“Relativistic” quantum criticality and the AdS/CFT correspondence

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Lecture notes
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PHYSICS
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J. Polchinski: TASI 2010

1.1 The greatest equation

A few years back, Physics World magazine had a reader poll to determine the Greatest Equation Ever, and came up with a two-way tie between Maxwell's equations

$$d * F = j\,, \quad dF = 0\,, \tag{1.1}$$

and Euler's equation

$$e^{i\pi} + 1 = 0\,. \tag{1.2}$$

The remarkable appeal of Euler's equation is that it contains in such a compact form the five most important numbers, $0, 1, i, \pi, e$, and the three basic operations, $+$, $\times$, ^.

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The remarkable appeal of Euler's equation is that it contains in such a compact form the five most important numbers, $0, 1, i, \pi, e$, and the three basic operations, $+, \times,$ ^. But my own choice would have been Maldacena's equation

$$\mathrm{AdS} = \mathrm{CFT} \,, \tag{1.3}$$

because this contains all the central concepts of fundamental physics: Maxwell's equations, to start with, and their non-Abelian extension, plus the Dirac and Klein-Gordon equations, quantum mechanics, quantum field theory and general relativity.

The Hubbard Model

$$H = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

$t_{ij} \rightarrow$ "hopping". $U \rightarrow$ local repulsion, $\mu \rightarrow$ chemical potential

Spin index $\alpha = \uparrow, \downarrow$

$$n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha}$$

$$c_{i\alpha}^\dagger c_{j\beta} + c_{j\beta} c_{i\alpha}^\dagger = \delta_{ij} \delta_{\alpha\beta}$$
$$c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0$$

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S.Sachdev: TASI 2010 (Baskaran 1987)

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The greatest equation

Outline

1. Quantum phase transitions of a semi-metal

   Honeycomb lattice, Dirac fermions and the Gross-Neveu model

2. Quantum critical transport

   Self-duality and the AdS/CFT correspondence

3. Quantum impurities and AdS_2

   Quantum spin coupled to a CFT

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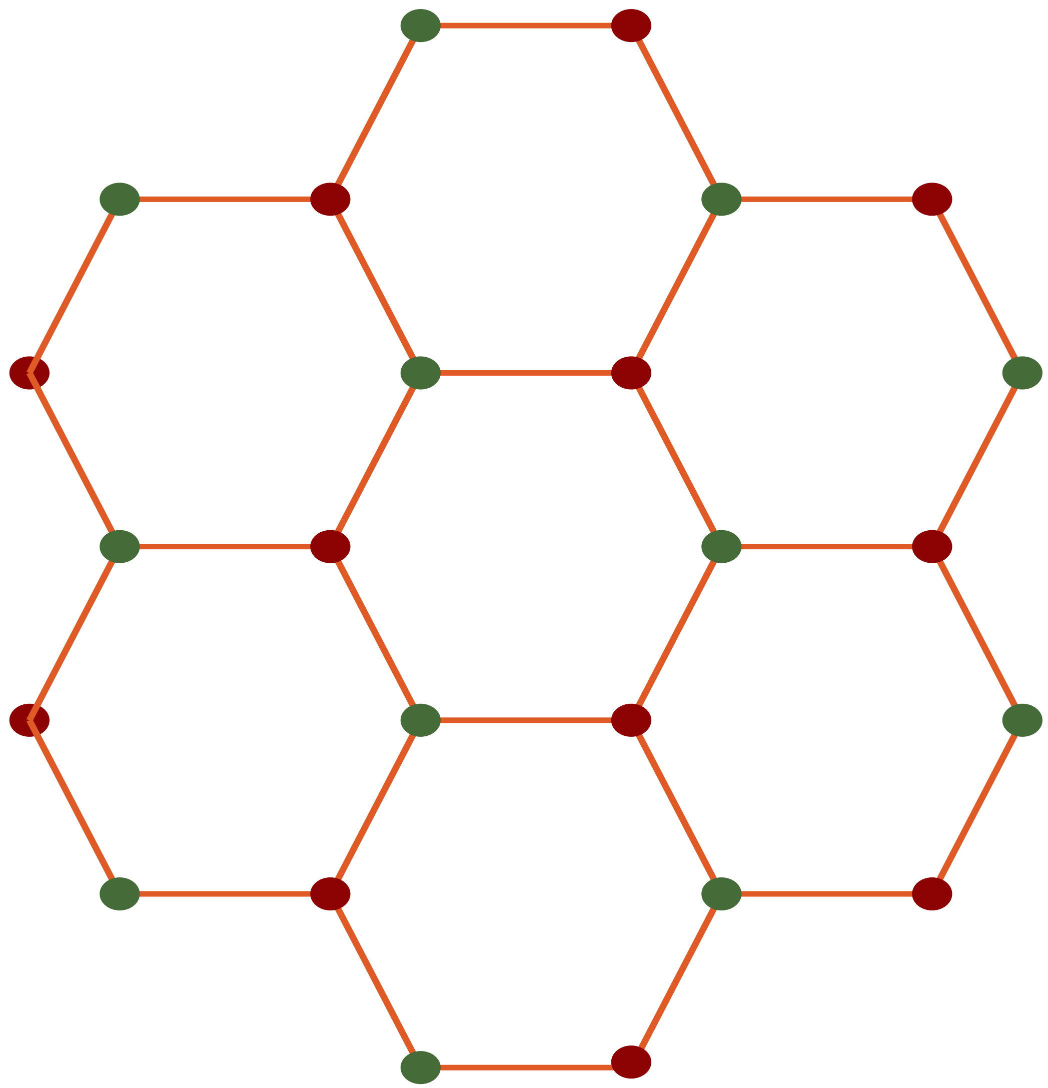
Honeycomb lattice

(describes graphene after adding long-range Coulomb interactions)



$$H = -t \sum_{\langle ij \rangle} c_{i\alpha}^{\dagger} c_{j\alpha} + U \sum_{i} \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right)$$

The Hubbard Model

$$H = -\sum_{i,j} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c^\dagger_{i\alpha} c_{i\alpha}$$

In the limit of large U, and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

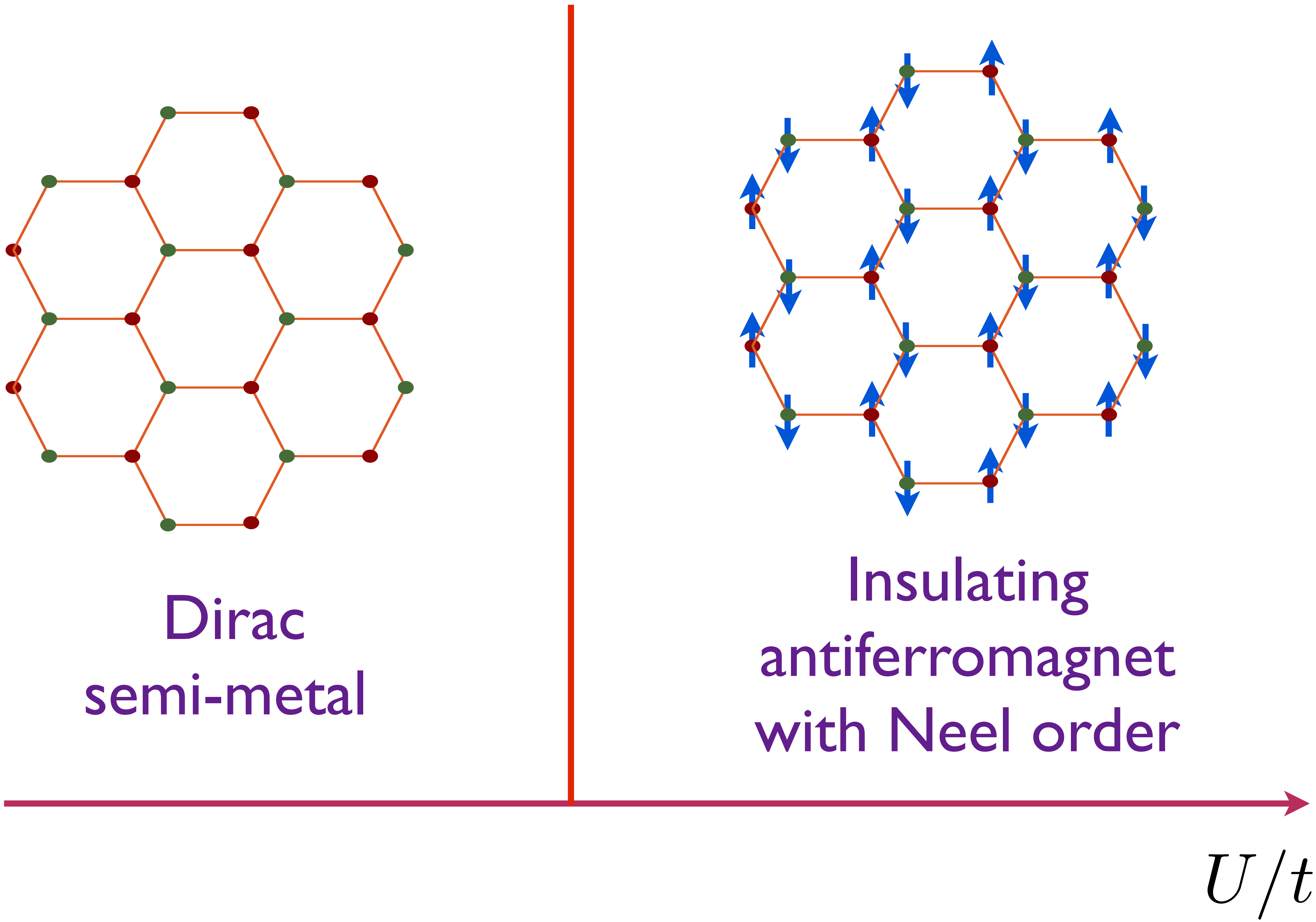
$$H_{AF} = \sum_{i<j} J_{ij} S^a_i S^a_j$$

where $a = x, y, z$,

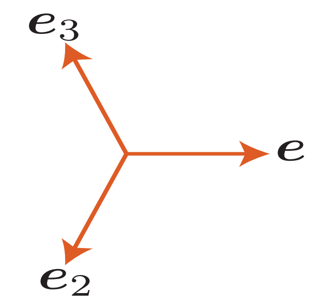
$$S^a_i = \frac{1}{2} c^{a\dagger}_{i\alpha} \sigma^a_{\alpha\beta} c_{i\beta},$$

with σ^a the Pauli matrices and

$$J_{ij} = \frac{4t^2_{ij}}{U}$$


Dirac
semi-metal
Insulating
antiferromagnet
with Neel order
U/t

Honeycomb lattice at half filling.



We define the unit length vectors

$$\mathbf{e}_1 = (1,0) \quad , \quad \mathbf{e}_2 = (-1/2, \sqrt{3}/2) \quad , \quad \mathbf{e}_3 = (-1/2, -\sqrt{3}/2). \tag{1}$$

Note that $\mathbf{e}_i \cdot \mathbf{e}_j = -1/2$ for $i \neq j$, and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$.
We take the origin of co-ordinates of the honeycomb lattice at the center of an *empty hexagon*. The A sublattice sites closest to the origin are at $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, while the B sublattice sites closest to the origin are at $-\mathbf{e}_1$, $-\mathbf{e}_2$, and $-\mathbf{e}_3$.
The reciprocal lattice is generated by the wavevectors

$$\mathbf{G}_1 = \frac{4\pi}{3}\mathbf{e}_1 \quad , \quad \mathbf{G}_2 = \frac{4\pi}{3}\mathbf{e}_2 \quad , \quad \mathbf{G}_3 = \frac{4\pi}{3}\mathbf{e}_3 \tag{2}$$

The first Brillouin zone is a hexagon whose vertices are given by

$$\mathbf{Q}_1 = \frac{1}{3}(\mathbf{G}_2 - \mathbf{G}_3) \quad , \quad \mathbf{Q}_2 = \frac{1}{3}(\mathbf{G}_3 - \mathbf{G}_1) \quad , \quad \mathbf{Q}_3 = \frac{1}{3}(\mathbf{G}_1 - \mathbf{G}_2), \tag{3}$$

and $-\mathbf{Q}_1$, $-\mathbf{Q}_2$, and $-\mathbf{Q}_3$.

