Physics of Dense Neutron Star Matter

Somenath Chakrabarty

Department of Physics, Visva-Bharati,

Santiniketan 731 235, India E-Mail: somenath.chakrabarty@visva-bharati.ac.in

For

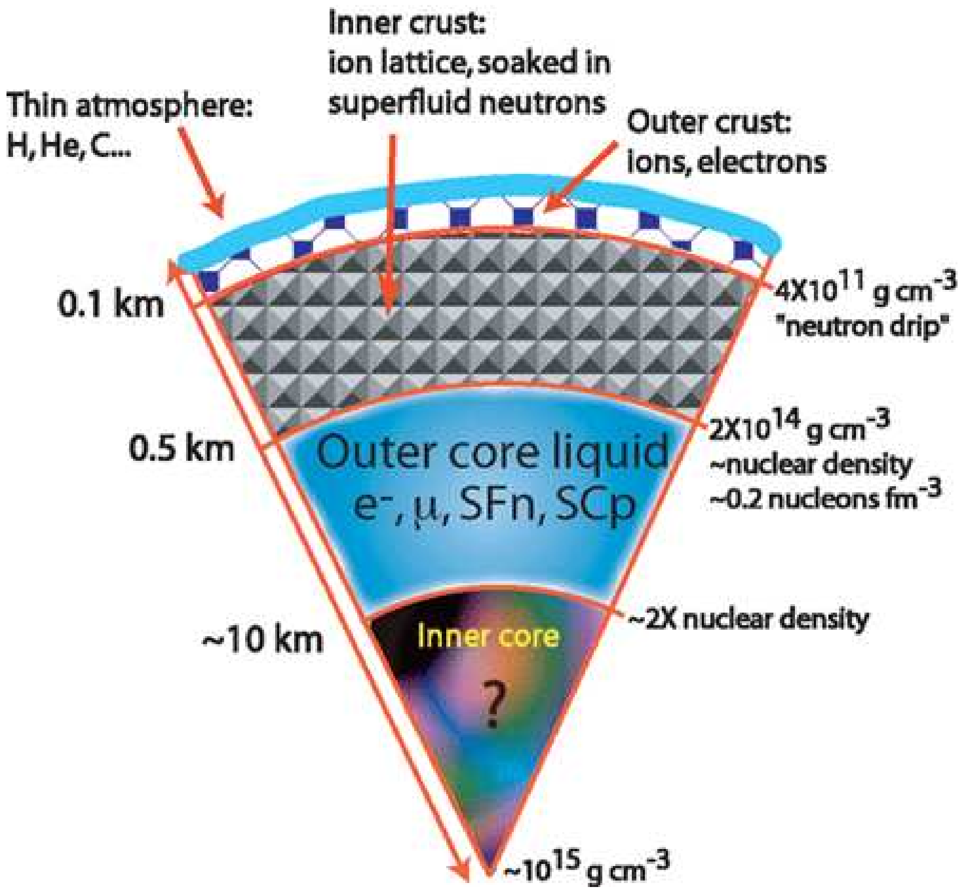
NN Interaction & the Nuclear Many-Body Problem,

Tata Institute of Fundamental Research, November 18-27, 2010.

SN1986J

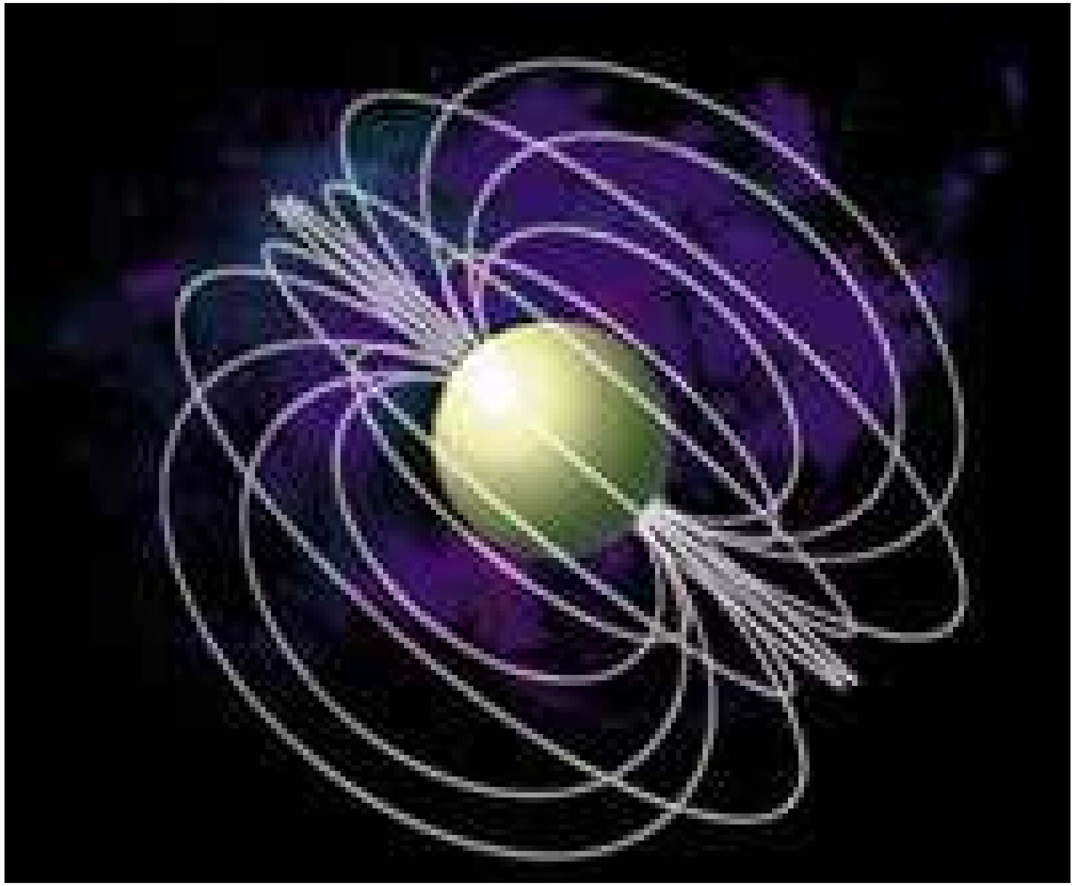
A Typical Neutron Star

Schematic Diagram for Neutron Star Internal Structure



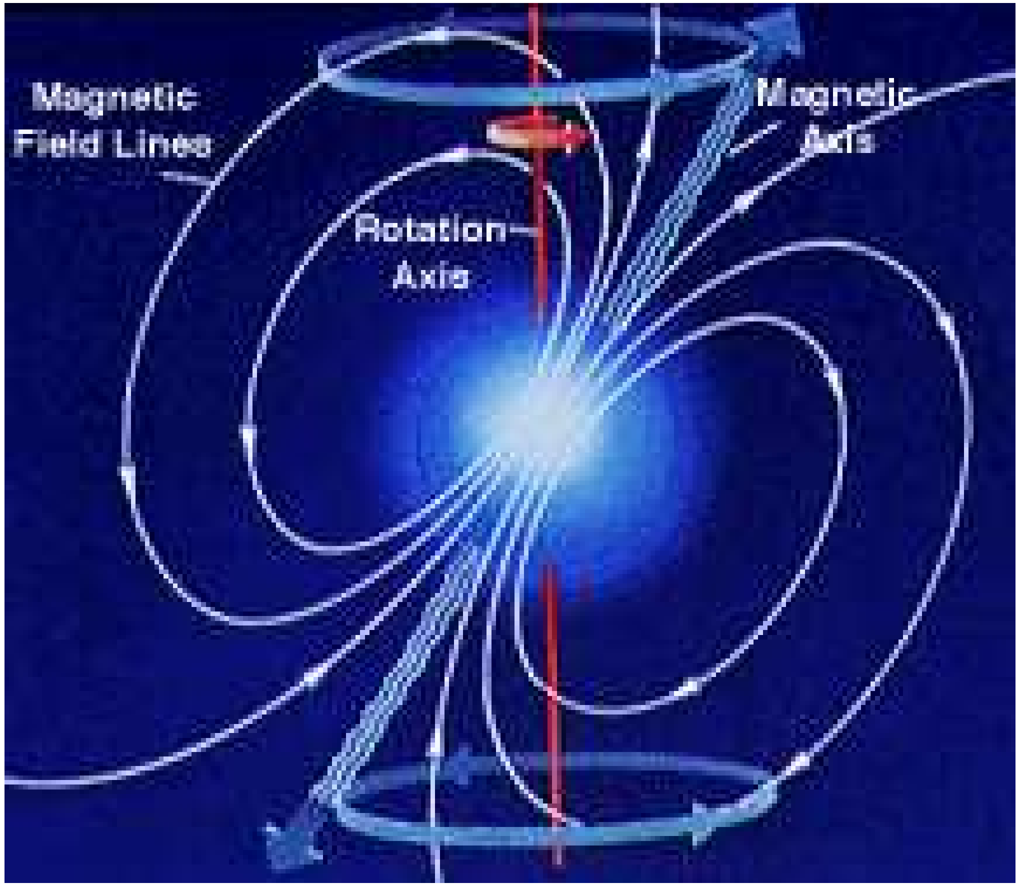
Mass: $M \sim 1.4 M_{\odot}$, Radius: $R \sim 10$km.

