilon)$
- And obtain an important relation

$$\left| \vec{p} \right\rangle^{\pm} = \lim_{\varepsilon \to 0} \pm i\varepsilon\, g(z) \left| \vec{p} \right\rangle$$

- Which marks a transition from time-dependent to

- Introducing the free resolvent

$$g_0(z) \equiv (z - H_0)^{-1}$$

- And using the relation

$$V = H - H_0 = g_0^{-1}(z) - g^{-1}(z)$$

- We get, by multiplying with $g(z)\, g_0(z)$
- The operator identities
- $$g(z) = g_0(z) + g(z)\, V\, g_0(z),$$
$$= g_0(z) + g_0(z) V\, g(z)$$

- With this identity and the relation.

$$\left|\vec{p}\right\rangle^{\pm} = \lim_{\varepsilon\to 0} \pm i\,\varepsilon\, g(E \pm i\,\varepsilon)\left|\vec{p}\right\rangle$$

$$= \lim_{\varepsilon\to 0} \pm i\,\varepsilon\left[g_0(E \pm i\,\varepsilon) + g_0(E \pm i\,\varepsilon)\,Vg(E \pm i\,\varepsilon)\,\right]\left|\vec{p}\right\rangle$$

- Now.

$$\lim_{\varepsilon\to 0} \pm i\,\varepsilon\, g_0(E \pm i\,\varepsilon)\left|\vec{p}\right\rangle = \lim_{\varepsilon\to 0} \pm i\,\varepsilon \frac{1}{E \pm i\,\varepsilon - H_0}\left|\vec{p}\right\rangle = \left|\vec{p}\right\rangle$$

- This gives the well-known L-S equation

$$\left|\vec{p}\right\rangle^{\pm} = \left|\vec{p}\right\rangle + g_0(E \pm i\,0)V\left|\vec{p}\right\rangle^{\pm}$$

- In configuration space representation

$$\langle \vec{r} | g_0(E \pm i0) | r' \rangle = -\frac{\mu}{2\pi} \frac{e^{\pm i\sqrt{2\mu E}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

Note that the free resolvent $g_0(E)$ has a cut along the positive real E-axis. On which side of the cut we should stay in order to fulfil the required boundary condition is decided by the $\varepsilon - \lim it$. The L-S equation in configuration space

$$\langle \vec{r} | \vec{p} \rangle^{(+)} = \langle \vec{r} | \vec{p} \rangle - \int d\vec{r}' \frac{\mu}{2\pi} \frac{e^{i\sqrt{2\mu E}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}') \langle \vec{r}' | \vec{p} \rangle^{(+)}$$

- This shows Schmidt norm can be finite when (i) the integral over absolute square of the potential exists –excluding Coulomb and hard core potentials;
- (ii) the imaginary part of $\sqrt{2\mu z}$ is not zero—excluding thereby the scattering energies'
- Reason? Recall: the domain of Moller operators
- has been extended!
- A finite Schmidt norm is only a sufficient condition not a necessary one

- In order that L-S equation has a unique solution, it is sufficient to investigate if Schmidt norm of the kernel is finite, viz.,

$$\|K\|_S = \left[Tr(K^+K)\right]^{1/2} = \left[\int\int d\vec{r}\, d\vec{r}\,' \left|K(\vec{r},\vec{r}\,')\right|^2\right]^{1/2}$$

- The square of the Schimdt norm

$$\|K\|_S^2 = \frac{\mu^2}{4\pi^2}\int\int d\vec{r}\, d\vec{r}\,' \frac{e^{-2\,\mathrm{Im}(\sqrt{2\mu z})|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|^2}\left|V(\vec{r},\vec{r}\,')\right|^2$$

$$= \frac{\mu^2}{4\pi^2} 4\pi \int dR\, e^{-2\,\mathrm{Im}(\sqrt{2\mu z})R} \int d\vec{r}\,' \left|V(\vec{r}\,')\right|^2$$

$$= \frac{\mu^2}{\pi}\frac{1}{2\,\mathrm{Im}(\sqrt{2\mu z})}\int d\vec{r}\,' \left|V(\vec{r}\,')\right|^2$$

- A necessary condition is that the kernel of the integral equation must be compact(completely continuous).