We define the Fourier transform of the fermions by

$$c_A(\mathbf{k}) = \sum_{\mathbf{r}} c_A(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

and similarly for c_B.
The hopping Hamiltonian is

$$H_0 = -t \sum_{\langle ij \rangle} \left(c^\dagger_{Ai\alpha} c_{Bj\alpha} + c^\dagger_{Bj\alpha} c_{Ai\alpha} \right) \tag{5}$$

where α is a spin index. If we introduce Pauli matrices τ^a in sublattice space ($a = x, y, z$), this Hamiltonian can be written as

$$\begin{aligned} H_0 \quad = \quad & \int \frac{d^2k}{4\pi^2} c^\dagger(\mathbf{k}) \Big[-t \Big(\cos(\mathbf{k}\cdot\mathbf{e}_1) + \cos(\mathbf{k}\cdot\mathbf{e}_2) + \cos(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^x \\ & + t \Big(\sin(\mathbf{k}\cdot\mathbf{e}_1) + \sin(\mathbf{k}\cdot\mathbf{e}_2) + \sin(\mathbf{k}\cdot\mathbf{e}_3) \Big) \tau^y \Big] c(\mathbf{k}) \end{aligned} \tag{6}$$

The low energy excitations of this Hamiltonian are near $\mathbf{k} \approx \pm \mathbf{Q}_1$.

In terms of the fields near $\mathbf{Q}_1$ and $-\mathbf{Q}_1$, we define

$$\begin{aligned} \Psi_{A1\alpha}(\mathbf{k}) &= c_{A\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\ \Psi_{A2\alpha}(\mathbf{k}) &= c_{A\alpha}(-\mathbf{Q}_1 + \mathbf{k}) \\ \Psi_{B1\alpha}(\mathbf{k}) &= c_{B\alpha}(\mathbf{Q}_1 + \mathbf{k}) \\ \Psi_{B2\alpha}(\mathbf{k}) &= c_{B\alpha}(-\mathbf{Q}_1 + \mathbf{k}) \end{aligned} \tag{7}$$

We consider Ψ to be a 8 component vector, and introduce Pauli matrices ρ^a which act in the $1, 2$ valley space. Then the Hamiltonian is

$$H_0 = \int \frac{d^2k}{4\pi^2} \Psi^\dagger(\mathbf{k}) \Big(v\tau^y k_x + v\tau^x \rho^z k_y \Big) \Psi(\mathbf{k}), \tag{8}$$

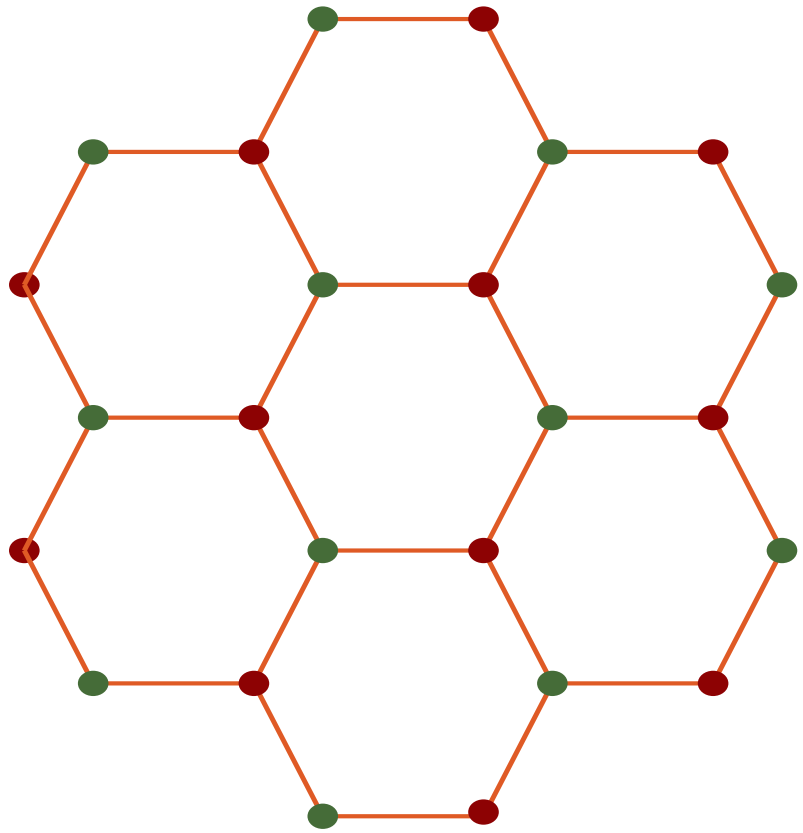
where $v = 3t/2$; below we set $v = 1$. Now define $\overline{\Psi} = \Psi^\dagger \rho^z \tau^z$. Then we can write the imaginary time Lagrangian as

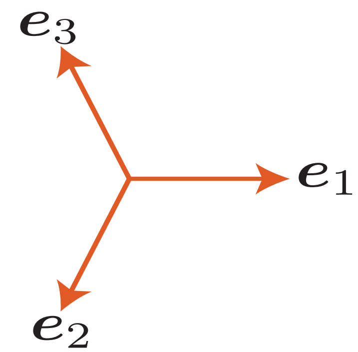
$$\mathcal{L}_0 = -i\overline{\Psi} \left(\omega\gamma_0 + k_x\gamma_1 + k_y\gamma_2 \right) \Psi \tag{9}$$

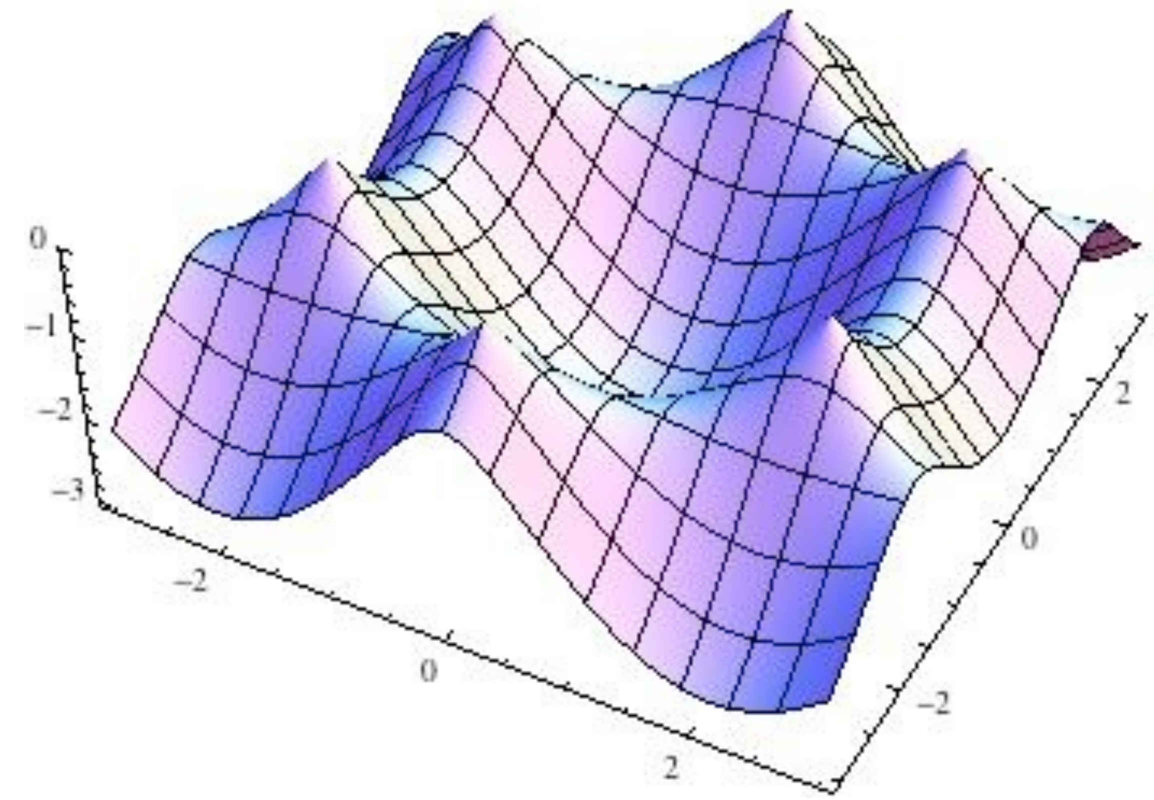
where

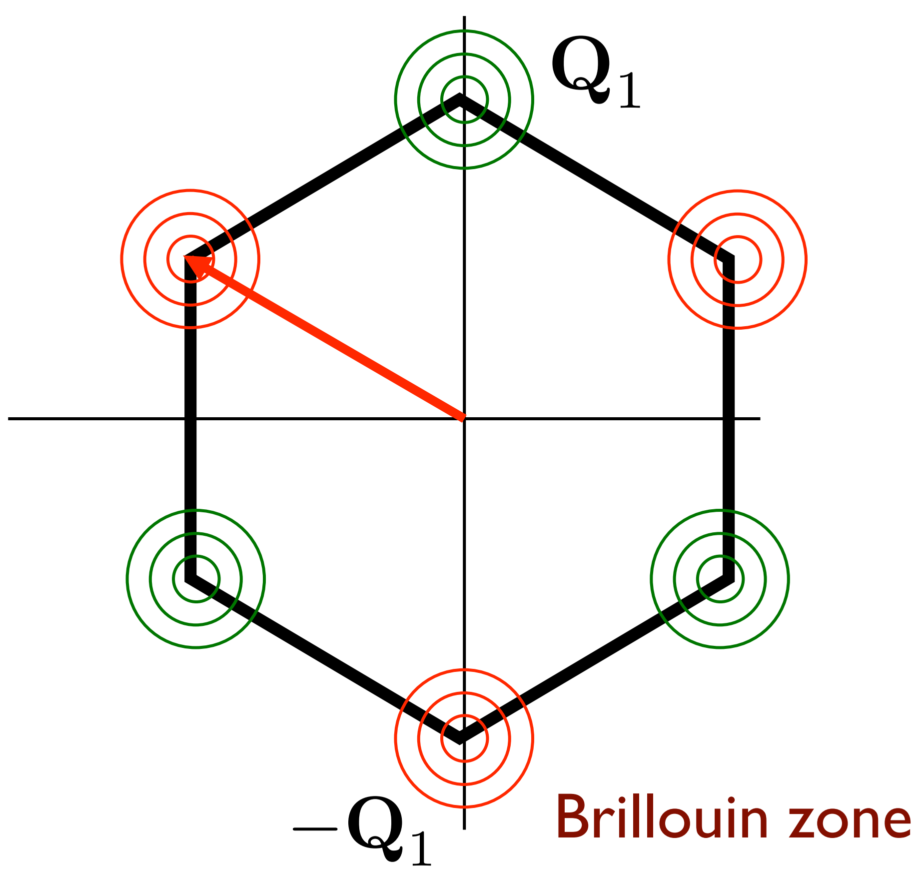
$$\gamma_0 = -\rho^z\tau^z \quad \gamma_1 = \rho^z\tau^x \quad \gamma_2 = -\tau^y \tag{10}$$

Graphene









Semi-metal with massless Dirac fermions

Exercise: Observe that $\mathcal{L}_0$ is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction U in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.