Schematic Diagram for Neutron Star as a Pulsar



Surface Magnetic Field:$B \sim 10^{12}$G $\longrightarrow$ Radio Pulsars, $B \sim 10^{8}$G $\longrightarrow$ milli-second Pulsars, $B \geq 10^{14}$G $\longrightarrow$ Magnetars.

Schematic Diagram Pulsar Emitting Synchrotron Radiation



References:

- Introductory Nuclear Physics, Samuel S.M. Wong, Prentice-Hall (1990).
- Nuclear Physics, R.R. Roy and B.P. Nigam, Wiley Eastern Ltd. (1979).
- R.D. Evans, The Atomic Nucleus, McGraw-Hill Book Co., NY (1955).
- A.E.S. Green, Rev. Mod. Phys. 30, 569 (1958).
- Atomic Data and Nuclear Data Tables 36, 495 (1987).
- H.B. Levy, Phys. Rev. 106, 1265 (1961).
- S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983) pp. 42 and 188.

- Y.S. Leung, Physics of Dense Matter, World Scientific, Singapore, (1984) pp. 35.

- Theoretical Nuclear and Sub-Nuclear Physics, J.D. Walecka, Oxford University Press (1995).

Inner-Crust Region

Density: $\rho \geq 10^{11}$gm cm^{-3} $\longrightarrow$ neutron drip ($\sim 4.3 \times 10^{11}$gm cm^{-3}). Matter:Nuclei, free neutron gas and electron gas (for overall charge neutrality).
Pressure: Neutron (beyond neutron drip density) and electron gas. Energy: Rest

mass of the nuclei (normal and neutron rich)
Equation of States: (i) Harrison-Wheeler (HW) ($10^7 \leq \rho \leq 4 \times 10^{11}$gm cm^{-3}),

(ii) Baym-Pethick-Sutherland (BPS) ($10^7 \leq \rho \leq 4.3 \times 10^{11}$gm cm^{-3}) (just onset of neutron drip)) and (iii) Baym-Bethe-Pethick (BBP) (neutron drip to nuclear density $\sim 10^{14}$ gm cm^{-3}).

HW EOS: Inner Crust (nuclei (normal and also neutron rich), electron gas and neutron gas (above neutron drip)):

How to get EOS?:

- Choose a value of $A > 56$.
- Get Z from A.
- Test whether neutron drip has been reached ($n_n > 0$)
- If $n_n > 0$, obtain ϵ_n and P_n- neutron matter energy density and pressure respectively.
- Then obtain x_e, the fractional abundance of electrons.
- Finally, obtain n_e, the electron density, ϵ'_e, the electron kinetic energy density and P_e, the electron pressure.

We start with the energy density of the system:

$$\epsilon = n_N M(A,Z) + \epsilon'_e(n_e) + \epsilon_n(n_n)$$

where $M(A,Z)$ is the energy of a nucleus $\Longleftarrow$ Nuclear Mass Formula. Here we define: $n_N \longrightarrow$ nuclei/vol, $n_n \longrightarrow$ neutrons/vol and $n_e \longrightarrow$ electrons/vol.

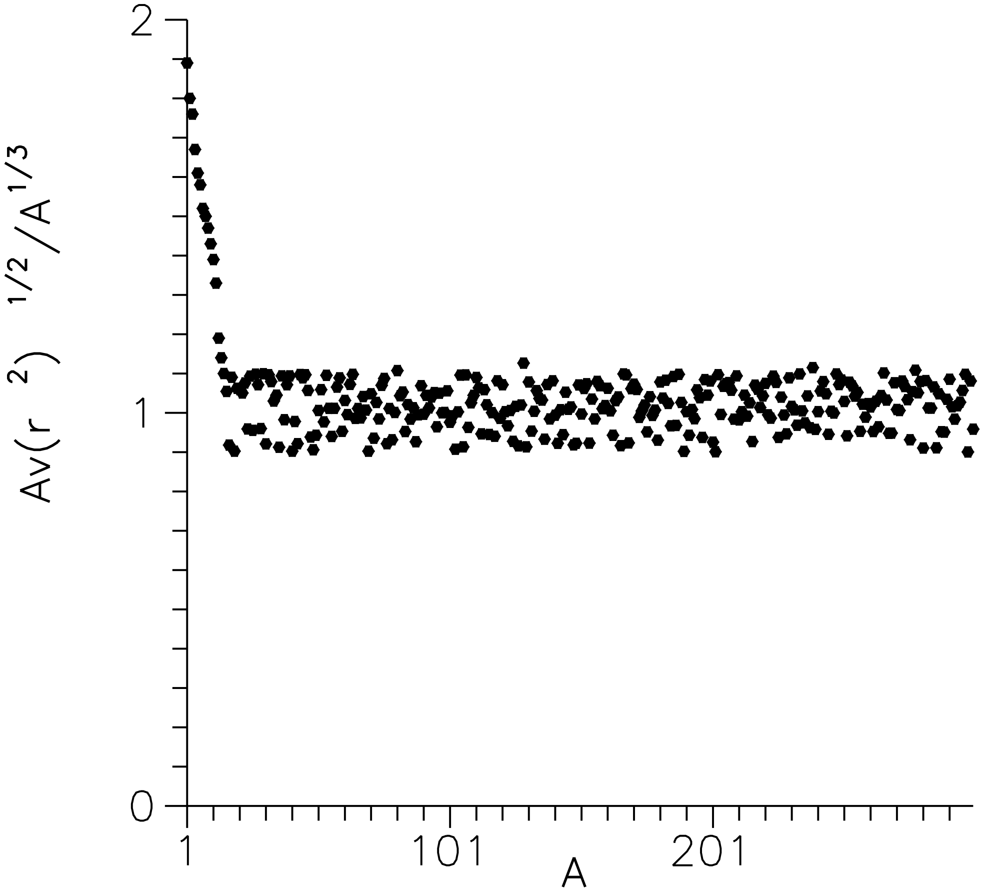
Semi-Empirical Mass Formula: It contains a number of terms. In HW EOS, nuclei are incompressible.

Various contributions to nuclear mass formula:

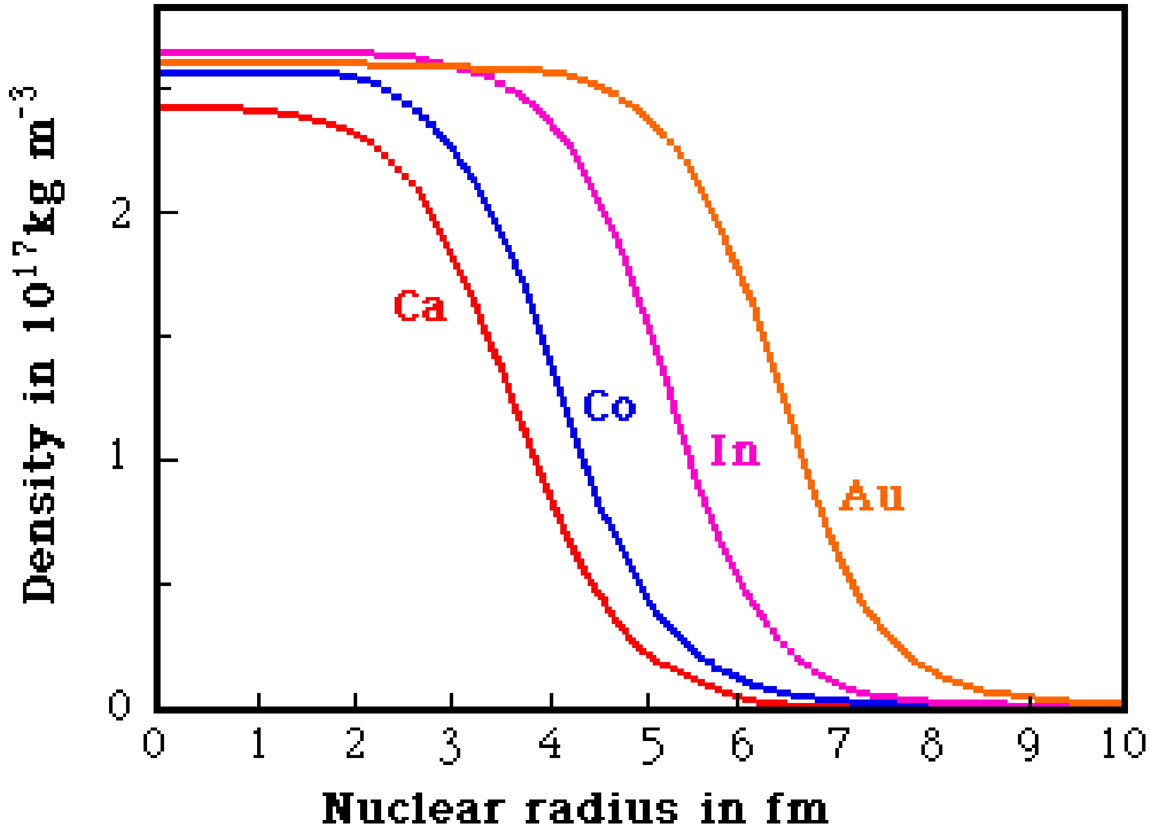
(1) Bulk part: $E_v = \alpha_1 A$. α_1 is an unknown parameter, to be obtained from the binding energy data.