- Lovelace (Phys.Rev.135, B1225 (‘64)) shows that the kernel remains compact in the Banach space of continuous bounded functions with continuous bounded derivatives.

- S-matrix: A link between the scattering states and the measurable data:

$$S_{ba} = \lim_{t\to\infty} \left\langle \phi_b(t) \middle| \Psi_a^{(+)}(t) \right\rangle = \lim_{t\to\infty} \left\langle e^{-iH_0 t}\phi_b \middle| e^{-iHt}\Psi_a^{(+)} \right\rangle$$

$$= \lim_{t\to\infty} \left\langle e^{iHt} e^{-iH_0 t}\phi_b \middle| \Psi_a^{(+)} \right\rangle = \left\langle \Omega^{(-)}\phi_b \middle| \Omega^{(+)}\phi_a \right\rangle$$

- This enables us to express

$$S_{ba} = \left\langle \Psi_b^{(-)} \middle| \Psi_a^{(+)} \right\rangle = \left\langle \phi_b \middle| \Omega^{(-)^+} \Omega^{(+)} \middle| \phi_a \right\rangle$$

- The S-matrix in operator form is $S = \Omega^{(-)^{+}}\Omega^{(+)}$
- With momentum eigen states, its matrix-element

$$S_{\vec{p}'\,\vec{p}} = \left\langle \vec{p}' \middle| \Omega^{(-)^{+}}\,\Omega^{(+)} \middle| \vec{p} \right\rangle = \left\langle \vec{p}' \middle| \Omega^{(-)^{+}} \middle| \vec{p} \right\rangle^{(+)}$$

$$= \lim_{t\to\infty} \left\langle \vec{p}' \middle| e^{iH_0 t} e^{-iHt} \middle| \vec{p} \right\rangle^{(+)}$$

$$= \lim_{t\to\infty} \left\langle e^{-iE't}\, \vec{p}' \middle| e^{-iEt} \middle| \vec{p}' \right\rangle^{(+)}$$

$$S_{\vec{p}'\,\vec{p}} = \lim_{t\to\infty} \lim_{\varepsilon\to 0} i\varepsilon\, e^{i(E'-E)t} \left\langle \vec{p}' \middle| g(E+i\varepsilon) \middle| \vec{p} \right\rangle$$

$$S_{\vec{p}'\,\vec{p}} = \lim_{t\to\infty} \lim_{\varepsilon\to 0} i\varepsilon\, e^{i(E'-E)t} \left\langle \vec{p}' \middle| g(E+i\varepsilon) \middle| \vec{p} \right\rangle$$

- Recall the Green's function satisfies L-S type equation. $$g(z) = g_0(z) + g_0(z) V g(z)$$

- This eqn can be recast as

$$g(z) = g_0(z) + g_0(z) V [g_0(z) + g(z) V g_0(z)]$$

$$= g_0(z) + g_0(z) V g_0(z) + g_0 V g(z) V g_0(z)$$

$$= g_0(z) + g_0(z) [V + V g(z) V] g_0(z)$$

$$= g_0(z) + g_0(z) t(z) g_0(z)$$

- The resolvent is thus related to the operator t(z), -- the t-matrix , which is less singular than g(z).

- In momentum representation, the kinematic singularities of g(z) show up explicitly:

$$\langle \vec{p}' | g(z) | \vec{p} \rangle = \frac{\delta(\vec{p}' - \vec{p})}{z - p^2/2\mu} + \frac{\langle \vec{p}' | t(z) | \vec{p} \rangle}{(z - p'^2/2\mu)(z - p^2/2\mu)}$$

- Using this result in S-matrix.