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Antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3}S^{a2} + \frac{U}{4} \tag{11}$$

Then we decouple the interaction via

$$\exp\left(\frac{2U}{3}\sum_i \int d\tau S_i^{a2}\right) = \int \mathcal{D}J_i^a(\tau)\exp\left(-\sum_i \int d\tau \left[\frac{3}{8U}J_i^{a2} - J_i^a S_i^a\right]\right) \tag{12}$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for J_i^a. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field φ^a where

$$J_A^a = \varphi^a \quad , \quad J_B^a = -\varphi^a \tag{13}$$

The coupling between the field φ^a and the Ψ fermions is given by

$$\begin{aligned}\sum_i J_i^a c_{i\alpha}^\dagger \sigma_{\alpha\beta}^a c_{i\beta} &= \varphi^a \left(c_{A\alpha}^\dagger \sigma_{\alpha\beta}^a c_{A\beta} - c_{B\alpha}^\dagger \sigma_{\alpha\beta}^a c_{B\beta} \right) \\ &= \varphi^a \Psi^\dagger \tau^z \sigma^a \Psi = -\varphi^a \overline{\Psi} \rho^z \sigma^a \Psi \end{aligned} \tag{14}$$

From this we motivate the low energy theory

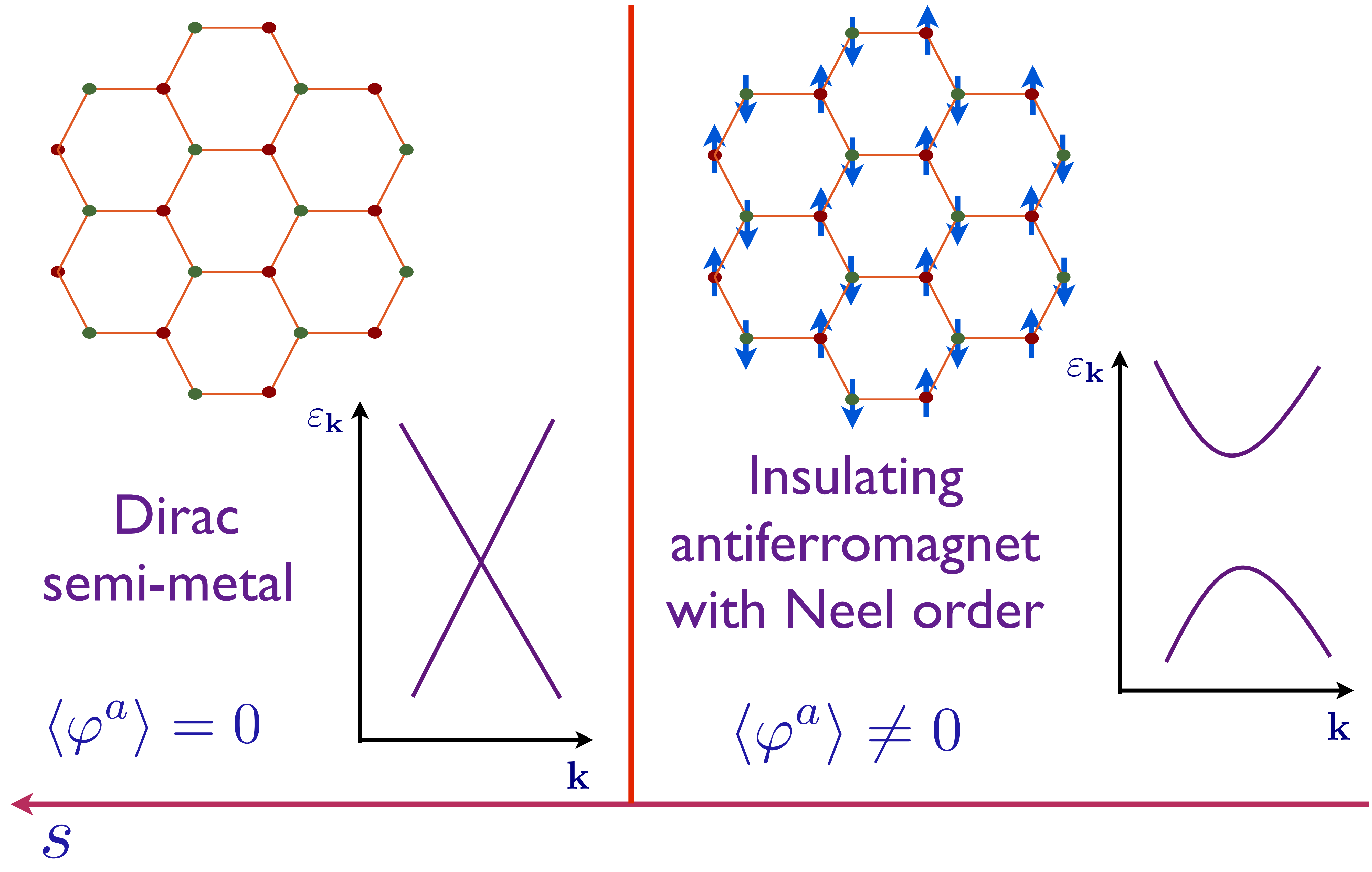
$$\mathcal{L} = \overline{\Psi} \gamma_\mu \partial_\mu \Psi + \frac{1}{2} \left[(\partial_\mu \varphi^a)^2 + s \varphi^{a2} \right] + \frac{u}{24} \left(\varphi^{a2} \right)^2 - \lambda \varphi^a \overline{\Psi} \rho^z \sigma^a \Psi \tag{15}$$

Note that the matrix $\rho^z \sigma^a$ commutes with all the γ_μ; hence $\rho^z \sigma^a$ is a matrix in "flavor" space. This is the Gross-Neveu model, and it describes the quantum phase transition from the Dirac semi-metal to an insulating Néel state In mean-field theory, the

Dirac semi-metal is obtained for $s > 0$ with $\langle \varphi^a \rangle = 0$. The Néel state obtains for $s < 0$, and we have $\varphi^a = N_0 \delta_{az}$ (say), and so the dispersion of the electrons is

$$\omega_k = \pm\sqrt{k^2 + \lambda^2 N_0^2} \tag{16}$$

near the points $\pm\mathbf{Q}_1$. These form the conduction and valence bands of the insulator.


Dirac
semi-metal
$\langle \varphi^a \rangle = 0$
$\varepsilon_{\mathbf{k}}$
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Insulating
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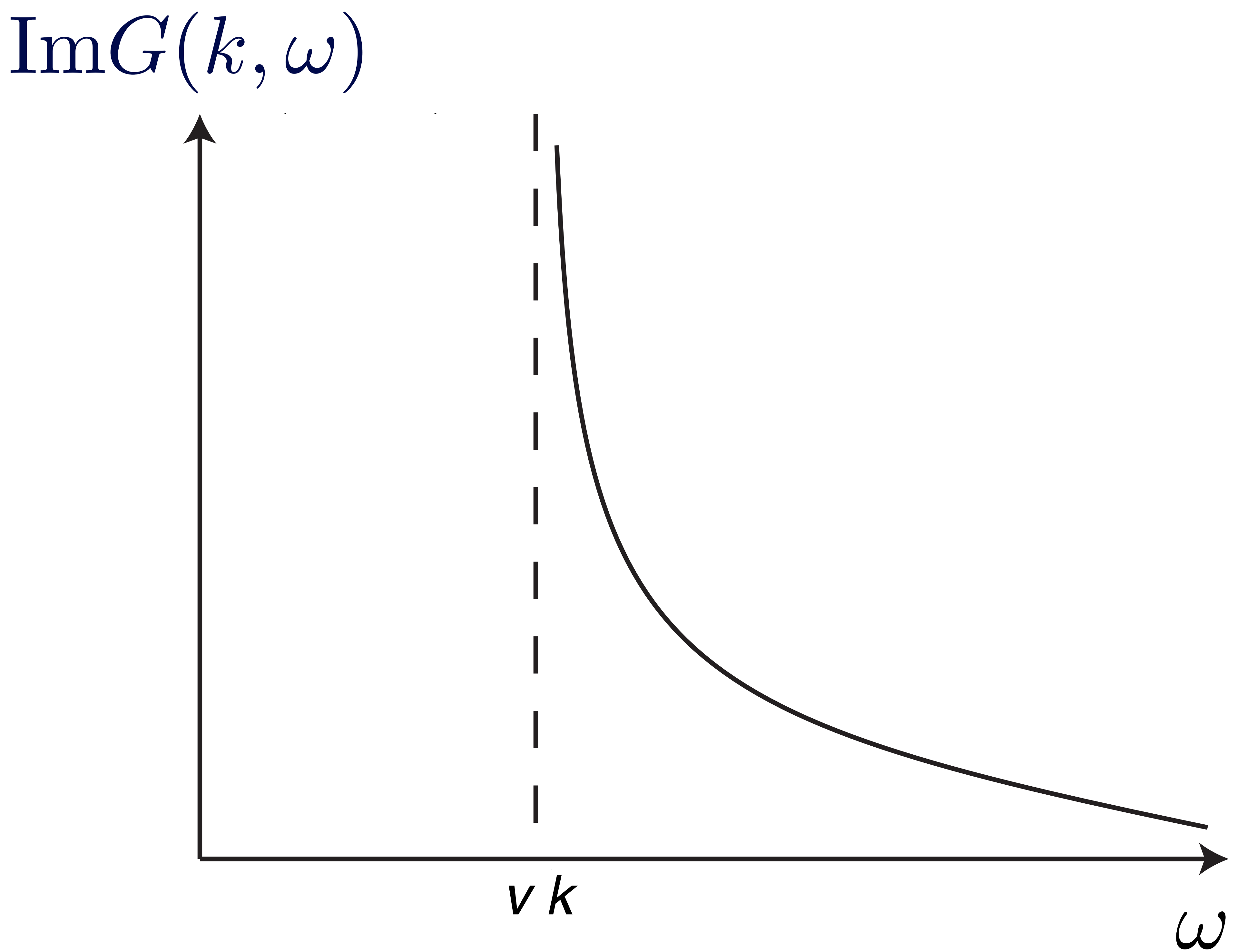
Exercise: Perform a tree-level RG transformation on $\mathcal{L}$. The quadratic gradient terms are invariant under $\Psi' = \Psi e^{\ell}$ and $\varphi' = \varphi e^{\ell/2}$. Show that this leads to $s' = se^{2\ell}$. Thus s is a relevant perturbation which drives the system into either the semi-metal or antiferromagnetic insulator. The quantum critical point is reached by tuning s to its critical value ($= 0$ at tree level). Show that the couplings u and λ are both relevant perturbations at this critical point. Thus, while interactions are irrelevant in the Dirac semi-metal (and in the insulator), they are strongly relevant at the quantum-critical point.

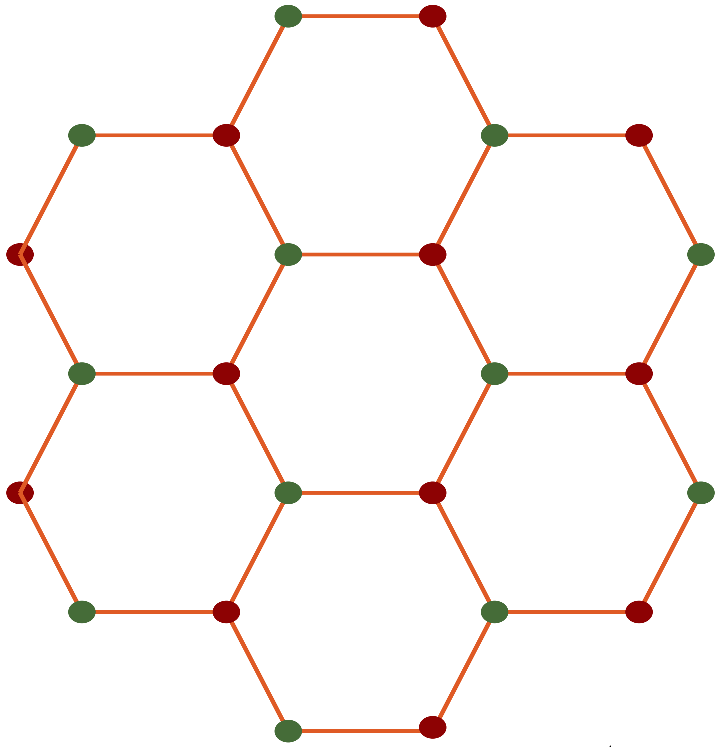
An analysis of this quantum critical point requires a RG analysis which goes beyond tree-level. Such an analysis can be controlled in an expansion in $1/N$ (where N is the number of fermion flavors) or $(3-d)$ (where d is the spatial dimensionality. Such analyses show that the couplings u and λ reach a RG fixed point which describes a conformal field theory (CFT).