Bulk energy comes from the saturation property of nuclear matter. Volume increases with the number of nucleons $\Longrightarrow$ nuclear radius $R = r_0 A^{1/3}$, with $r_0 = 1.12$fm.

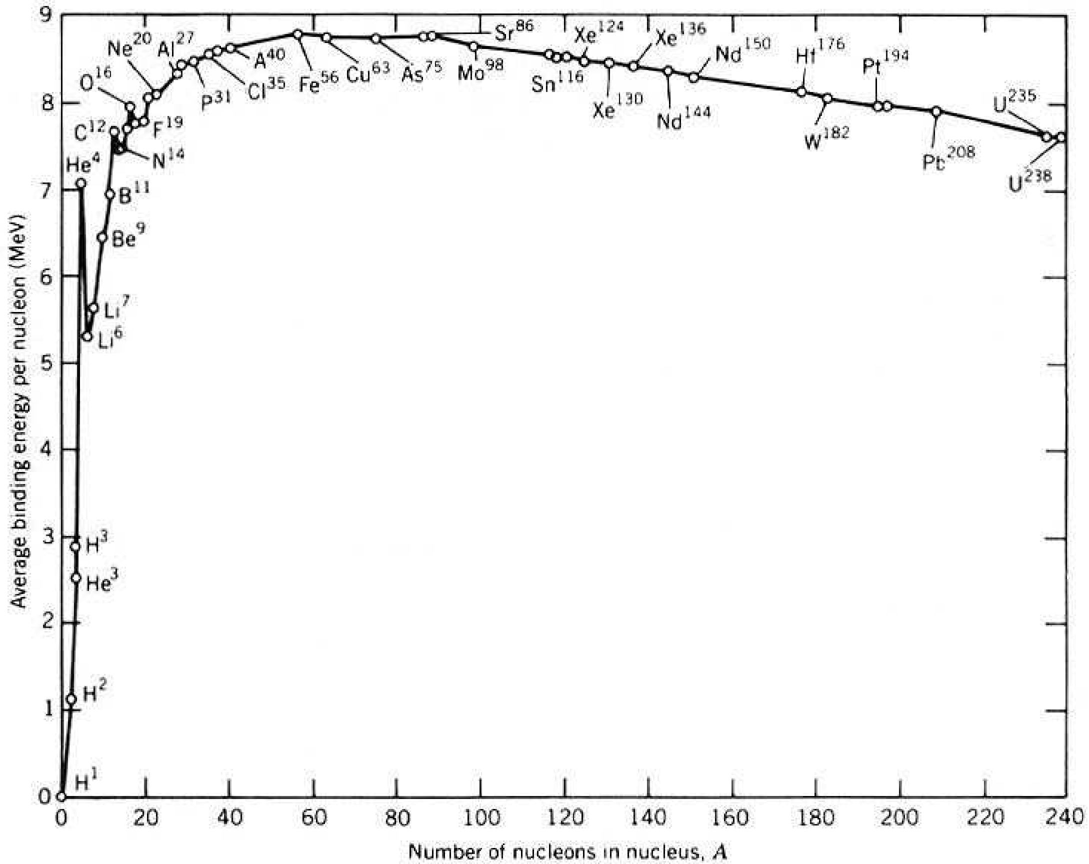
Nuclear radius $R \propto A^{1/3}$



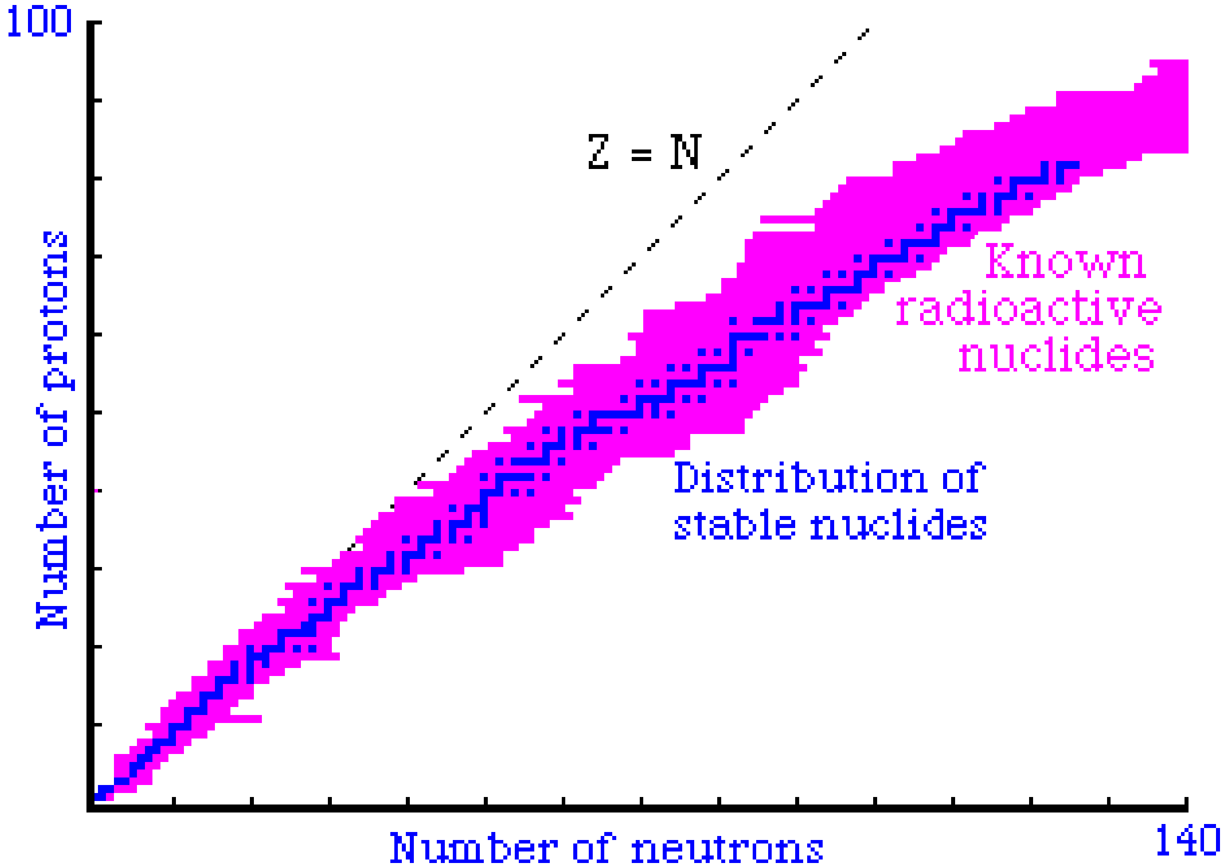
Nuclear Matter Distribution Inside a Nucleus



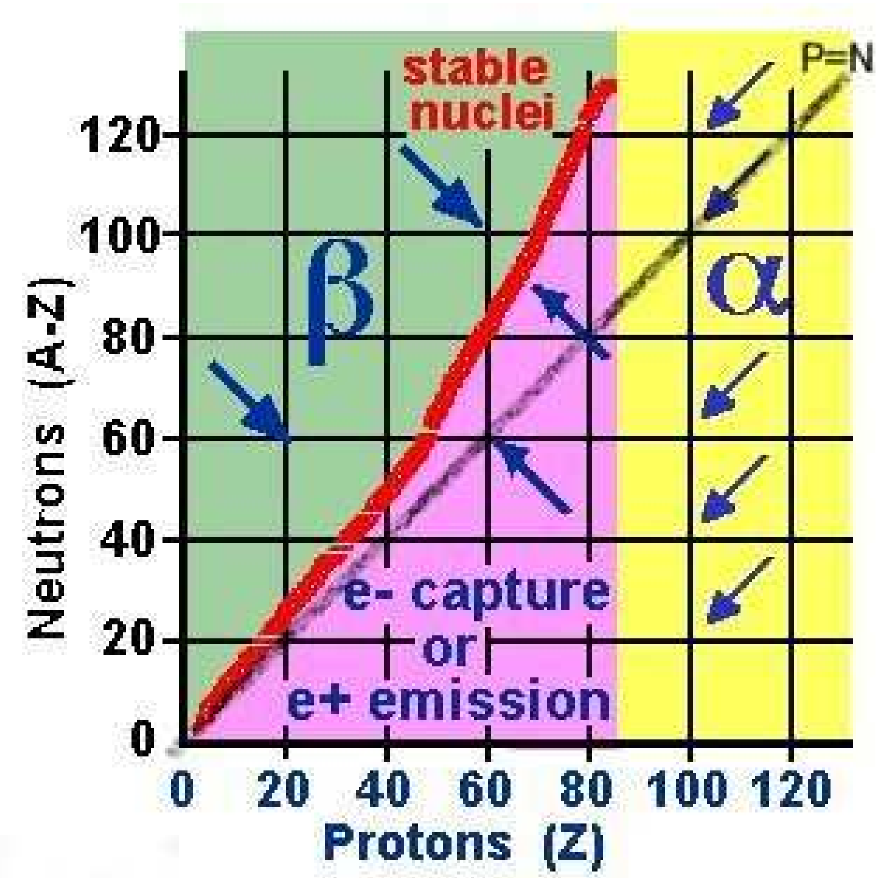
Nuclear Binding Energy Curve



Nuclear Binding Energy Curve



β-Stability Curve



(2) Surface Energy:- Reduces the binding energy. In HW, effect of the surroundings on nuclear surface energy has not been considered. Nuclei are placed in vacuum. $E_s = -\alpha_2 A^{2/3}$. $\alpha_2 = \sigma\pi R^2 = -\sigma\pi r_0^2 A^{2/3}$, where σ is the nuclear surface energy density.