$$S_{\vec{p}'\vec{p}} = \lim_{t\to\infty} \lim_{\varepsilon\to 0} i\varepsilon e^{i(E'-E)t} \left[\frac{\delta(\vec{p}' - \vec{p})}{i\varepsilon} + \frac{\langle \vec{p}' | t(E+i\varepsilon) | \vec{p} \rangle}{i\varepsilon(E + i\varepsilon - E')} \right]$$

$$= \delta(\vec{p}' - \vec{p}) - \lim_{t\to\infty} \lim_{\varepsilon\leftarrow 0} \frac{e^{i(E'-E)t}}{E' - E - i\varepsilon} \langle \vec{p}' | t(E+i\varepsilon) | \vec{p} \rangle$$

- Here we use $z = E + i\varepsilon;\ E = p^2/2\mu$ in the expression for the S=matrix . In the second term,

- Using the identity

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \frac{e^{i\omega t}}{\omega - i\varepsilon} = 2\pi i\,\delta(\omega), \quad we \quad get$$

$$S_{\vec{p}'\vec{p}} = \delta(\vec{p}' - \vec{p}) - 2\pi i\,\delta(p'^2/2\mu - p^2/2\mu)\langle \vec{p}' | t(E+i\varepsilon) | \vec{p} \rangle$$

- Note the two $\delta -$ functions in the S-matrix. One refers to "no scattering (since $\vec{p}' = \vec{p}$)" and the other stands for energy conservation. All information about scattering is contained in t-matrix, related to

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \mu^2 \left| \langle \vec{p}' | t(E+i\varepsilon) | \vec{p} \rangle \right|^2$$

Unitarity Relation & Optical Theorem

- For a Hermitian potential, unitarity of S-matrix, i.e

$$S^+S = I$$

- ensures conservation law for probability flux. In operator form,

$$S = 1 - 2\pi i t \quad and \quad S^+ = 1 + 2\pi i t^+$$

$$leads\ \ to$$

$$S^+S = 1 - 2\pi i t + 2\pi i t^+ + 4\pi^2 t^+ t = 1$$

- Writing explicitly

$$t(E+i0) - t^+(E+i0) = -2\pi i t(E+i0)\delta(E-H_0)t^+(E+i0)$$

- It follows

$$\mathrm{Im}\langle \vec{p} | t(E+i0) | \vec{p} \rangle = -\pi \int d\,\vec{p}''\,\delta(E - p''^2/2\mu) \left| \langle \vec{p} | t(E+i0) | \vec{p}'' \rangle \right|^2$$

$$= -\pi \frac{1}{(2\pi)^4 \mu^2} \int p''^2 d\,p'' d\Omega_{p''} \delta(p^2/2\mu - p''^2/2\mu) \frac{d\sigma}{d\Omega_{p''}}$$

$$= -\frac{p}{16\pi^3 \mu} \sigma_{tot}$$

- This is the well-known optical theorem.

- Before studying three-body scattering, let us recall the relations between the two body t-matrix and the resolvent $g(z) = g_0 + g_0 \, t \, g_0$
- and comparing it with

$$g(z) = g_0 + g_0 V \, g(z)$$

$$= g_0 + g_0 V [g_0 + g_0 \, t \, g_0]$$

$$= g_0 + g_0 [V + V \, g_0 \, t] g_0$$

- leads to the relation $t = V + V \, g_0 \, t$
- Or ,equivalently $t = V + t \, g_0 V$

Three Body Scattering

- Introduction: Consider three particles 1,2,3 having masses $m_1,\ m_2\ \ and\ \ m_3$ interacting through two body potentials, denoted by

$$V^1 = V_{23},\ V^2 = V_{31},\ V^3 = V_{12}$$

- We write its Hamiltonian

$$H = H_0 + \sum_{i=1}^{3} V^i$$

- In three particle space, two body sub-systems have an important role. Recognizing this, we introduce channel Hamiltonians:

$$H^i = p_i^2 / 2\mu_i + q_i^2 / 2M_i + V^i$$

$$H = H_0 + \sum_{i=1}^{3} V^i$$

- Note that because of translational invariance of two body potentials, three body C.M motion becomes a momentum eigenstate and can be factored out resulting in two independent momenta, $\vec{p}^{\,i}$ *and* $\vec{q}^{\,i}$
- Asymptotic states with two particles in a bound state, as for example, in elastic scattering are eigen states of H^i
- For the full Hamiltonian