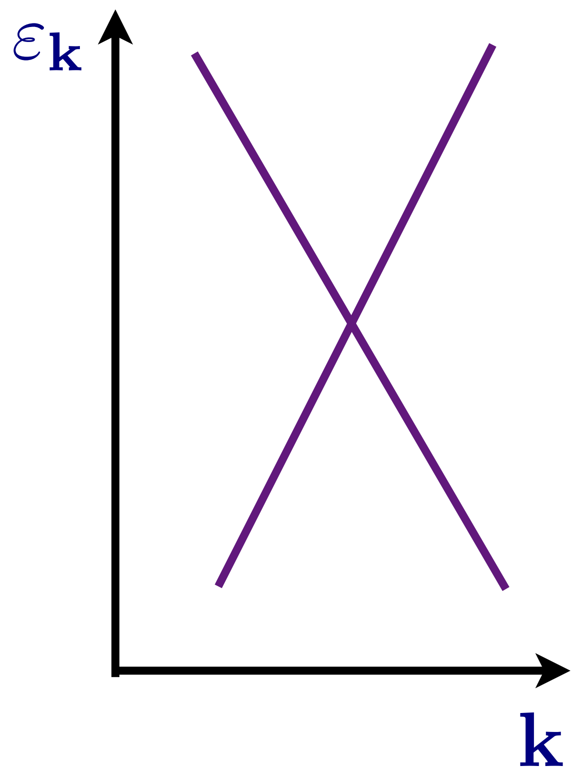
An important result of such an analysis is the following structure in the electron Green's function:

$$G(k,\omega) = \left\langle \Psi(k,\omega); \Psi^\dagger(k,\omega) \right\rangle \sim \frac{i\omega + vk_x\tau^y + vk_y\tau^x\rho^z}{(\omega^2 + v^2k_x^2 + v^2k_y^2)^{1-\eta/2}} \quad (17)$$

where $\eta > 0$ is the *anomalous dimension* of the fermion. Note that this leads to a fermion spectral density which has no quasiparticle pole: thus the quantum critical point has no well-defined quasiparticle excitations.

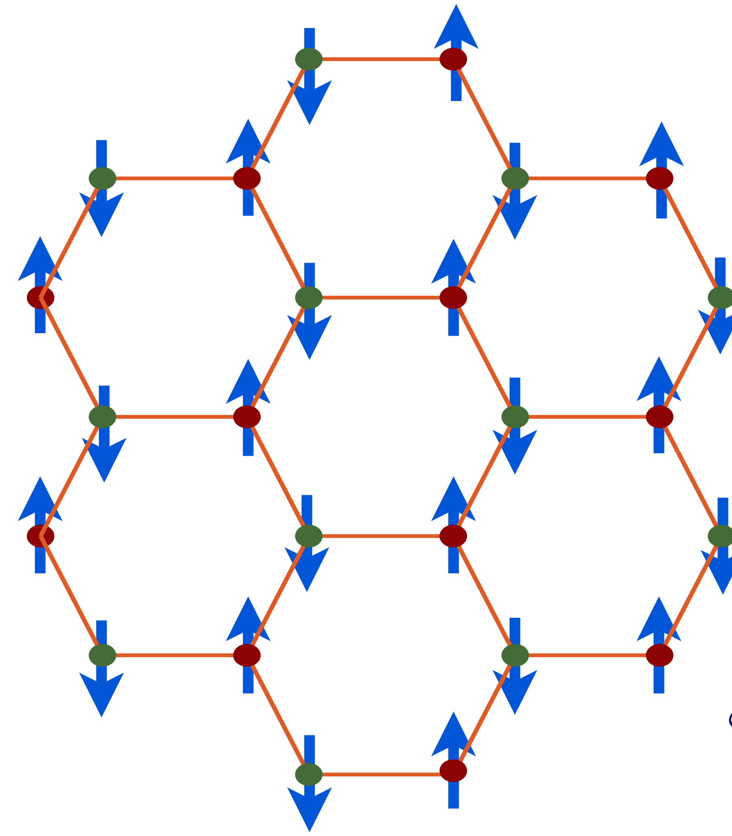

$\mathrm{Im}G(k,\omega)$
v k
ω

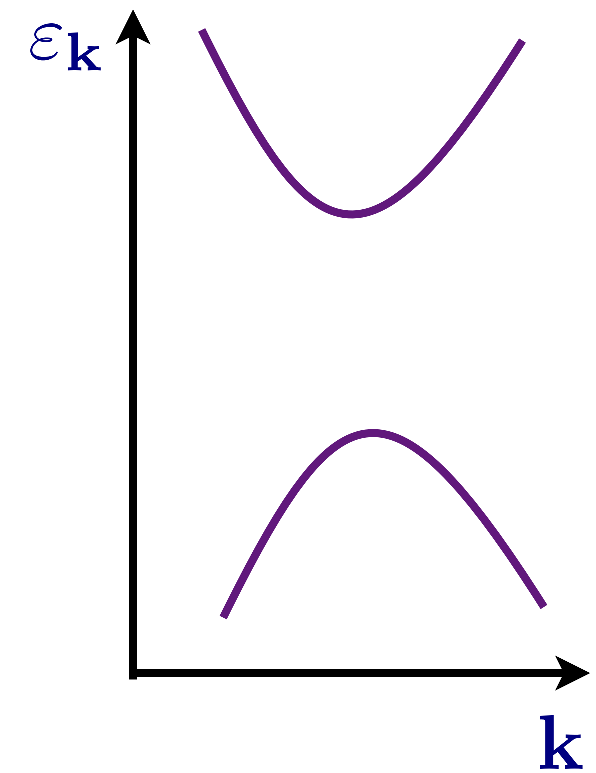




Dirac
semi-metal

$$\langle \varphi^a \rangle = 0$$





Insulating
antiferromagnet
with Neel order

$$\langle \varphi^a \rangle \neq 0$$

s

Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles

Outline

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   Honeycomb lattice, Dirac fermions and the Gross-Neveu model

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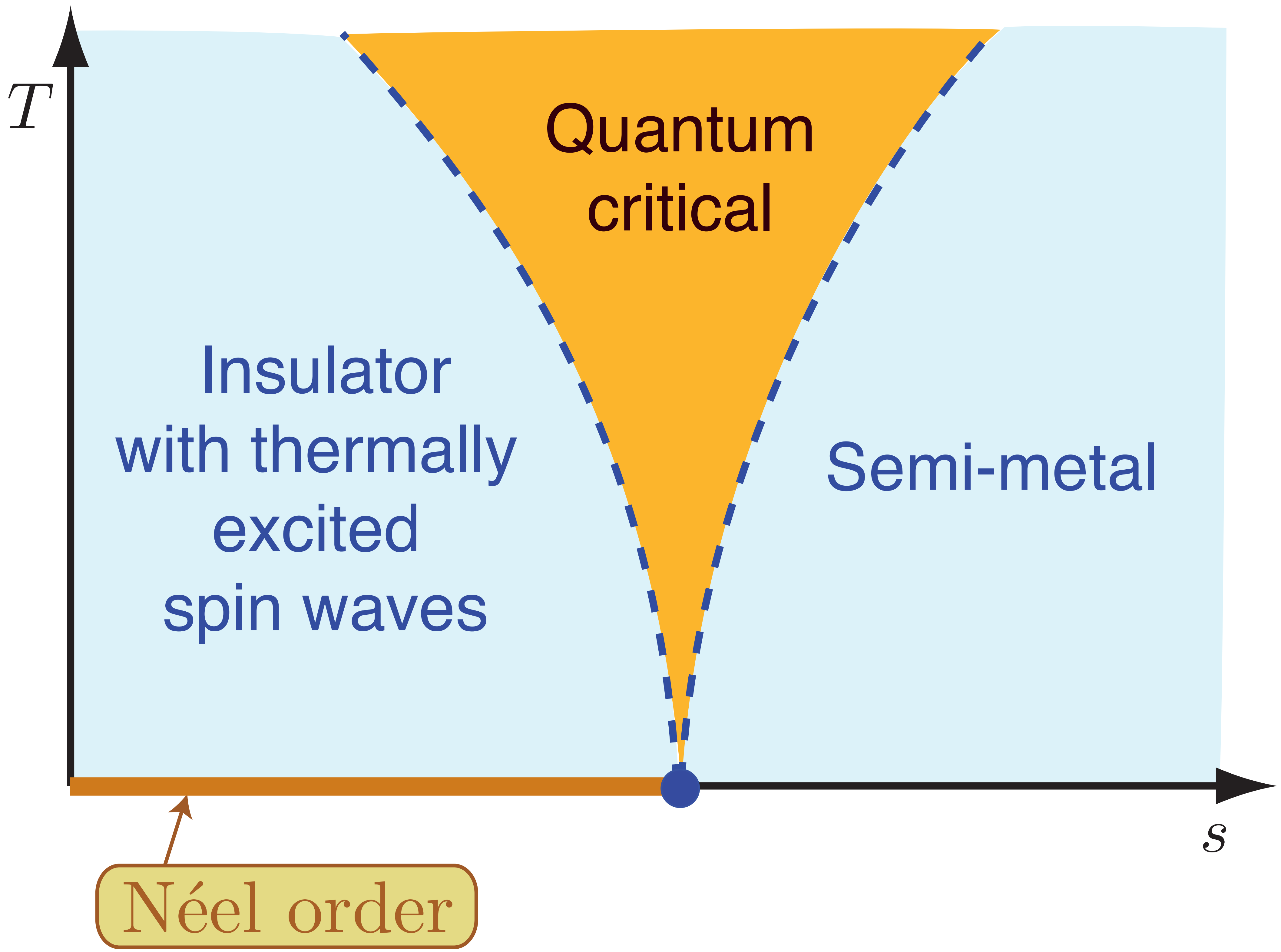

T
Quantum critical
Insulator with thermally excited spin waves
Semi-metal
s
Néel order

Electrical transport

The conserved electrical current is

$$J_\mu = -i\overline{\Psi}\gamma_\mu\Psi. \tag{1}$$

Let us compute its two-point correlator, $K_{\mu\nu}(k)$ at a spacetime momentum k_μ at $T = 0$. At leading order, this is given by a one fermion loop diagram which evaluates to

$$\begin{aligned} K_{\mu\nu}(k) &= \int \frac{d^3p}{8\pi^3} \frac{\mathrm{Tr}\left[\gamma_\mu(i\gamma_\lambda p_\lambda + m\rho^z\sigma^z)\gamma_\nu(i\gamma_\delta(k_\delta + p_\delta) + m\rho^z\sigma^z)\right]}{(p^2 + m^2)((p+k)^2 + m^2)} \\ &= -\frac{2}{\pi}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^1 dx \frac{k^2 x(1-x)}{\sqrt{m^2 + k^2 x(1-x)}}, \end{aligned} \tag{2}$$

where the mass $m = 0$ in the semi-metal and at the quantum critical point, while $m = |\lambda N_0|$ in the insulator. Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

$$k_\mu K_{\mu\nu} = 0. \tag{3}$$

Of particular interest to us is the K_{00} component, after analytic continuation to Minkowski space where the spacetime momentum k_μ is replaced by (ω, k). The conductivity is obtained from this correlator via the Kubo formula

$$\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} K_{00}(\omega, k). \tag{4}$$

In the insulator, where $m > 0$, analysis of the integrand in Eq. (2) shows that that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (4) vanishes. These properties are as expected in any insulator.
In the metal, and at the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (2) and the conductivity in Eq. (4) evaluate to the simple universal results

$$\begin{aligned} K_{00}(\omega, k) &= \frac{1}{4} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= 1/4. \end{aligned} \tag{5}$$

Going beyond one-loop, we find *no change* in these results in the semi-metal to all orders in perturbation theory. At the quantum critical point, there are no anomalous dimensions for the conserved current, but the amplitude does change yielding

$$\begin{aligned} K_{00}(\omega, k) &= \mathcal{K} \frac{k^2}{\sqrt{k^2 - \omega^2}} \\ \sigma(\omega) &= \mathcal{K}, \end{aligned} \tag{6}$$

where $\mathcal{K}$ is a universal number dependent only upon the universality class of the quantum critical point. The value of the $\mathcal{K}$ for the Gross-Neveu model is not known exactly, but can be estimated by computations in the $(3-d)$ or $1/N$ expansions.