(3) Coulomb Energy: Repulsive in nature- reduces the binding energy. To assemble a uniform sphere of Z protons require energy. It is given by

$$E_c = \frac{3}{5} e^2 \frac{Z(Z-1)}{R}$$

Expressing R in terms of A, we have

$$E_c = -\alpha_3 \frac{Z(Z-1)}{A^{1/3}}$$

So far all the effects are purely classical in nature.

(4) Iso-spin Effect or Symmetry Energy: Except for Coulomb repulsion, $N \approx Z$ nuclei are more stable. Symmetry energy is given by: $E_{sym} = -\alpha_4 \frac{(N-Z)^2}{A}$. Division by A: to make E_{sym} independent of A. This energy also reduces the binding energy. Maximum value is 0.

(5) Pairing Energy: For A-even: (a) N-even Z-even or (b) N-odd Z-odd. Even-even nuclei are more tightly bound than odd-odd nuclei for same A.

$$\Delta \longrightarrow \text{ pairing energy } = \begin{array}{rl} + & \delta \ \text{even} - \text{even} \\ & 0 \ \text{odd} \ A \\ - & \delta \ \text{odd} - \text{odd} \end{array}$$

in MeV. From binding energy data:$\alpha_1 \approx 16, \alpha_2 \approx 17, \alpha_3 =\approx 0.6, \alpha_4 \approx 25, \delta \approx 25/A$, all in MeV.

Alternative form of pairing energy:

$$\Delta \longrightarrow \text{ pairing energy } = \begin{array}{rl} + & 33A^{-3/4} \ \text{even} - \text{even} \\ & 0 \ \text{odd} \ A \\ - & 33A^{-3/4} \ \text{odd} - \text{odd} \end{array}$$

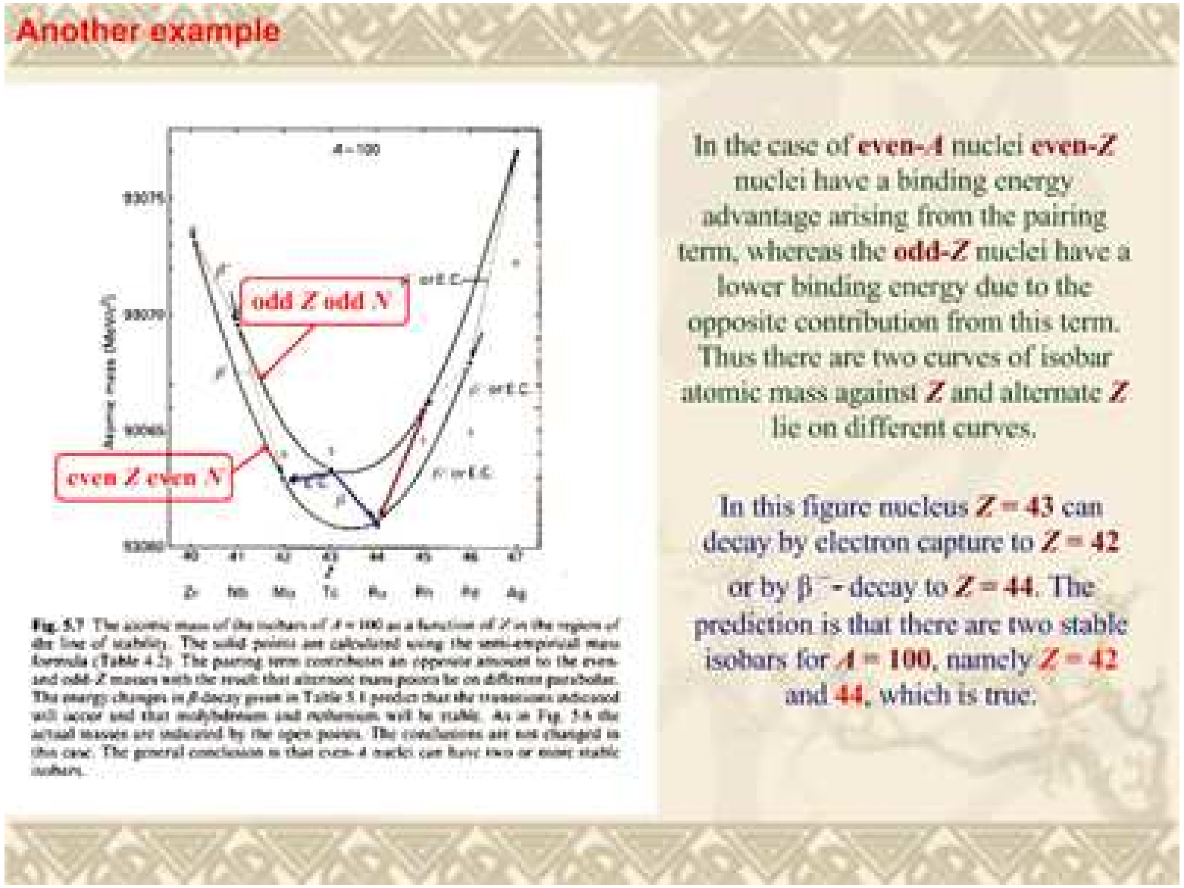
in MeV. Then the binding energy:

$$E_B = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} + \Delta$$

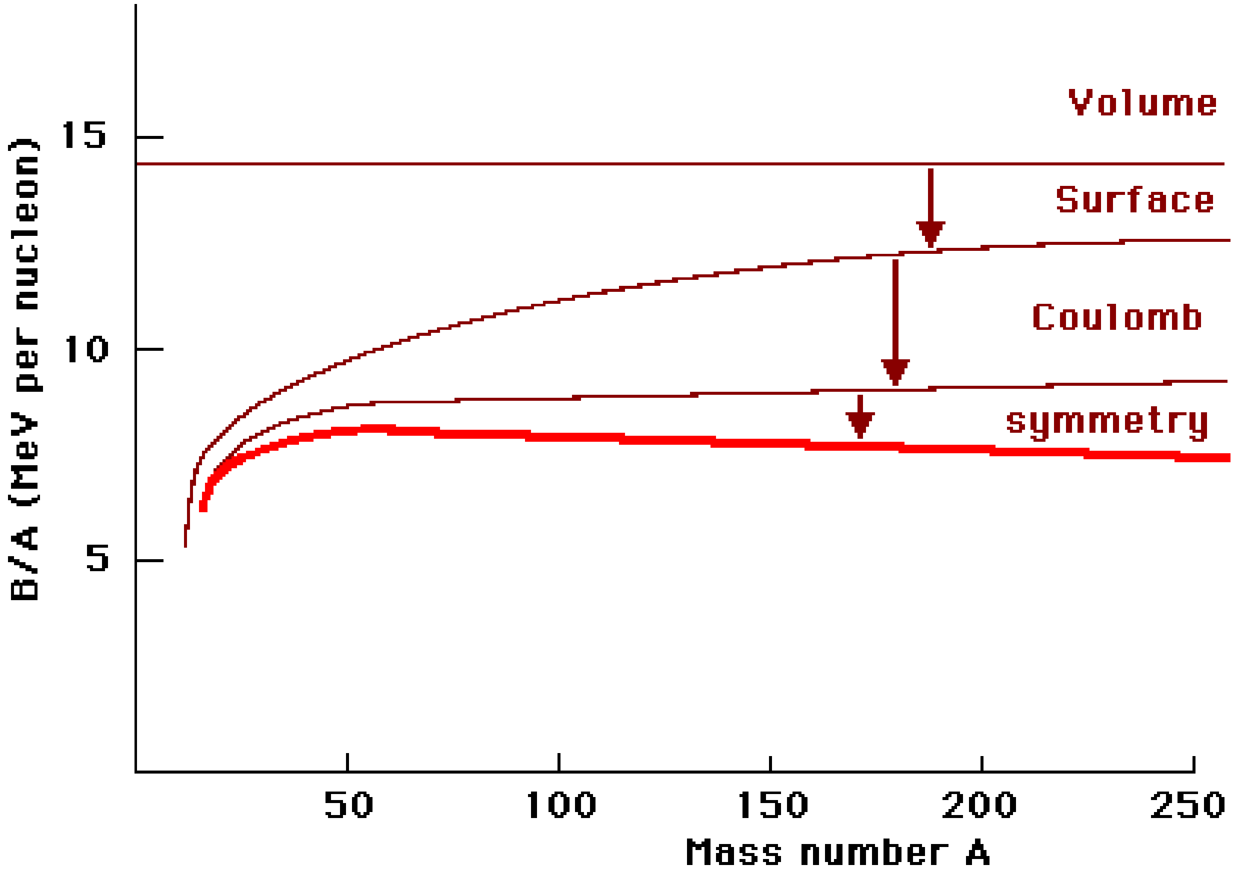
Semi-empirical mass:

$$M(A, Z) = ZM_H + (A - Z)M_M + E_B = ZM_H + (A - Z)M_N + A\overline{E}_B$$

Nuclear Pairing Energy Curve



Various Contributions in Semi-Empirical Mass Formula



Baryon density:$n = n_A A + n_n$, electron density: $n_e = n_N Z$, then the fractional abundances are related by:$Y_e = Y_N Z$ and $Y_N A + Y_n = 1$.