$$H = H^i + \bar{V}^i, \ where \ \bar{V}^i = V - V^i$$

- Generalizing the definition of Moller operators and boundary conditions to three particle scattering, we write time-development of a 3-particle wavepacket
- While reference wavepackets develop according to

$$\Psi_m^{i(+)}(t) = e^{-iHt}\Psi_m^{i(+)}$$

$$\phi_m^{i(+)}(t) = e^{-iH^i t}\phi_m^i$$

$$\phi_m^i$$

- The wavepacket ϕ^0 describes the free motion of particle i (i=1,2,3) relative to the other two particles which are in their mth bound state. For i=0, we can have a wavepacket with three free particles.

- The boundary condition is formulated by requiring

$$\lim_{t\to-\infty}\left\| e^{-iHt}\Psi_m^{i(+)} - e^{-iH^i t}\phi_m^i \right\| = 0$$

- This leads to the representation

$$\Psi_m^{i(\pm)} = s - \lim_{t\to\mp\infty} e^{iHt}e^{-iH^i t}\phi_m^i \equiv \Omega^{i(\pm)}\phi_m^i$$

- And to the definition of Moller operators

$$\Omega^{i(\pm)} = s - \lim_{t\to\mp\infty} e^{iHt}e^{-iH^i t} \qquad (i = 0,1,2,3)$$

- Note that here a separate Moller operator is needed for every channel i--- signalling the complications in 3- body problem. Indeed, one is required to prove that Moller operators exist and

- That they can be applied to states of sharp energy. The study of these mathematical details has been carried out by Faddeev.
- The domain of Moller operators defined above is the space of channel states:

$$\left|\Psi_m^{i(\pm)}\right\rangle = \Omega^{i(\pm)}\left|\phi_m^i\right\rangle$$

- Transition from time limit to Euler limit is also possible in 3-particle scattering:

$$\left|\Psi_m^{i(\pm)}\right\rangle = \lim_{\varepsilon\to 0} \pm i\,\varepsilon\, G(E \mp i\,\varepsilon)\left|\phi_m^i\right\rangle$$

- G(z) is the full resolvent satisfying

$$G(z) = G^i(z) + G^i(z)\bar{V}^i G(z)$$

$$= G^i(z) + G(z)\bar{V}^i G^i(z)$$

Difficulty with the L-S Equation

- In terms of the channel resolvents (i).
- The L-S equation for the scattering state

$$\left|\Psi_m^{i(\pm)}\right\rangle = \left|\phi_m^i\right\rangle + G^i(E \pm i0)\bar{V}^i\left|\Psi_m^{i(\pm)}\right\rangle$$

- Note that

$$\lim_{\varepsilon \to 0} \pm i\varepsilon G^i(E \pm i\varepsilon)\left|\phi_m^i\right\rangle = \left|\phi_m^i\right\rangle$$

- Using this result, we get the L-S equation

$$\left|\Psi_m^{i(\pm)}\right\rangle = \left|\phi_m^i\right\rangle + G^i(E \pm i\varepsilon)\bar{V}^i\left|\Psi_m^{i(\pm)}\right\rangle$$

- Solution to the L-S equation is not uniquely determined.

- Because the homogeneous equation

$$\left|\Psi\right\rangle = G^{i}(E+i0)\overline{V}^{i}\left|\Psi\right\rangle$$

- has solutions for energies in the scattering region. Consider, for instance, another channel state

$$\left|\phi_n^j\right\rangle \quad j \neq i.$$

- For such a state

$$\lim_{\varepsilon\to 0} \pm i\varepsilon G^{i}(E+i\varepsilon)\left|\phi_n^j\right\rangle = 0$$