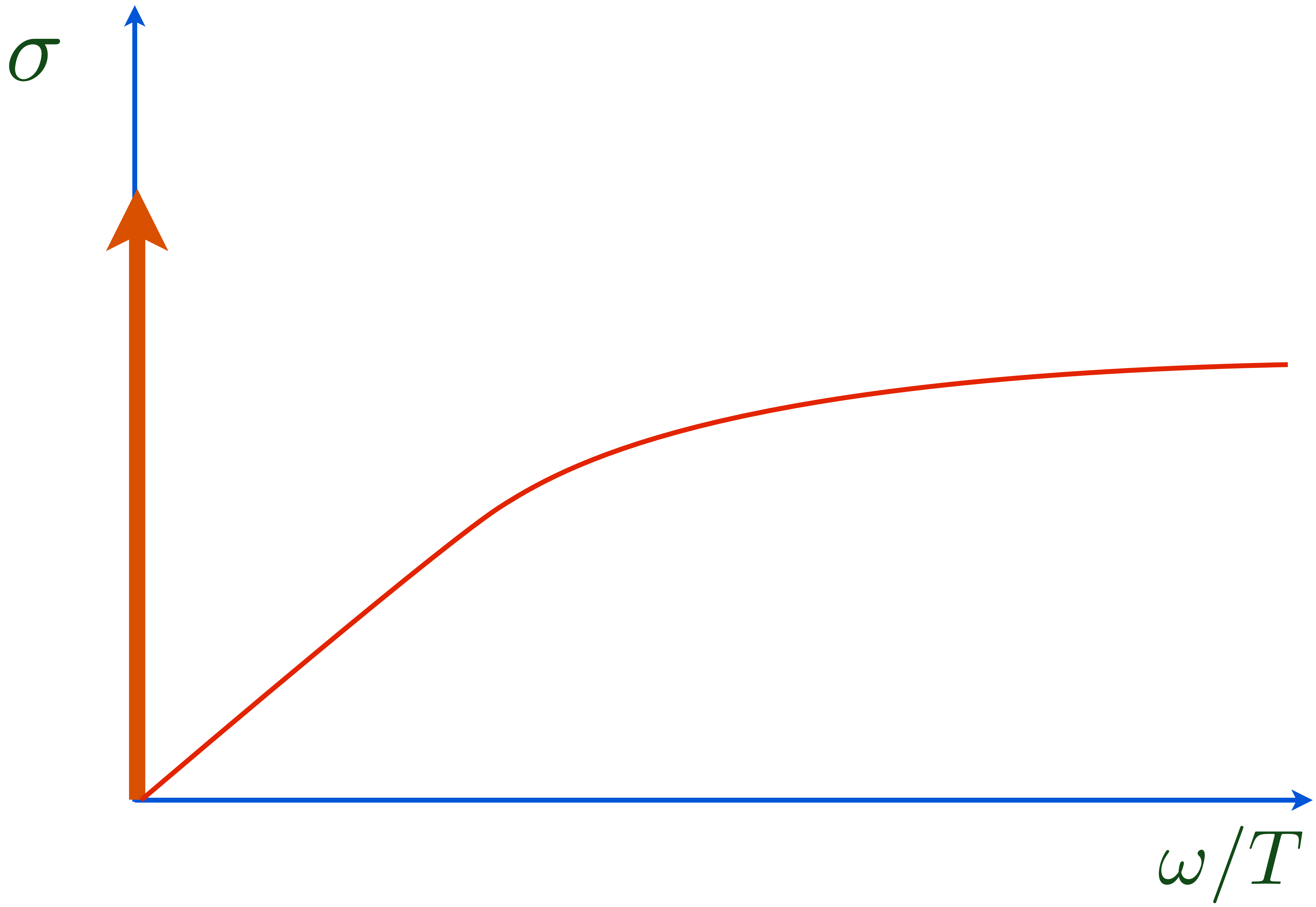
Non-zero temperatures

At the quantum-critical point at one-loop order, we can set $m = 0$, and then repeat the computation in Eq. (2) at $T > 0$. This only requires replacing the integral over the loop frequency by a summation over the Matsubara frequencies, which are quantized by odd multiples of πT. Such a computation, via Eq. (4) leads to the conductivity

$$\mathrm{Re}[\sigma(\omega)] = (2T \ln 2)\, \delta(\omega) + \frac{1}{4} \tanh\left(\frac{|\omega|}{4T}\right) ; \tag{7}$$

the imaginary part of $\sigma(\omega)$ is the Hilbert transform of $\mathrm{Re}[\sigma(\omega)] - 1/4$. Note that this reduces to Eq. (5) in the limit $\omega \gg T$. However, the most important new feature of Eq. (7) arises for $\omega \ll T$, where we find a delta function at zero frequency in the real part. Thus the d.c. conductivity is infinite at this order, arising from the collisionless transport of thermally excited carriers.

Electrical transport in a free-field theory for *T > 0*



Collisions between carriers invalidate the form in Eq. (7) for the density correlation function, and we instead expect the form dictated by the hydrodynamic diffusion of charge. Thus for K_{00}, Eq. (6) applies only for $\omega \gg T$, while

$$K_{00}(\omega, k) = \chi \frac{Dk^2}{Dk^2 - i\omega} \quad , \quad \omega \ll T. \tag{8}$$

Here χ is the charge susceptibility (here it is the compressibility), and D is the charge diffusion constant. These have universal values in the quantum critical region:

$$\chi = \mathcal{C}_\chi T \quad , \quad D = \frac{\mathcal{C}_D}{T}, \tag{9}$$

where again $\mathcal{C}_\chi$ and $\mathcal{C}_D$ are universal numbers. For the conductivity, we expect a crossover from the collisionless critical dynamics at frequencies $\omega \gg T$, to a hydrodynamic collision-dominated form for $\omega \ll T$. This entire crossover is universal, and is described by a universal crossover function

$$\sigma(\omega) = \mathcal{K}_\sigma(\omega/T). \tag{10}$$

The result in Eq. (6) applies for $\omega \gg T$, and so

$$\mathcal{K}_\sigma(\infty) = \mathcal{K}. \tag{11}$$

For the hydrodynamic transport, we apply the Kubo formula in Eq. (4) to Eq. (8) and obtain

$$\mathcal{K}_\sigma(0) = \mathcal{C}_\chi \mathcal{C}_D \tag{12}$$

which is a version of Einstein's relation for Brownian motion.

More generally, at $T > 0$, we do not expect $K_{\mu\nu}$ to be relativistically covariant, and so can only constrain it by spatial isotropy and density conservation. These two constraints, along with dimensional analyses, lead to the most general form

$$K_{\mu\nu}(\omega, k) = \sqrt{k^2 - \omega^2}\Big(P^T_{\mu\nu} K^T(\omega, k) + P^L_{\mu\nu} K^L(\omega, k)\Big), \qquad (13)$$

where $K^{L,T}$ are dimensionless functions of the arguments, and depend upon ω and the magnitude of the 2-vector k. Also $P^T_{\mu\nu}$ and $P^L_{\mu\nu}$ are orthogonal projectors defined by

$$P^T_{00} = P^T_{0i} = P^T_{i0} = 0 \ , \ P^T_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2} \ , \ P^L_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - P^T_{\mu\nu}, \qquad (14)$$

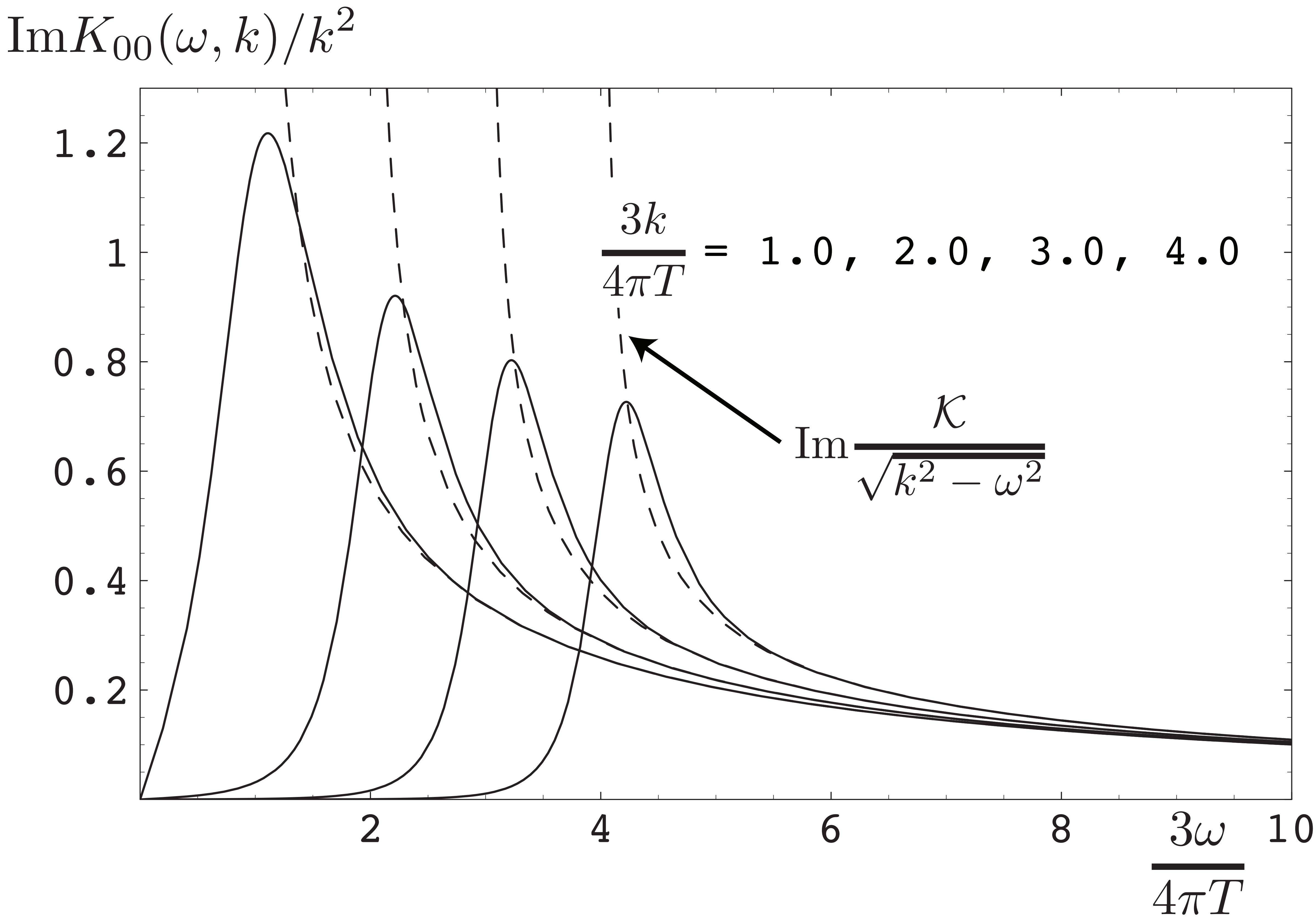
with the indices i, j running over the 2 spatial components. Thus, in the general case at $T > 0$, the full density and current responses are described in terms of two functions $K^{L,T}(k, \omega)$, representing current fluctuations longitudinal and transverse to the momentum. These two functions are not entirely independent. At $T > 0$, we expect all correlations to be smooth functions at $k = 0$: this is because all correlations are expected to decay exponentially to zero as a function of spatial separation. However, this is only possible from (13) if we have the additional relation

$$K^T(\omega, 0) = K^L(\omega, 0). \tag{15}$$

The relations of the previous paragraph are completely general and apply to any theory. We now compute the charge correlations by the holographic Maxwell theory

$$\mathcal{S}_{EM} = \frac{1}{g_4^2} \int d^4x \sqrt{-g} \left[-\frac{1}{4} F_{ab} F^{ab} \right] . \tag{16}$$