Energy density can be re-expressed as:

$$\epsilon = n(1 - Y_n)\frac{M(A,Z)}{A} + \epsilon_e'(n_e) + \epsilon_n(n_n)$$

We also have $n_e = n(1 - Y_n)Z/A$ and $n_n = nY_n$.

Let Fermi momentum for the ith. species ($i = e$ and n):p_{F_i}, defining $x_i = p_{F_i}/m_i$ with $c = 1$, we have:

$$n_i = \frac{g_i}{(2\pi)^3}\int d^3p = \frac{1}{\pi^2}\int_0^{p_{F_i}} p^2 dp = \frac{1}{3\pi^2\lambda_i^3}x_i^3$$

where $\lambda_i = 1/m_i$- Compton wave length and in natural units $\hbar = c = k_B = 1$.

Energy density:

$$\epsilon_i = \frac{g_i}{(2\pi)^3}\int d^3p(p^2 + m_i^2)^{1/2} = \frac{m_i}{\lambda_i^3}\chi(x_i)$$

and Kinetic Pressure:

$$P_i = \frac{1}{3}\frac{g_i}{(2\pi)^3}\int d^3p\frac{p^2}{(p^2 + m_i^2)^{1/2}} = \frac{m_i}{\lambda_i^3}\phi(x_i)$$

where

$$\chi(x_i) = \frac{1}{8\pi^2}\left[x_i(1+x_i^2)^{1/2}(1+2x_i^2) - \ln\left\{x + (1+x^2)^{1/2}\right\}\right]$$

and

$$\phi(x_i) = \frac{1}{8\pi^2}\left[x_i\left(1+x_i^2\right)^{1/2}\left(\frac{2x_i^2}{3} - 1\right) + \ln\left\{x_i + (1+x_i^2)^{1/2}\right\}\right]$$

For electron the kinetic energy density:

$$\epsilon_e' = \epsilon_e - n_e m_e$$

Therefore the semi-empirical nuclear mass may be written in the form:

$$M(A,Z) = [(A-Z)m_n + Z(m_p + m_e) - A\overline{E_B}]$$

where $\overline{E_B} \longrightarrow$ mean binding energy per baryon. Considering all kinds of contributions:

$$M(A,Z) = m_u\left[b_1 A + b_2 A^{2/3} - b_3 Z + b_4 A\left(\frac{1}{2} - \frac{Z}{A}\right)^2 + \frac{b_5 Z^2}{A^{1/3}}\right]$$

where $b_1 = 0.991749, b_2 = 0.01911, b_3 = 0.000840, b_4 = 0.10175, b_5 = 0.000763$ and $m_u = 1.66057 \times 10^{-24}$gm (atomic mass unit)- average baryon mass.

Assuming A and Z are continuous variables, we have:

$$\frac{\partial \epsilon}{\partial Z} = \frac{\partial}{\partial Z}\left[n_N M(A,Z) + \epsilon_e' + \epsilon_n\right] = 0$$

$\Longrightarrow$

$$\frac{\partial M}{\partial Z} = -(\mu_e - m_e)$$

$\Longrightarrow$

$$b_3 + b_4\left(1 - \frac{2Z}{A}\right) - 2b_5\frac{Z}{A^{1/3}} = \left[(1 + x_e^2)^{1/2} - 1\right]\frac{m_e}{m_u}$$

$\longrightarrow$ continuous limit of the β-stability condition. $M(Z-1,A)$ is in equilibrium with $M(Z,A)$, the free electron being at the top of the Fermi level. Here,

$$\mu_n = \frac{\partial \epsilon_n}{\partial n_e} \ \text{and}\ \ \mu_e - m_e = \frac{\partial \epsilon_e'}{\partial n_e}$$

Again

$$\frac{\partial \epsilon}{\partial A} = \frac{\partial}{\partial A}\left[n_N M(A,Z) + \epsilon_e' + \epsilon_n\right] = 0$$

gives

$$A\frac{\partial M}{\partial A} - M = Z(\mu_e - m_e)$$

Hence

$$Z\frac{\partial M}{\partial Z} + A\frac{\partial M}{\partial A} - M = 0$$

$\Longrightarrow$

$$Z = \left(\frac{b_2}{2b_5}\right)^{1/2} A^{1/2} = 3.54A^{1/2}$$

Finally,

$$\frac{\partial M}{\partial A} = \mu_n$$

$\Longrightarrow$

$$b_1 + \frac{2b_2A^{-1/3}}{3} + b_4\left(\frac{1}{4} - \frac{Z^2}{A^2}\right) - \frac{b_5Z^2}{3A^{4/3}} = (1 + x_n^2)^{1/2}\frac{m_n}{m_u}$$

Hence Z increases with A ($Z \sim A^{1/2}$), but Z/A decreases with A

How to get EOS?:

- Choose a value of $A > 56$.
- Get Z from A.
- Test whether neutron drip has been reached ($n_n > 0$)
- If $n_n > 0$, obtain ϵ_n and P_n- neutron matter energy density and pressure respectively.
- Then obtain x_e, the fractional abundance of electrons.
- Finally, obtain n_e, the electron density, ϵ'_e, the electron kinetic energy density and P_e, the electron pressure.

Then mass density or the energy density:

$$\rho = \epsilon = n_e \frac{M(A,Z)}{Z} + \epsilon_e' + \epsilon_n$$

Kinetic pressure: $P = P_e + P_n$ and the baryon density:

$$n = n_e \frac{A}{Z} + n_n$$

hence the equation of state $P \equiv P(\rho)$.

Neutron drip: $\rho \sim 3.18 \times 10^{11}\text{gm cm}^{-3}$ at $(Z, A) = (122, 39.1) \longrightarrow$ Yttrium and in this density $\mu_e \sim 23.6$MeV.

At $\rho \sim 4.54 \times 10^{12}\text{gm cm}^{-3}$, $(A, Z) = (187, 48.7)$. At this density $P_n/P \sim 0.6$.

Above this density $\longrightarrow$ free $n - p - e$ mixture in β-equilibrium.

$$\begin{aligned} n_p &= n_e \\ \mu_n &= \mu_p + \mu_e \\ n &= n_p + n_n \end{aligned}$$

References:

- B.K. Harrison, M. Wakano and J.A. Wheeler, ”Matter at High Density; End Point of Thermonuclear Evolution”, Brussels, Belgium, (1958) pp. 124.
- B.K. Harrison, K.S. Thorne, M. Wakano and J.A. Wheeler, Gravitation theory and gravitational collapse, University of Chicago Press, Chicago, (1965).
- S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983) pp. 42 and 188.
- Y.S. Leung, Physics of Dense Matter, World Scientific, Singapore, (1984) pp. 35.
- Nandini Nag, Sutapa Ghosh and Somenath Chakrabarty, Euro. J. Phys. A45, (2010) 99.
- Theoretical Nuclear and Sub-Nuclear Physics, J.D. Walecka, Oxford University Press (1995).

BPS EOS: Inner Crust (nuclei (normal and also neutron rich), electrons in Wigner-Seitz cells, free electron gas (at high density) and neutron gas (above neutron drip)):

Energy Density of the System:

$$\epsilon = n_N M(A,Z) + \epsilon_e'(n_e) + \epsilon_n(n_n) + \epsilon_L$$

$\epsilon_L \longrightarrow$ Lattice energy.

Nuclei are at regular lattice points. Around each nuclei a charge neutral cell, known as Wigner-Seitz (WS) cell is considered. Lattice energy:

$$\epsilon_L = n_e \frac{E_c}{Z} = n_e \frac{E_{ei} + E_{ee}}{Z} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3} = a n_e^{4/3} \approx -1.45079 n_e^{4/3}$$

for Fe-nucleus. For BCC type lattice: $\epsilon_L \approx -1.44423$. The arrangement is almost BCC type.

Lattice contribution of pressure:

$$P_L = -\frac{d(E_c/Z)}{d(1/n_e)} = n_e^2 \frac{d}{dn_e}\left(\frac{E_c}{Z}\right) = \frac{1}{3}\epsilon_L$$

Modified form of the Basic Equations:

$$\frac{\partial M}{\partial Z} = -(\mu_e - m_e) - 2aZ^{2/3}n_e^{1/3}$$

$$\frac{\partial M}{\partial A} = \mu_n - \frac{4}{3}aZ^{5/3}n_e^{1/3}$$

and

$$Z\frac{\partial M}{\partial Z} + A\frac{\partial M}{\partial A} - M = -\frac{2}{3}aZ^{5/3}n_e^{1/3}$$

Results: A increases with $n \longrightarrow$. Z also increases with n.

In the mass formula, the extra effect, which is quite important, the local increase in binding energy for nuclei near closed shell- known as shell effect has been taken into account:

References:

- G. Baym, C.J. Pethick and P. Sutherland, Astrophys. Jour. **170**, (1971) 299.
- S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983) pp. 42 and 188.
- Y.S. Leung, Physics of Dense Matter, World Scientific, Singapore, (1984) pp. 35.
- Nandini Nag, Sutapa Ghosh and Somenath Chakrabarty, Euro. J. Phys. A45, (2010) 99.