- Because the state $\left|\phi_n{}^j\right\rangle j\neq i$ is not an eigenstate of H^{i}
- Therefore, we get $\left|\Psi_n^{j(\pm)}\right\rangle = G^{i}(E+i0)\overline{V}^{i}\left|\Psi_n^{j(\pm)}\right\rangle$
- This equation tells us that the homogeneous eqn which belongs to the inhomogeneous L-S eqn. has non trivial solutions of the scattering states
- in channels $j \neq i$

Kernel Disconnected

$$\left\langle \vec{k}_1 \vec{k}_2 \vec{k}_3 \middle| g_0 \bar{V}^i \middle| \vec{k}_1' \vec{k}_2' \vec{k}_3' \right\rangle = \left(z - \sum_{i=1}^{3} k_i^2 / 2m_i \right)^{-1} \sum_{k \neq i} \delta(\vec{q}_k - \vec{q}_k') \left\langle \vec{p}_k \middle| V^k \middle| \vec{p}_k' \right\rangle$$

Here the $\delta-$ function belonging to the total C.M motion is omitted . However, the $\delta-$ functions associated with the two body potentials signify that the third particle remains free when the other two are interacting –the kernel $g_0 V$ of the integral equation thus becomes disconnected . As a result the matrix-element of the kernel of L-S equation is not an $£^2$ operator.

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- . In momentum representation, the matrix element of the kernel

$$\left\langle \vec{k}'_1 \vec{k}'_2 \vec{k}'_3 \middle| K \middle| \vec{k}_1 \vec{k}_2 \vec{k}_3 \right\rangle = \frac{\delta(\vec{k}'_1+\vec{k}'_2+\vec{k}'_3-\vec{k}_1+\vec{k}_2+\vec{k}_3)}{E+i\varepsilon-\sum_{i=1}^{3}(k_i^2/2m_i)} \times$$

$$\left[\delta(\vec{k}'_1-\vec{k}_1)\langle \vec{p}'_1|V^1|\vec{p}_1\rangle+\delta(\vec{k}'_2-\vec{k}_2)\langle \vec{p}'_2|V^2|\vec{p}_2\rangle+\delta(\vec{k}'_3-\vec{k}_3)\langle \vec{p}'_3|V^3|\vec{p}_3\rangle\right]$$

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- In the C.M of the 3-particle system, one can get rid of the total momentum conserving $\delta -$ function.But the integrand still contains terms like $\left[\delta(\vec{k}'_1-\vec{k}_1)\right]^2$
- As a result, its $Tr(K K^+)$ is infinite and the kernel K is no longer £^{2}

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The Faddeev Equations

- In order to remove these difficulties, Faddeev's starting point is to study the L-S equation for the three body T-matrix:

$$T = \sum_{i=1}^{3} V^{i} + \sum_{i=1}^{3} V^{i} G_0 T$$

- This equation also suffers from the same disadvantage as the L-S equation for the three particle states $\left|\Psi_m^{i\pm}\right\rangle$, viz.,the kernel is no longer Hilbert-Scmidt type together with the non-uniqueness of the L-S equation.

- The turning point in Faddeev's approach is to suggest that the scattering operator can be split into three parts as

$$T(E) = T^{(1)}(E) + T^{(2)}(E) + T^{(3)}(E) \ ,$$

- where $T^{(1)}(E)$ represents the sum of all the diagrams contributing to T(E) in which particle 2 and 3 are the last to interact. Diagramatically, each of the three parts are as shown.
- Splitting the basic L-S equation into 3-equations

$$T^{(i)} = V^{(i)} + V^{(i)} G_0 T \qquad\qquad (i = 1,2,3)$$

- Substituting for the T operator in terms of its components, we write

$$T^{(1)} - V^{(1)} G_0 T^{(1)} = V^{(1)} + V^{(1)} G_0 \left(T^{(2)} + T^{(3)} \right)$$

- Multiplying by $\left(1 - V^{(1)} G_0\right)^{-1}$ from the left, yields

$$T^{(1)} = \left(1 - V^{(1)} G_0\right)^{-1} V^{(1)} + \left(1 - V^{(1)} G_0\right)^{-1} V^{(1)} G_0 \left(T^{(2)} + T^{(3)} \right)$$