Density correlations in the holographic Maxwell theory



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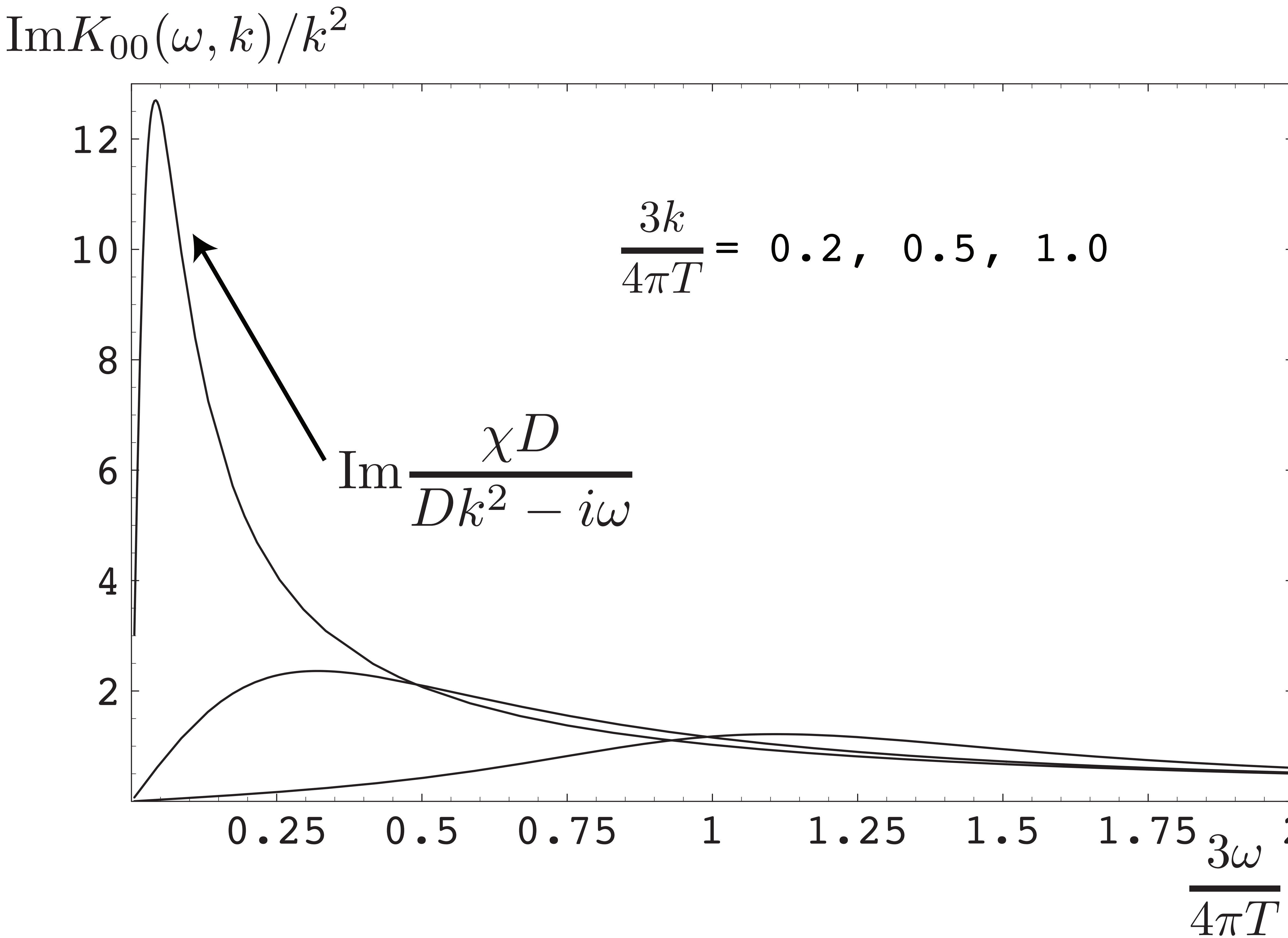


The self-duality of the 4-dimensional Maxwell theory leads to the simple and remarkable identity:

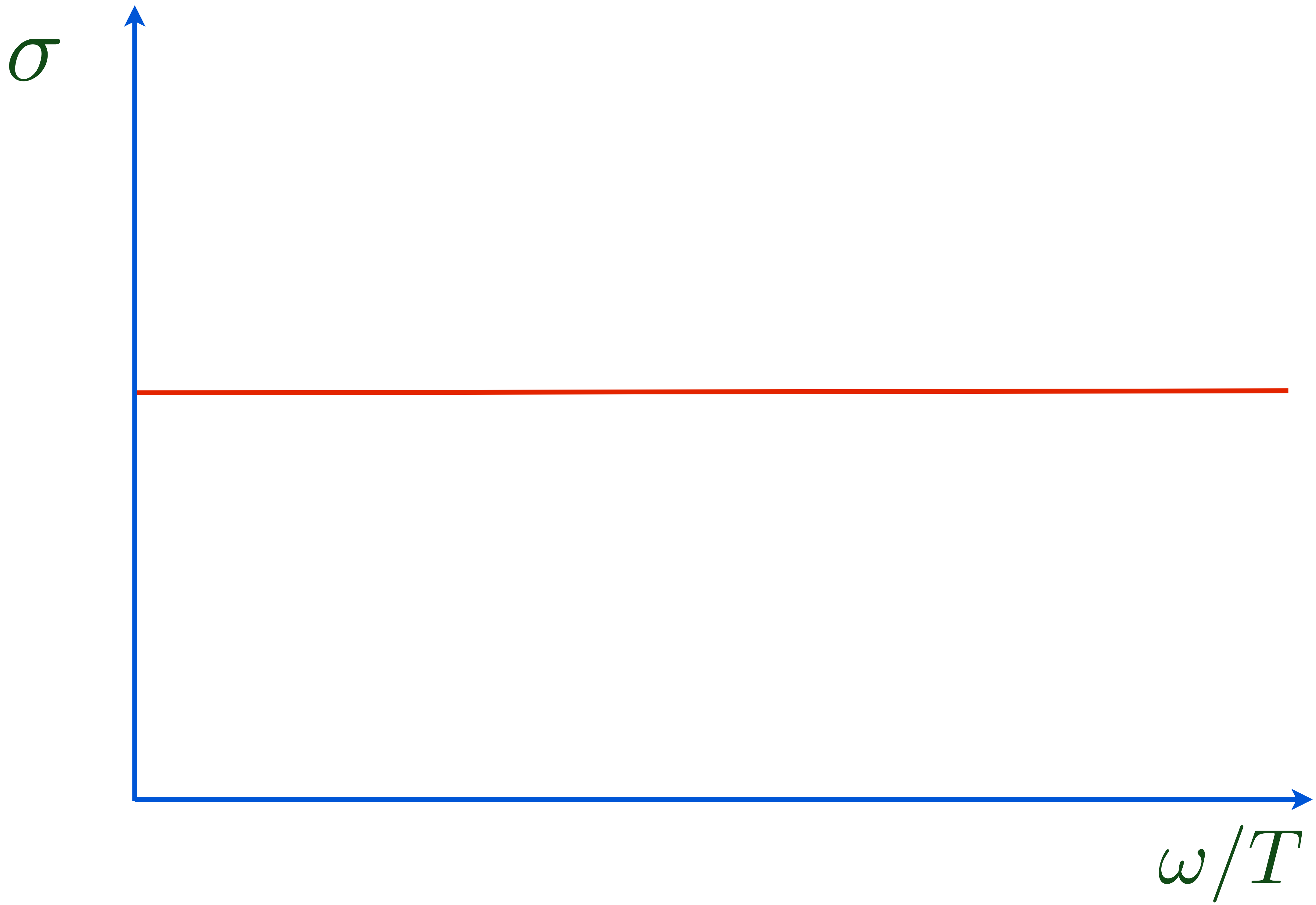
$$K^L(\omega, k)K^T(\omega, k) = \mathcal{K}^2 \tag{17}$$

where $\mathcal{K}$ is a known pure number, independent of ω and k. The combination of (17) and (15) now fully determine the response functions at zero momenta: $K^L(\omega, 0) = K^T(\omega, 0) = \mathcal{K}$. Computing the conductivity from Eq. (4), we then have

$$\sigma(\omega) = \mathcal{K}_\sigma(\omega/T) = \mathcal{K}; \tag{18}$$

i.e. the scaling function in Eq. (10) is independent of ω and equal to the value in Eq. (11). This result is an important surprise and the result is a direct consequence of the self-duality of the U(1) Maxwell theory on AdS_4.

Electrical transport in the holographic Maxwell theory



Let us now go beyond the Maxwell theory, and include all possible 4-derivative terms

$$\mathcal{S}_4 = \int d^4x\sqrt{-g}\left[\alpha_1 R^2 + \alpha_2 R_{ab}R^{ab} + \alpha_3\left(F^2\right)^2 + \alpha_4 F^4 \right. \quad (19)$$

$$\left. + \alpha_5 \nabla^a F_{ab}\nabla^c F_c{}^b + \alpha_6 R_{abcd}F^{ab}F^{cd} + \alpha_7 R^{ab}F_{ac}F_b{}^c + \alpha_8 RF^2\right]$$

where $F^2 = F_{ab}F^{ab}$, $F^4 = F^a{}_b F^b{}_c F^c{}_d F^d{}_a$ and the $\alpha_{\rm i}$ are some coupling constants. After using field redefinitions, and dropping terms of order F^4 which do not influence the linear conductivity, we obtain a form which has only *one* dimensionless constant γ (L is the radius of AdS$_4$):

$$\mathcal{S}_{vec} = \frac{1}{g_4^2}\int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{ab}F^{ab} + \gamma\, L^2 C_{abcd}F^{ab}F^{cd}\right], \quad (20)$$