BBP EOS: Inner Crust (nuclei (normal and also neutron rich), electrons in Wigner-Seitz cells, free electron gas (at high density) and neutron gas (above neutron drip)):

Basic Assumptions:

1. Nuclei are compressible liquid drops.
2. Pressure equilibrium: Internal pressure = external pressure.
3. Chemical equilibrium inside and out side matter.
4. Effect of external matter on surface energy: surface energy vanishes when the external density of neutron matter just reaches the internal nuclear density $\longrightarrow$ the nuclei just dissolve to uniform neutron matter (with a small fraction of protons and electrons).

Total energy density:
$$\epsilon = \epsilon(A, Z, n_N, n_n, V_N) = n_N(W_N + W_L) + \epsilon_n(n_n)(1 - V_N n_N) + \epsilon_e(n_e)$$

where n_N: nuclei/volume, n_n: free neutrons/volume, V_N: volume of a nucleus (decreases as the outside pressure by n or e increases); V_N is such that $V_N n_N$ is the fraction of unit volume occupied by the nuclei, W_N: energy of a nucleus, including the rest mass, W_L: lattice energy, ϵ_n: energy of free neutron/volume and ϵ_e: energy of electrons/volume.

Consider an unit volume: $1 - V_N n_N$: the fraction occupied by neutron gas. Electron density $n_e = Z n_N$. Baryon density $n = n_N A + n_n$. Let V: volume containing N_n free neutrons and $n_n V$ nuclei. Hence

$$n_n = \frac{N_n}{V_n} = \frac{N_n}{V(1 - V_N n_N)}$$

Equilibrium: Minimization of energy for a fixed n:
Energy/nucleon inside the nuclei must be minimum $\Longrightarrow$

$$\frac{\partial}{\partial A}\left(\frac{W_N + W_L}{A}\right)_{Z, n_N A, n_N V_N, n_n} = 0$$

Chemical potentials:μ_e- electrons, $\mu_n^{(N)}$- neutrons inside the nuclei, $\mu_n^{(G)}$- neutrons in the neutron matter and $\mu_p^{(N)}$- protons inside the nuclei.

Stable to β-decay:

$$\mu_e = \frac{1}{Z}\frac{\partial \epsilon}{\partial Z} = -\frac{\partial}{\partial Z}(W_N + W_L)_{A, n_N, V_N, n_n}$$

Again

$$\mu_n^{(N)} = \frac{\partial}{\partial A}(W_N + W_L)_{A-Z,n_n,V_N,n_n}$$
$$= \frac{\partial}{\partial Z}(W_N + W_L)_{A,n_N,V_N,n_n} + \frac{\partial}{\partial A}(W_N + W_L)_{Z,n_N,V_N,n_e}$$

since $\left(\frac{\partial A}{\partial Z} = 1\right)$

Hence we have $\mu_p^{(N)} = \mu_e + \mu_n^{(N)}$

Writing

$$\frac{\partial}{\partial A}|_{Z,n_N,V_N,n} = \frac{\partial}{\partial A}|_{Z,n_N,V_N,n_n} + \frac{\partial n_n}{\partial A}\frac{\partial}{\partial n_n}|_{Z,n_N,V_N,A},$$

we have $\mu_n^{(N)} = \mu_n^{(G)}$: It must cost no energy to transfer a neutron from the gas to the nucleus and vice-versa.

Minimizing ϵ w.r.t. V_N for fixed Z, A, n_N and $N_n/V = n_n(1 - V_N n_N)$, we have $P_n^{(N)} = n_n\mu_n^{(G)} - \epsilon_n$, i.e., $P_n^{(N)} = P_n^{(G)} \longrightarrow$ Pressure equilibrium.

To obtain EOS, one has to know the functional forms for: W_N, W_L, ϵ_n and ϵ_e.

Form of W_N:

$$W_N = A[(1-x)m_n + xm_p + W(k,x)] + W_c + W_s$$

where $x = Z/A$- determines $n-p$ asymmetry of the system, $W(k,x)$- bulk energy of the nuclear matter / nucleon, k is the Fermi momentum, W_c is the Coulomb energy and W_s is the surface energy per nucleon. Baryon density:

$$n = \frac{2k^3}{3\pi^2} \quad \text{(bulk matter)} \quad \text{and} \quad = \frac{A}{V_n} \quad \text{(inside the nuclei)}$$

Bulk energy density inside the nucleus:$\epsilon_N = n_N[W(k,x) + (1-x)m_n + xm_p]$.
Keeping consistency, same outside the nuclei:$\epsilon_n = n_n[W(k_n,0) + m_n]$.

Evaluation of $W(k,x)$:

1. Parameters can be obtained by fitting nuclear data. This is equivalent with the semi-empirical mass formula.
2. Nuclear potential approach- fitted from scattering data.
3. Many-body theory for various k and x ranges.

Evaluation of W_s:
Must vanish explicitly when the density of neutron gas and the density of the nucleus becomes exactly equal. The surface energy used by BBP is constructed to vanish explicitly at the matter density mentioned above.

In BBP EOS, the total surface energy is given by

$$W_s = \frac{\sigma (W_0 - W_i)^{1/2}}{w_0^{1/2}} \frac{(n_i - n_0)^{3/2}}{n_s^{3/2}} \frac{k_0^2}{k^2} A^{2/3}$$

where $\sigma \sim 20$MeV, $w_0 = 16.5$MeV, $k_0 = 1.43\text{fm}^{-3}$, $W_0 = W(n_0)$- bulk energy outside the nucleus and $W_i = W(n_i)$, bulk energy inside the nucleus.

Evaluation of W_c:

$$W_c = \frac{3}{5} \frac{Z^2 e^2}{r_N}$$

-the energy of a uniformly charged sphere of radius r_N ($V_N = 4\pi r_N^3/3$- the volume of a nucleus).

Evaluation of W_L:
BBP result:

$$W_{c+L} = \frac{3}{5} \frac{Z^2 e^2}{r_N} \left(1 - \frac{r_N}{r_c}\right)^2 \left(1 - \frac{r_N}{2r_c}\right)$$

where r_c is given by $4\pi r_c^3 n_N/3 = 1$. In $W_c + W_L$, the Coulomb energy for electron gas is included. ϵ_e is known.

In BBP-model, for $\rho \sim 1.5 \times 10^{12}$gm cm^{-3}, $P_n/P \sim 0.20 \longrightarrow 20\%$, whereas, for $\rho \sim 1.5 \times 10^{12}$gm cm^{-3}, $P_n/P \sim 0.80 \longrightarrow 80\%$

In BBP EOS, the adiabatic index Γ drops sharply ($\approx 4/3$) near neutron drip density and rises above $4/3$ beyond $\rho = 7 \times 10^{12}$gm cm^{-3}.

There is no stars for which ρ_c is in this region ($\Gamma > 4/3$ from GR). Neutron star surface density can be within or less than these values- it is the average Γ that matters.

References:

- G. Baym, H.A. Bethe and C.J. Pethick, Nucl. Phys. **A175**, (1971) 225.
- S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983) pp. 42 and 188.
- Y.S. Leung, Physics of Dense Matter, World Scientific, Singapore, (1984) pp. 35.
- Nandini Nag, Sutapa Ghosh and Somenath Chakrabarty, Euro. J. Phys. A45, (2010) 99.

EOS with Yukawa Potential

A oversimplified model calculation.

Potential:

$$\phi(r) = V(r) = \pm g^2 \frac{\exp(-\mu r)}{r}$$

$\mu \sim m_\pi \sim 1.4$fm, $m_\pi = 140$MeV, exchange quanta (π-mesons) mass. $\alpha_s = g^2/4\pi \sim 10$, strong coupling constant (electromagnetic coupling or fine structure constant $\alpha_c = e^2/4\pi \sim 1/137$)

Inside nuclei $1/\mu \ll R \Longrightarrow$ number of particle is sufficiently large.

Interaction energy in a volume V:

$$E_V = \frac{1}{2}\sum_{i \neq j} V_{ij} = \pm \frac{1}{2} n^2 g^2 \int\int \frac{\exp(-\mu r_{12})}{r_{12}} d^3r_1 d^3r_2$$

To evaluate, assume r_2 is the origin, take spherical polar coordinate:$\Longrightarrow r_{12} = r$. Since nuclear interaction range is small enough, we can integrate from 0 to ∞ without any appreciable error. Hence

$$E_V = \pm \frac{1}{2} n^2 g^2 \frac{4\pi}{\mu^2}$$

Then the energy density

$$\epsilon = \epsilon_{kin} + \epsilon_V$$

Kinetic part:

$$\epsilon_{kin} = n\ m + \frac{3}{10m}(3\pi^2)^{2/3} n^{5/3}\ \ (\mathrm{NR})\ \ = \frac{(9\pi)^{2/3} i}{4} n^{4/3}\ \ (\mathrm{ER})$$

Crude approximation for bulk energy:

$$W = \frac{\epsilon}{n} - m$$

Hence,

$$\epsilon = \frac{\rho}{A}\left(M(A,Z) - Zm_e - \frac{9}{10}\frac{(Ze)^2}{R} \right) + \epsilon_e$$

As a first approximation, one can replace $M(A,Z)$ by the atomic masses of the nuclei.