- Note the first term, which by iteration can be shown to be a two body t-operator

$$t_1 = \left(1 - V^{(1)} G_0\right)^{-1} V^{(1)}$$

- so is the first part of the second term. One can

$$T^{(1)} = t_1 + t_1 G_0 \left(T^{(2)} + T^{(3)} \right)$$

Faddeev Equations

- Similarly the other two components $T^{(2)}$ and $T^{(3)}$ can be transformed and the set of three coupled T-matrices can be expressed in the matrix form

$$\begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + \begin{pmatrix} 0 & t_1 & t_1 \\ t_2 & 0 & t_2 \\ t_3 & t_3 & 0 \end{pmatrix} G_0 \begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix}$$

- These coupled equations are the Faddeev Eqns.

- These new equations exhibit two strking features: (i) while the two body t-matrices appearing in the kernel are indeed disconnected, but since only the off-diagonal elements of $K_{i\,j}$ are present, any iteration of the kernel will suppress the trouble some δ- -functions. On iterating once the Faddeev equations take the form

$$\begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + [K] \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + [K^2] \begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix}$$

- With

$$[K^2] = \begin{bmatrix} t_1 G_0 t_2 + t_1 G_0 t_3 & t_1 G_0 t_3 & t_1 G_0 t_2 \\ t_2 G_0 t_3 & t_2 G_0 t_1 + t_2 G_0 t_3 & t_2 G_0 t_1 \\ t_3 G_0 t_2 & t_3 G_0 t_1 & t_3 G_0 t_1 + t_3 G_0 t_2 \end{bmatrix}$$

- We see that the kernel $[K^2]$ contains only the connected elements. As for example the element
- K^2_{12} can be interpreted through a diagram as:
- The second feature is that now the two particle T-matrices appear as operators in 3-particle space. The evaluation of the operator product

- $t_i G_0 T^i$ implies the integration over all possible intermediate states $|\vec{p}_i''\rangle$, $|\vec{q}_i''\rangle$ involving the independent momenta, $\vec{p}_i'', \vec{q}_i''$
- where now $p_i^2/2\mu_i \neq z - q_i''^2/2M_i \neq p_i''^2/2\mu_i$
- As a result the two body t- matrices appearing in the Faddeev kernel are now off-the-energy shell.
- The next important work of Faddeev was to show that the operator K^2 is compact or even square integrable (with some restrictions on the two body potentials) for all real (physical) values of z. Note that

$$t_i(z) G_0(z) t_j(z) \qquad for \quad i \neq j$$

Compactness of the Kernel

- diverges for $\operatorname{Im} z \to 0$ and $\operatorname{Re} z > E^B_{\min}$ because the t-matrices (showing poles at bound states) as well as the free propagator are singular. For example

$$\left\langle \vec{p}_{i}, \vec{q}_i \left| G_0(z) \right| \vec{p}'_i, \vec{q}'_i \right\rangle = \frac{\delta(\vec{q}_i - \vec{q}'_i)\,\delta(\vec{p}_i - \vec{p}'_i)}{z - q_i^2/2M_i - p_i^2/2\mu_i}$$

- The limit as z becomes real has been studied in great detail by Faddeev. He shows that for real z the fifth power of the kernel is a compact operator in a certain Banach space ensuring that the solutions are unique. For interactions involving superposition of Yukawa potentials, analysis becomes considerably simplified.

Banach Spaces

- A vector space V over the real or complex numbers with a norm $\| \;\|$ such that every Cauchy sequence (w.r.t metric d(x,y)= $\| x-y \|$) in V has a limit in V.
- The familiar Eucldean space K, where the norm of
- x= $(x_1, x_2, \ldots\ldots x_n)$
- is given by $$\|x\| = \left(\sum |x_i|^2 \right)^{1/2}$$
- is a Banach space.
- The space of all continuous functions f : $[a,b] \to K$ defined on a closed interval [a,b] becomes a Banach space if we can define the norm of such a function.