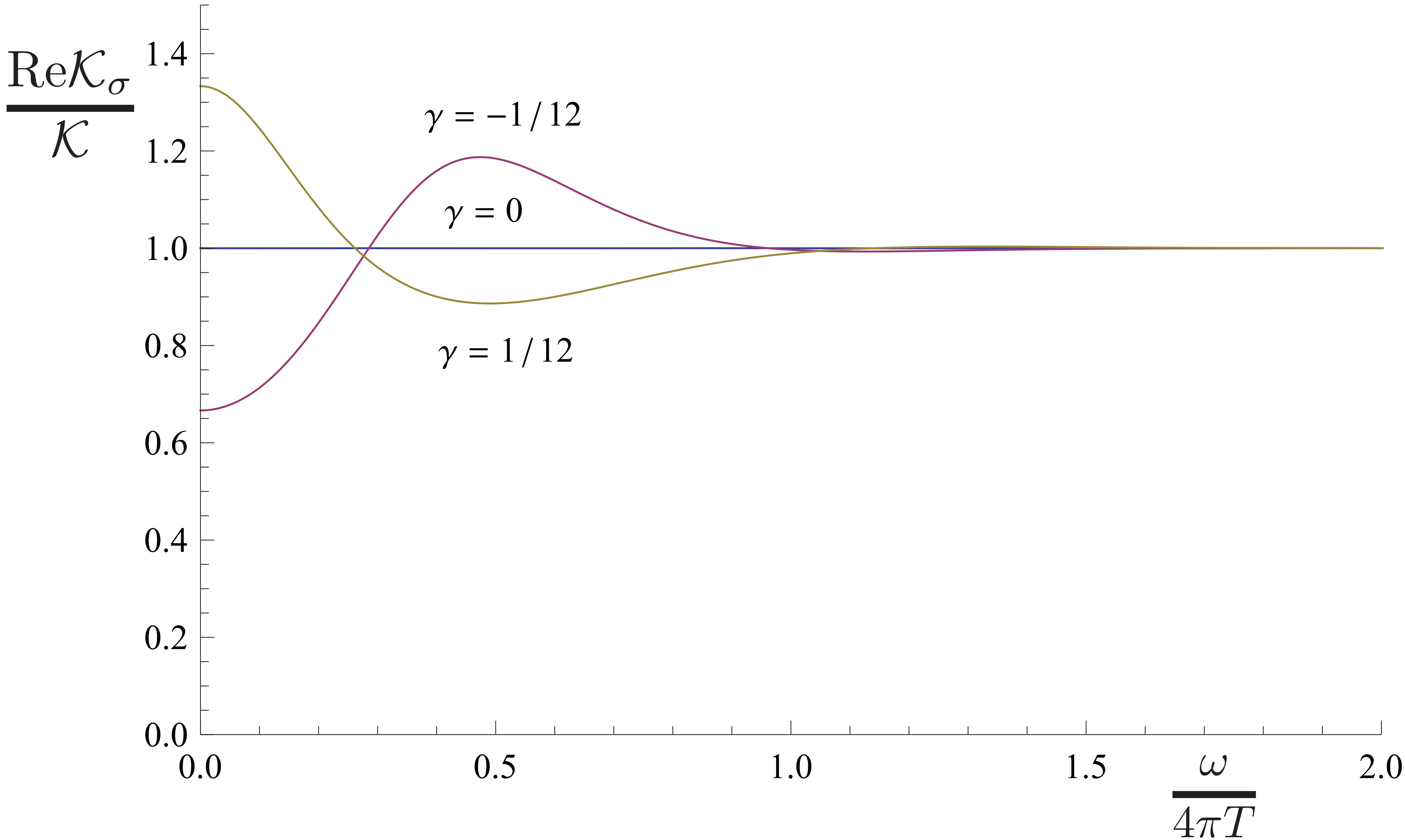
where we have formulated the extra four-derivative interaction in terms of the Weyl tensor C_{abcd}. *Stability and causality constraints on the effective theory restrict* $|\gamma| < 1/12$.

A generalized duality relation applies also to $\mathcal{S}_{vec}$. However this is *not* a *self*-duality. The dual CFT has current correlation functions which were characterized by functions $\widetilde{K}^{L,T}(\omega, k)$ which are distinct from those of the direct CFT $K^{L,T}(\omega, k)$, and the self-duality relation of Eq. (17) take the less restrictive form

$$K^L(\omega, k)\widetilde{K}^T(\omega, k) = \mathcal{K}^2 \quad , \quad K^T(\omega, k)\widetilde{K}^L(\omega, k) = \mathcal{K}^2. \qquad (21)$$

These duality relation determines the correlators of the dual CFT in terms of the direct CFT, but do not fix the latter. Identical relations apply under **particle-vortex duality** to the theory of complex scalar field, and to SQED3. Determination of the functions $K^L(\omega, 0) = K^T(\omega, 0)$ requires explicit computation using the extended theory $\mathcal{S}_{vec}$.

Electrical transport in the extended holographic theory



Lessons from AdS/CFT

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- For systems with particle-hole symmetry, a frequency dependent conductivity is obtained upon considering corrections to the effective Einstein-Maxwell theory. Stability conditions on the effective theory strongly restrict the range of frequency dependence.

Lessons from AdS/CFT

- Such quantum perfect fluids also have universal momentum transport. By extending the scaling arguments to momentum transport we would conclude that the ratio of the shear viscosity to the entropy density η/s should equal a universal number characterizing the collision-dominated regime. This number was computed in the Einstein-Maxwell theory by KSS and found to equal $\hbar/(4\pi k_B)$.

Outline

1. Quantum phase transitions of a semi-metal
   Honeycomb lattice, Dirac fermions and the Gross-Neveu model

2. Quantum critical transport
   Self-duality and the AdS/CFT correspondence

3. Quantum impurities and AdS_2
   Quantum spin coupled to a CFT

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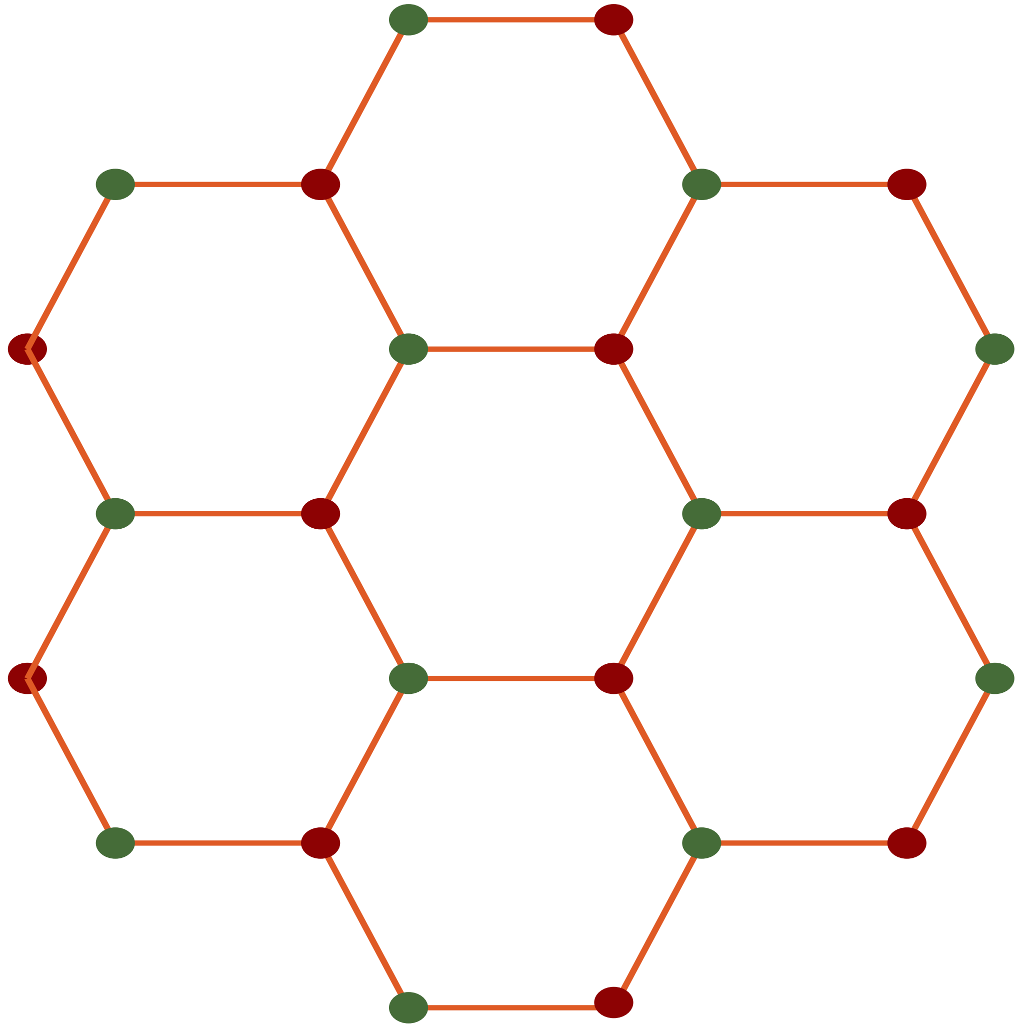
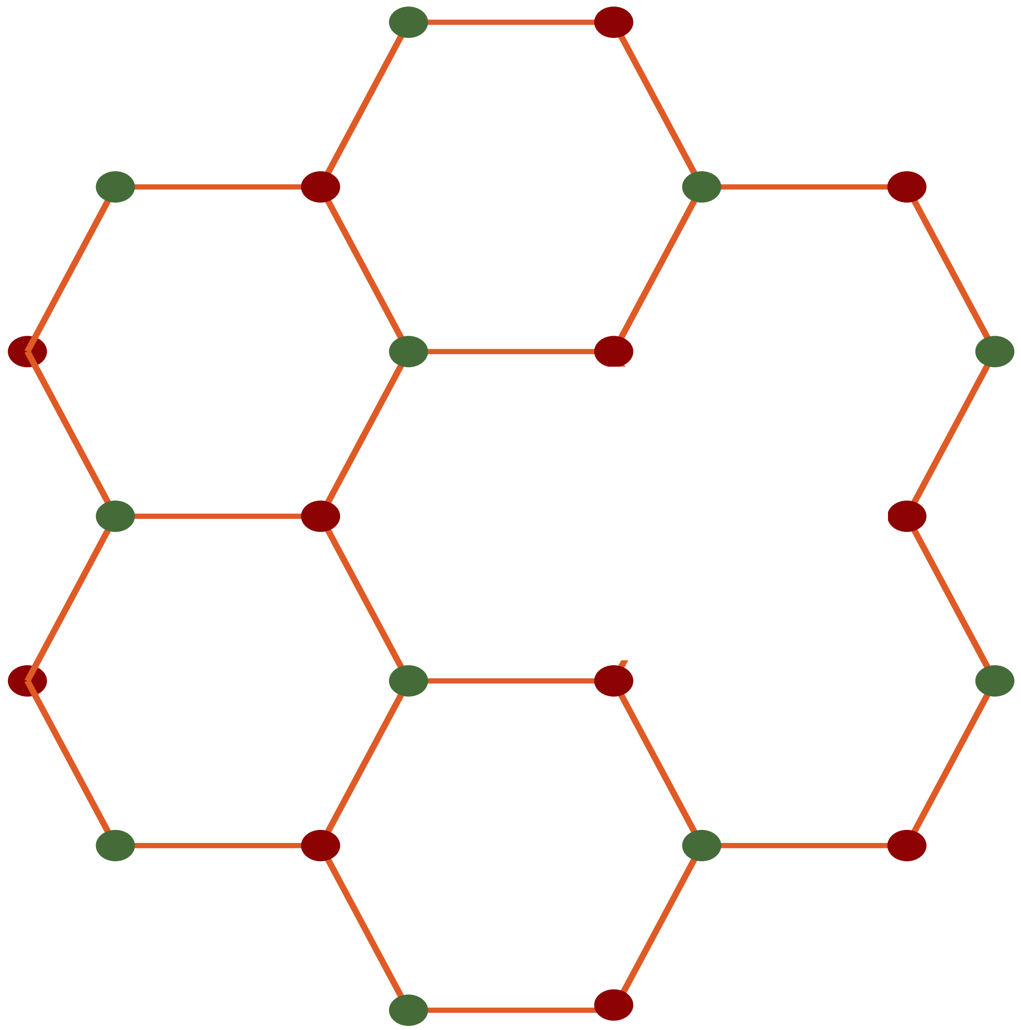
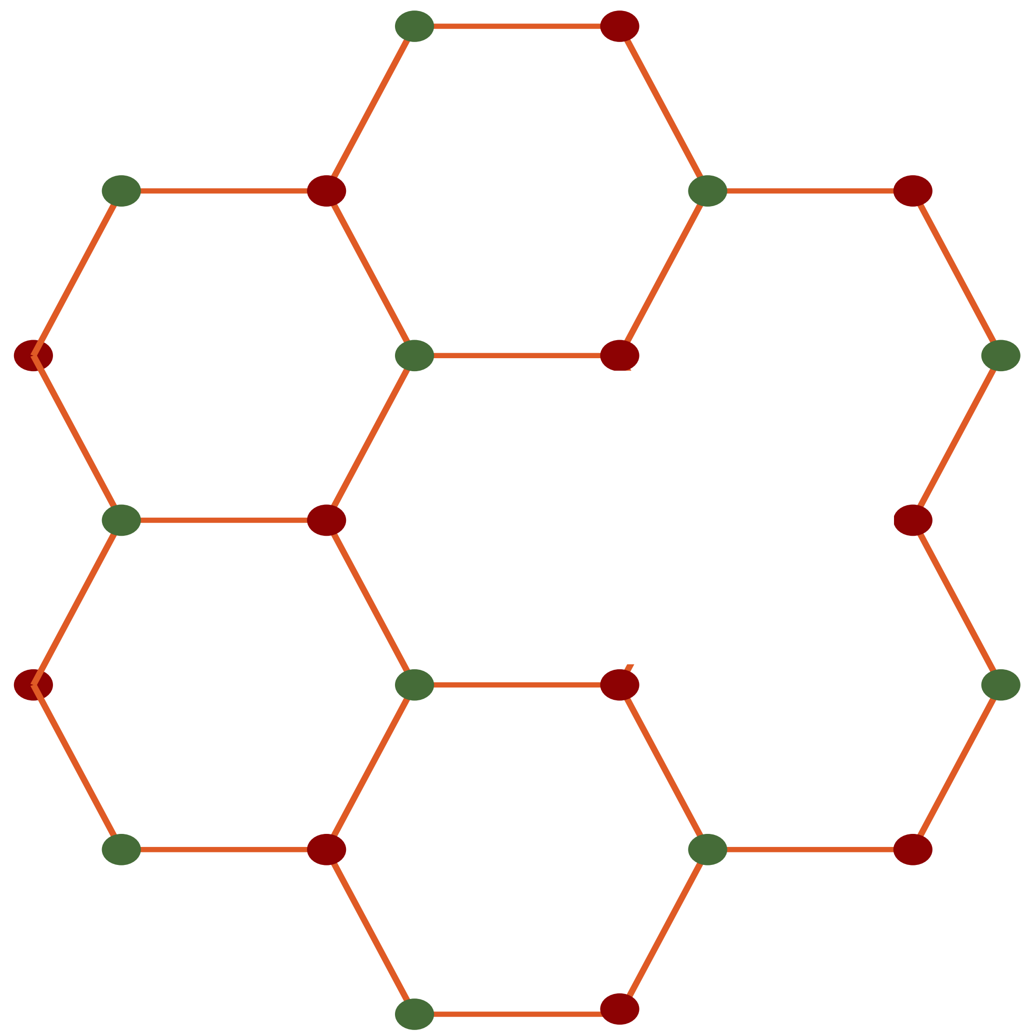