Pressure:

$$P = n^2 \frac{d}{dn}\left(\frac{\epsilon}{n}\right) = P_{kin} \pm \frac{2\pi n^2 g^2}{\mu^2}$$

If we write

$$P_{kin} = K n^{\Gamma}\ \ (\text{Polytropic form})$$

$\Gamma = 5/3$ (NR) and $= 4/3$ (ER). $\Longrightarrow$ for $\rho \leq \rho_{nucl} \longrightarrow$ nuclear force is attractive $\longrightarrow P$ is less (softer). For very high density (repulsive core) P is greater (EOS is hardened).

Hartree Analysis: (Non-Relativistic)

Zeroth order quantum mechanical generalization- gives classical result: Start with the Hamiltonian:

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\nabla_i^2 + \sum_{i=1}^{N} V(r_i) + \sum_{i<j=1}^{N} V(r_i - r_j)$$

Many body system: $\psi(r_1, r_2, ..., r_N) = u_1(r_1)u_2(r_2)...u_N(r_N)$ and N is large enough. No need to anti-symmetrize and there is no spin. Background potential term is omitted. Then

$$\begin{aligned} < H > &= < \psi|H|\psi > \\ &= \sum_{i=1}^{N} \int d^3 r u_i^*(r) \left(-\frac{\hbar^2}{2m}\nabla^2 \right) u_i(r) \\ &+ \sum_{i<j=1}^{N} d^3 r_i d^3 r_j V_{ij} |u_i(r_i)|^2 |u_j(r_j)|^2 \end{aligned}$$

here

$$V_{ij} = \pm g^2 \frac{\exp(-\mu r_{ij})}{r_{ij}}$$

Normalization conditions: $< \psi|\psi >= 1$ and $< u_i|u_i >= 1$.

variational principle:

$$-\frac{\hbar^2}{2m}\nabla^2 u_i + V_i u_i = \varepsilon_i u_i$$

where

$$V_i(r_i) = \sum_{j\neq i=1}^{N} \int d^3 r_j V_{ij}(r_{ij}) |u_j(r_j)|^2$$

Now to solve N-number of coupled Schrödinger equation self-consistently is not so easy- has to be done numerically with some initial guess basis functions.

Alternatively, assume u_i's for free particles- plane waves:

$$u(r) = \frac{1}{V^{1/2}} \exp(i\vec{k}.\vec{r})$$

and then make perturbative (time independent) calculation.

Further, although we have not anti-symmetrize the N-body wave functions, still assume that they satisfy Fermi statistics. The system is degenerate Fermi gas and occupies energy levels up to the Fermi level. Then the sum over particle numbers $\Longrightarrow$ integral over momentum within the limit 0 to k_F. We have replaced:

$$\frac{1}{V}\sum_i \longrightarrow \frac{2}{(2\pi)^3}\int d^3k$$

Then we have from

$$< H >= \sum_i \frac{p^2}{2m} \pm \frac{1}{2V^2} \sum_{i,j} \int d^3r_1 d^3r_2 \frac{\exp(-\mu r_{12})}{r_{12}}$$

or

$$< H >= \sum_i \frac{p^2}{2m} \pm \sum_{i,j} \frac{2\pi g^2}{V\mu^2}$$

we have

$$\epsilon = \frac{< H >}{V} + nm = \epsilon_{kin} \pm \frac{2\pi n^2 g^2}{\mu^2}$$

where

$$\epsilon_{kin} = \frac{3}{10} m (3\pi^2)^{2/3} n^{5/3}$$

So the Hartree result exactly coincides with the classical one.

Hartree-Fock Analysis: (Non-Relativistic)

N-fermion system. N-body wf's are represented by the Slatter determinant of a $N \times N$ matrix:

$$\psi = \frac{1}{(N!)^{1/2}} \det \begin{pmatrix} u_1(1) & u_1(2) & . & . & . & . & u_1(N) \\ u_2(1) & u_2(2) & . & . & . & . & u_2(N) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ u_N(1) & u_N(2) & . & . & . & . & u_N(N) \end{pmatrix}$$

Spin wf:$\chi(\sigma)$ and $\chi_i(\sigma)\chi_i(\sigma') = \delta(\sigma, \sigma')$. Single particle wf: $u_i(j) = u_i(r_j)\chi_i(\sigma)$.

Orthonormality condition:

$$\sum_{\sigma_1} \int d^3 r_i u_i^*(1) u_j(1) = \delta_{ij}$$

Variational condition:

$$\delta < \psi \mid H \mid \psi >= 0$$

gives Hartree-Fock equation, which is far more complicated than Hartree equation. We shall not solve Hartree-Fock equation:

Alternative approach:

Consider an operator

$$F = \sum_i f_i = \sum_i -\frac{\hbar^2}{2m}\nabla^2 \quad \text{(say)}$$

Then

$$< \psi|F|\psi >= \sum_{i=1}^{N} < u_i|f_i|\psi >$$

Evaluation with plane wave approximation gives exactly Hartree result. Consider another operator:

$$G = \sum_{i<j=1}^{N} g_{ij} = \sum_{i<j=1}^{N} V_{ij} \text{ or } V(r_i - r_j)$$

Here g_{ij} is symmetric two fermion operator. Then

$$< \psi|G|\psi >= \sum_{i<j} [< ij|g|ij > - < ij|g|ji >]$$

The first term is just that obtained in Hartree analysis.

Exchange Term:

$$
\begin{aligned}
I &= -\frac{1}{2}\sum_{i,j} < ij|g|ji > \\
&= -\frac{1}{2}\sum_{i,j}\sum_{\sigma_1,\sigma_2}\int d^3r_1 d^3r_2 u_i^*(r_1)u_j^*(r_2)V_{12}u_i(r_2)u_j(r_1) \\
&\times \quad \chi_i^*(\sigma_1)\chi_j^*(\sigma_2)\chi_i(\sigma_2)\chi_j(\sigma_1)
\end{aligned}
$$

Now

$$
\sum_{\sigma}\chi_i^*(\sigma)\chi_j(\sigma) = \delta(m_{s_i}, m_{s_j})
$$

where $m_s = \pm 1/2 \longrightarrow z$-component of spin. Then

$$
\begin{aligned}
I &= -\frac{1}{2}\sum_{i,j}\delta(m_{s_i}, m_{s_j})\int d^3r_1 d^3r_2 u_i^*(r_1)u_j^*(r_2)V_{12}u_i(r_2)u_j(r_1) \\
&= -\frac{1}{2}\times 2\int d^3r_1 d^3r_2 V_{12} \mid \rho(r_1, r_2) \mid^2
\end{aligned}
$$

With plane wave states:

$$\begin{aligned}\rho(r_1, r_2) &= \frac{1}{V}\sum_k \exp[i\vec{k}.(\vec{r}_1 - \vec{r}_2)] \\ &= \frac{1}{(2\pi)^3} d^3k \exp[i\vec{k}.(\vec{r}_1 - \vec{r}_2)] \\ &= \frac{1}{2\pi^2}\frac{1}{r_{12}^3}(\sin k_F r_{12} - k_F r_{12}\cos k_F r_{12})\end{aligned}$$

With Yukawa two-body potential and defining $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$ and $\vec{r}_{12} = \vec{r} = \vec{r}_1 - \vec{r}_2$, we have

$$I = I(\alpha) = \frac{1}{4} - \frac{\alpha^2}{24} - \frac{\alpha}{3}\tan^{-1}\left(\frac{2}{\alpha}\right) + \left(\frac{\alpha^2}{8} + \frac{\alpha^4}{06}\right)\ln\left(1 + \frac{4}{\alpha^2}\right)$$

where $\alpha = \mu/k_F \Longrightarrow$

$$\alpha \sim \frac{\text{interparticle separation}}{\text{interaction range}}$$

Perturbation calculation is valid for $\alpha \gg 1$.

For $\alpha \to 0$, $I(\alpha) \to 1/4$. Whereas for $\alpha \to \infty$,

$$I(\alpha) \to \frac{1}{9\alpha^2},$$

which gives

$$I = \mp \frac{g^2 \pi n^2}{\mu^2} V$$

This is opposite in sign and $1/2$ of the direct contribution.

EOS in HF Model:

$$\epsilon = \rho = n\,m + \frac{3}{10m}(3\pi^2)^{2/3} n^{5/3} \pm \frac{\pi n^2 g^2}{\mu^2}$$

$$P = K n^{5/3} \pm \frac{\pi n^2 g^2}{\mu^2}$$

For $\alpha \ll 1 \Longrightarrow$ Yukawa potential $\longrightarrow$ Coulomb potential.

Relativistic Mean Field Theory: σ-ω Model of Nuclear Matter

Scalar field:σ couples with baryon scalar density $\rho_s = g_\sigma \overline{\psi}\psi$.

Vector field ω^μ ($\mu = 0, 1, 2, 3$) couples with baryon four-current $j^\mu = g_\omega \overline{\psi}\gamma^\mu \psi$.
g_i with $i = \sigma$ and ω are the coupling constants.