- Let us now see how Faddeev eqns resolve the problem of non-uniqueness of the solutions. In order to see this, let us recast these eqns in terms of the three body resolvent G. Starting from the equation

$$G(z) = G_0(z) + G_0(z)T(z)G_0(z)$$

- Splitting the T-matrix into three terms

$G(z) = G_0(z) + G_0(z)T(z)G_0(z)$

$$G(z) = G_0 + G^{(1)} + G^{(2)} + G^{(3)}$$

$$with \quad G^{(i)} = G_0 T^{(i)} G_0$$

- The integral equations for the components $G^{(i)}$ are derived by inserting the Faddeev equations to get

$$G^{(i)} = G_0 t_i G_0 + G_0 \sum_{j=1}^{3} (t)_{ij} G_0 T^{(j)} G_0$$

- Recall that $G_i - G_0 = G_0 t_i G_0$ Substituting it in the above, we get

$$\begin{pmatrix} G^{(1)} \\ G^{(2)} \\ G^{(3)} \end{pmatrix} = \begin{pmatrix} G_1 - G_0 \\ G_2 - G_0 \\ G_3 - G_0 \end{pmatrix} - G_0 \begin{pmatrix} 0 & t_1 & t_1 \\ t_2 & o & t_2 \\ t_3 & t_3 & 0 \end{pmatrix} \begin{pmatrix} G^{(1)} \\ G^{(2)} \\ G^{(3)} \end{pmatrix}$$

- Consider an eigenstate $\Psi_n = \Psi_{i,n}$ of the total

Hamiltonian H corresponding to an initial state $\phi_n = \phi_{i,n}$

Here the subscript 'i' refers to the channel index which may take on the values i=0,1,2,3 and 'n' contains additional information on the bound state

- For the particular case i=1, we have a situation in which particle 1 is free and the pair (2,3) is bound. This corresponds to the case when the Hamiltonian , $H = H^{(1)} + \bar{V}^{(1)}$ where

$$\bar{V}^{(1)} = V_{13} + V_{12} \quad and \quad H^{(1)} = H_0 + V_{23}$$

- $$H^{(1)}\phi_{1,n} = E_{1,n}\phi_{1,n} \ ; \ G^{(1)}(z) = \frac{1}{z - H^{(1)}}$$

- Thus for the eigenstate $\phi_{1,n}$

$$\Psi_{1,n} = \Psi_{1,n}^{(1)} + \Psi_{1,n}^{(2)} + \Psi_{1,n}^{(3)}$$

- Operating the first of the resolvent equations

$$G^{(1)} = G_1 - G_0 + G_0 t_1 (G^{(2)} + G^{(3)})$$

- on the initial state $\phi_{1,n}$

- Using the definition $\lim_{\varepsilon\to 0} i\varepsilon G^{(i)}\phi_{1,n} = \Psi_{1,n}^{(i)}$
- We write

$$\Psi_{1n}^{(1)} = \lim_{\varepsilon\to 0} i\varepsilon\left[G_1(E+i\varepsilon) - G_0(E+i\varepsilon) + \sum_{j=1}^{3} G_0(E+i\varepsilon)t_j\, G^{(j)}\right]\left|\phi_{1,n}\right\rangle$$

- Now

$$G_1(E+i\varepsilon) = \frac{1}{E_{1,n} + i\varepsilon - H_0 - V_{23}} \quad \therefore \quad \lim_{\varepsilon\to 0} i\varepsilon G_1(E+i\varepsilon)\left|\phi_{1,n}\right\rangle = \left|\phi_{1,n}\right\rangle$$

- But

$$\lim_{\varepsilon\to 0} i\varepsilon G_1(E+i\varepsilon)\left|\phi_{j,n}\right\rangle = 0 \quad for \;\; j \neq 1$$