Honeycomb lattice with vacancy.

At $U = 0$, find a zero energy quasi-bound state with $|\psi(r)| \sim 1/r$.



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\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\Psi(\mathbf{r},\tau)\mathcal{D}\varphi^a(\mathbf{r},\tau)\mathcal{D}\chi_\alpha(\tau) \\
&\qquad \exp\left(-\int d^2r d\tau \mathcal{L}_{\rm GN} - \int d\tau \mathcal{L}_{\rm imp}\right) \\
\mathcal{L}_{\rm GN} &= \overline{\Psi}\gamma_\mu\partial_\mu\Psi + \frac{1}{2}\left[(\partial_\mu\varphi^a)^2 + s\varphi^{a2}\right] + \frac{u}{24}\left(\varphi^{a2}\right)^2 - \lambda\varphi^a\overline{\Psi}\rho^z\sigma^a\Psi \\
\mathcal{L}_{\rm imp} &= \chi_\alpha^\dagger \frac{\partial\chi_\alpha}{\partial\tau} - \kappa\,\chi_\alpha^\dagger\sigma^a_{\alpha\beta}\chi_\beta\,\varphi^a(\mathbf{r}=0,\tau)
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The fermion number $\chi_\alpha^\dagger\chi_\alpha$ commutes with the Hamiltonian, and the ground state is found in the sector $\chi_\alpha^\dagger\chi_\alpha = 1$.

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Exercise: Perform a tree-level RG transformation on $\mathcal{L}_{\rm GN} + \mathcal{L}_{\rm imp}$. The quadratic gradient terms are invariant under $\Psi' = \Psi e^{\ell}$ and $\varphi' = \varphi e^{\ell/2}$. Show that the coupling κ is relevant perturbations at the free field fixed point.

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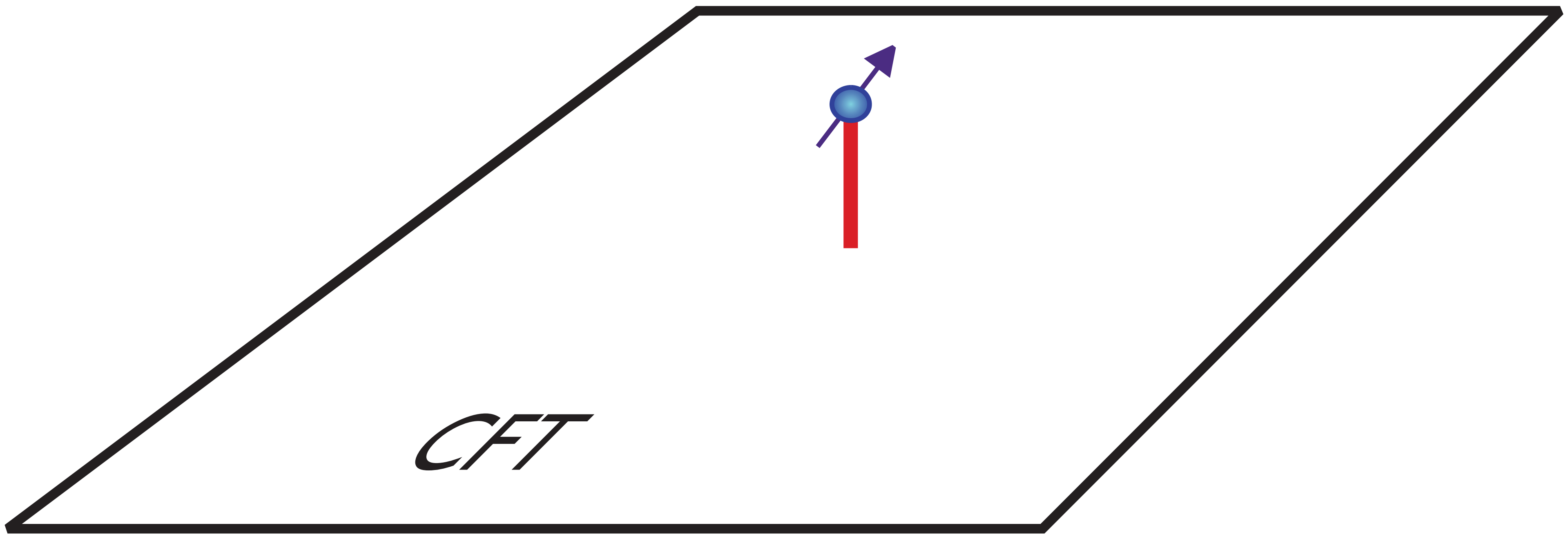
Under RG, we find that the coupling κ flows to a fixed point value. Then we have universal theory describing the dynamics near the impurity at the quantum-critical point between the semi-metal and the insulating antiferromagnet.

Quantum impurity coupled to a CFT



$$\mathcal{Z} = \int \mathcal{D}\Psi(\mathbf{r},\tau)\mathcal{D}\varphi^a(\mathbf{r},\tau)\mathcal{D}\chi_\alpha(\tau)\exp\left(-\int d^2r d\tau \mathcal{L}_{\rm GN} - \int d\tau \mathcal{L}_{\rm imp}\right)$$

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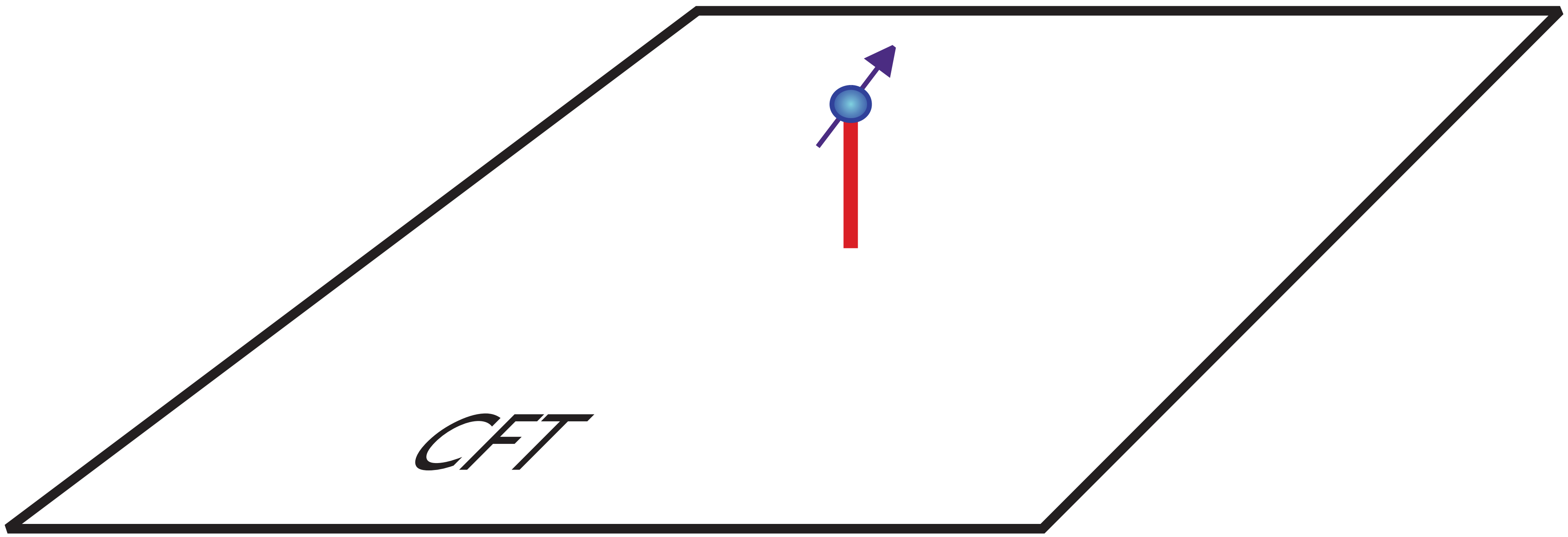
χ_α: spinful localized fermion describing impurity

κ: flows to a fixed point $\kappa \to \kappa^*$.

Quantum superspin coupled to SYM4



$$\mathcal{S} = \int d^3r d\tau \, \mathcal{L}_{\mathrm{SYM}} + \int d\tau \, \mathcal{L}_{\mathrm{imp}}$$

$$\mathcal{L}_{\mathrm{imp}} = \chi_b^\dagger \frac{\partial \chi^b}{\partial \tau} + i\chi_b^\dagger \left[\left(A_\tau(0,\tau)\right)^b_c + v^I \left(\phi_I(0,\tau)\right)^b_c \right] \chi^c$$

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Holographic lattices, dimers, and glasses

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We holographically engineer a periodic lattice of localized fermionic impurities within a plasma medium by putting an array of probe D5-branes in the background produced by N D3-branes. Thermodynamic quantities are computed in the large N limit via the holographic dictionary. We then dope the lattice by replacing some of the D5-branes by anti-D5-branes. In the large N limit, we determine the critical temperature below which the system dimerizes with bond ordering. Finally, we argue that for the special case of a square lattice our system is glassy at large but finite N, with the low temperature physics dominated by a huge collection of metastable dimerized configurations without long-range order, connected only through tunneling events.

Common features

- The correlations of the impurity fermion, or impurity spin, decay with a power-law in time, wtih non-trivial ‘impurity’ exponents.

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The SYM case is related in the large *N* limit to a AdS_2 geometry