- Thus

$$\Psi_{1,n}^{(1)} = \phi_{1,n} + G_0\, t_1(\Psi_{1,n}^{(2)} + \Psi_{1,n}^{(3)})$$

$$\Psi_{1,n}^{(2)} = G_0\, t_2(\Psi_{1,n}^{(1)} + \Psi_{1,}^{(3)})$$

$$\Psi_{1,n}^{(3)} = G_0\, t_3(\Psi_{1,n}^{(1)} + \Psi_{1,n}^{(2)})$$

- Is the solution of the inhomogeneous eqn unique? To see, write the homogeneous eqn corresponding to the first eqn:

$$\Psi_n^{(1)} = G_0(E)\, t_1(E)\left[\Psi_n^{(2)} + \Psi_n^{(3)}\right]$$

- Now

$$G_0\, t_1(E) = G_1^{+} V^1 \quad with \quad G_1 = (E - H_0 - V^1)^{-1}$$

- This gives $\quad G_1^{-1}\ \Psi_n^{(1)} = G_1 V^1 (\Psi_n^{(2)} + \Psi_n^{(3)})$

- Multiplying by from left, we get

$$(E - H_0)\Psi_n^{(1)} = V^{(1)}(\Psi_n^{(2)} + \Psi_n^{(3)}) + V^1 \Psi_n^{(1)}$$

$$= V^1 \Psi_n \qquad with \quad V^1 = V_{23}$$

$$(E - H_0)\Psi_n^{(2)} = V^2 \Psi_n^{(2)} \quad and \quad (E - H_0)\Psi_n^{(3)} = V^3 \Psi_n^{(3)}$$

- Similarly

- Adding the three eqns for the three components, $, \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}$ of Ψ
- we get the Schrodinger eqn,

$$(E - H_0 - V)\Psi = 0$$

- which is a solution of the homogeneous part of the Faddeev three coupled eqns for Ψ and clearly corresponds to the three body bound state.

Lovelace Equations

- A slightly different version of Faddeev Equations leading directly to the transition matrix elements for a process $i\alpha \rightarrow j\beta$
- Define

$$U_{fi} = V_i + V_f G V_i$$

$$and \quad \overline{U}_{fi} = V_f + V_f G V_i$$

$$V_i = V - V^i; V_1 = V - V^1 = V_{12} + V_{31}$$

$$H_i = V - V^i; \; V_f = V - V^f; \;\; i, f = 0,1,2,3$$

- Here $V_f = V - V^f$ is the interaction between the outgoing particles in the channel f. If, for example, f=2, then V_f is the interaction between particle 2 and the bound pair (1,3) ,i.e, 21+23 .
- The off-shell transition matrix-elements for a process i α f β are given by

$$\langle f\beta | T^{(+)} | i\alpha \rangle = \langle \Phi_{f\beta}(E_{f\beta}) | V_f + V_f G(E_{i\alpha} + i\varepsilon) V_i | \Phi_{i\alpha}(E_{i\alpha}) \rangle$$

$$= \langle \Phi_{f\beta}(E_{\beta}) | \overline{U}_{fi}(E_{i\alpha}) | \Phi_{i\alpha}(E_{i\alpha}) \rangle$$

and

$$\left\langle f\,\beta \left| T^{(-)} \right| i\,\alpha \right\rangle = \left\langle \Phi_{f\beta}(E_{f\beta}) \left| V_i + V_f G(E_{f\beta} + i\varepsilon) V_i \right| \Phi_{i\alpha}(E_{i\alpha}) \right\rangle$$

$$= \left\langle \Phi_{f\beta}(E_{fi}) \left| U_{f\beta}(E_{f\beta}) \right| \Phi_{i\alpha}(E_{i\alpha}) \right\rangle$$

The physical on-shell T-matrix is clearly obtained by setting and taking the limit $E_{i\alpha} = E_{f\beta} = E$ To establish the relationship between the Faddeev operators $T^{(i)}$ and the Lovelace operators and $\overline{U}_{fi}$ we use the Faddeev equations and obtain the Lovelace equations

$$\overline{U}_{fi} = V_f + \sum_{k \neq i} \overline{U}_{fk} G_0 t_k$$

$$U_{fi} = V_i + \sum_{k \neq f} t_k G_0 U_{ki}$$