Then we have the Lagrangian density

$$
\begin{aligned}
\mathcal{L} &= \overline{\psi}[i\gamma_\mu(\partial^\mu + ig_\omega \omega^\mu) - (n - g_\sigma \sigma)]\psi \\
&+ \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2) - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}m_\omega^2 \omega_\mu \omega^\mu
\end{aligned}
$$

where $\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$-vector field tensor.

EL-equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

where ϕ:$\sigma, \omega^\mu, \psi, \overline{\psi}$ $\Longrightarrow$ we have with $\partial_\mu \omega^\mu = 0$ (comes automatically since $\partial_\mu j^\mu = 0$)

$$
\begin{aligned}
[\Box + m_\sigma^2]\sigma &= g_\sigma \overline{\psi}\psi \\
[\Box + m_\omega^2]\omega_\mu &= g_\omega \overline{\psi}\gamma_\mu \psi
\end{aligned}
$$

and finally

$$[\gamma_\mu(i\partial^\mu - g_\omega \omega^\mu) - (m - g_\sigma \sigma)]\psi(x) = 0$$

Set of equations are coupled, non-linear and hence extremely difficult to solve numerically. $\Longrightarrow$ Introduced an approximation, called mean field approximation: Matter is assumed to be static and uniform in ground state and mean fields or the mean values of the scalar and vector fields are considered:
$\sigma(x) \longrightarrow < \sigma(x) > = \sigma$ and $\omega(x) \longrightarrow < \omega(x) > = \omega$ (we are using same symbols for the mean fields). $\Longrightarrow$

$$m_\sigma^2 \sigma = g_\sigma < \overline{\psi}\psi >$$
$$m_\omega^2 \omega_0 = g_\omega < \psi^\dagger \psi >$$
$$m_\omega^2 \omega_k = g_\omega < \overline{\psi}\gamma_k \psi >$$

With mean fields, Dirac eqn. is is given by:

$$[\gamma_\mu(i\partial^\mu - g_\omega \omega^\mu) - (m - g_\sigma \sigma)]\psi(x) = 0$$

Now σ and ω are treated as background field.

With $\psi(x) \sim \psi(k) exp(-ik.x)$, we have

$$[\gamma_\mu(k^\mu - g_\omega \omega^\mu) - (m - g_\sigma \sigma)]\psi(k) = 0$$

Define:$K^\mu = k^\mu - g_\omega \omega^\mu$ and effective baryon mass $m^* = m - g_\sigma \sigma$. Then the energy eigen value $\varepsilon(k) = k_0 = K_0 + g_\omega \omega_0$, with $K_0 = [(\vec{k} - g_\omega \vec{\omega})^2 + m^{*2}]^{1/2}$.

Spatial Component of ω-Field $= 0$

Let Γ is any operator. Define single-particle expectation value: $< \overline{\psi} | \Gamma | \psi >_{k,s,\tau}$.

Subscripts: k-momentum, s-spin and τ-isospin. Expectation value in the ground state of many nucleon system:

$$< \overline{\psi} | \Gamma | \psi > = \sum_{s,\tau} \frac{1}{(2\pi)^3} \int d^3k < \overline{\psi} | \Gamma | \psi >_{k,s,\tau} \Theta(\mu - \varepsilon(k))$$

where μ-Fermi energy $\equiv$ chemical potential (at $T = 0$).

From Dirac equation:

$$k_0 \psi(k) = \gamma_0 (\vec{\gamma}.\vec{k} + g_\omega \gamma_\mu \omega^\mu + m^*) \psi = H_D \psi$$

where H_D is the Dirac Hamiltonian. Consider any variable ξ, such that

$$\frac{\partial}{\partial \xi} < \psi^\dagger | H_D | \psi >_{k,s,\tau} = < \psi^\dagger \left| \frac{\partial H_D}{\partial \xi} \right| \psi >_{k,s,\tau} + k_0 \frac{\partial}{\partial \psi} < \psi^\dagger \psi >$$

The last term on rhs is zero.

$$\rho = < \psi^\dagger \psi > = \frac{4}{(2\pi)^3} \int d^3k \Theta(\mu - \varepsilon(k))$$

Hence by $\xi \longrightarrow k^i$ and taking $E(k)$ as the single-particle eigen value, we have

$$\frac{\partial}{\partial k^i} E(k) = < \overline{\psi} | \gamma^i | \psi >_{k,s,\tau}$$

Then

$$\begin{aligned} < \overline{\psi} | \gamma^i | \psi > &= \frac{4}{(2\pi)^3} \int d^3k \left[\frac{\partial}{\partial k^i} E(k) \right] \Theta(\mu - \varepsilon(k)) \\ &= \frac{4}{(2\pi)^3} \int dk^i dk^j dk^k \left[\frac{\partial}{\partial k^i} E(k) \right] \Theta(\mu - \varepsilon(k)) \\ &= \frac{4}{(2\pi)^3} \int dk^j dk^k \int dE(k^j, k^k) \end{aligned}$$

The last integral explicitly becomes zero since at any point on the Fermi surface the energy value is the Fermi energy$-g_\omega \omega_0$ (rotational invariance). Therefore, $< \overline{\psi} | \gamma^i | \psi >$, the baryon three-current in the medium vanishes identically.

Hence

$$\omega^i = \frac{g_\omega}{m_\omega^2} j^i = 0$$

Only $\omega_0 \neq 0$. Further, the single-particle energy $E(k) = (k^2 + m^{*2})^{1/2}$.

Baryon density (vector density):

$$\rho =< \psi^\dagger|\psi >= \frac{4}{(2\pi)^3}\int d^3k\Theta(\mu-\varepsilon(k)) = \frac{2k_F^3}{3\pi^2}$$

Scalar density:

Now

$$< \overline{\psi}|\psi >_{k,s,\tau}= \frac{\partial E(k)}{\partial m} = \frac{m^*}{(k^2+m^{*2})^{1/2}}$$

Then

$$\rho_s =< \overline{\psi}|\psi >= \frac{2}{\pi^2}\int_0^{k_F} k^2 dk \frac{m^*}{(k^2+m^{*2})^{1/2}}$$

Energy density:

$$\epsilon = - < \mathcal{L} > + < \overline{\psi}\gamma_0 k_0 \psi >$$

Pressure:

$$P =< \mathcal{L} > + \frac{1}{3} < \overline{\psi}\gamma_i k_i \psi >$$

where $i = 1, 2, 3$ Hence

$$\epsilon = \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_\omega^2\omega_0^2 + \frac{2}{\pi^2}\int_0^{k_F}(k^2+m^{*2})^{1/2}k^2 dk$$

and

$$P = -\frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_\omega^2\omega_0^2 + \frac{1}{3}\frac{2}{\pi^2}\int_0^{k_F}\frac{k^2}{(k^2+m^{*2})^{1/2}}k^2dk$$

then the EOS:$P \equiv P(\epsilon)$

Role of σ and ω fields are opposite in nature: σ-decreases the energy of the system, where is ω_0 increases the energy. At a particular density, σ and ω_0 will be such that energy will be minimum $\longrightarrow$ saturation energy at saturation density. Saturation energy gives saturation binding energy. Binding energy/nucleon:

$$\frac{B}{A} = \left(\frac{\epsilon}{n_B}\right)_0 - m$$

Note: $g_\sigma\sigma$ has a lower limit ($m^* \geq 0$). $g_\omega\omega_0$ grows with ρ.

In $\sigma - \omega$ model, there are only two parameters (g_σ/m_σ and g_ω/m_ω), the saturation density and the binding energy per nucleon can be fitted exactly.

Unfortunately, in this model, (i) the effective mass at saturation density is $m^* \sim 0.5m$ (the semi-empirical value is $0.74m - 0.82m$[†]) and (ii) the compressibility $K \approx 550$MeV, about two times larger than it shall be. Further, the EOS is too hard because of repulsive ω-meson part.

[†]References:

- N.K. Glendenning, Phys. Rev. C37, 2733 (1988).
- C. Mahaux and R. Sartor, Nucl. Phys. A475, 247 (1987).
- M. Jaminon and C. Mahaux, Phys. Rev. C40, 354 (1989).
- C.H. Johnson, D.J. Horen and C. Mahaux, Phys. Rev. C36, 2252 (1987).
- Theoretical Nuclear and Sub-Nuclear Physics, J.D. Walecka, Oxford University Press (1995).

- M.H. Johnson and E. Teller, Phys. Rev. 98, 783 (1955).
- J.D. Walecka, Ann. Phys. 83, 491 (1974).
- B.D. Serot, Phys. Lett. B86, 146 (1979).

Asymmetry parameter:

$$a_{sym} = \frac{k_F^2}{6(k_F^2 + m^{*2})^{1/2}}$$

For $k_F \approx 1.4\text{fm}^{-1}$, $m^* = 0.75m \approx 3.5686\text{fm}^{-1}$, we have $a_{sym} = 14.8\text{MeV}$, where as the semi-empirical value is 32.5MeV.

The model is therefore not successful at high density (symmetric or asymmetric) inside NS.

The plus point of the model: It has provided the opportunity of introducing in a simple manner the techniques that can be improved for dense neutron star matter.

EOS with Scalar Self-interaction:

Non-linear part:

$$U(\sigma) = \frac{1}{3} bm (g_\sigma \sigma)^3 + \frac{1}{4} (g_\sigma \sigma)^4$$

here b and c are two parameters to be determined from saturation data. $m = 938$, free average nucleon mass.

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_I + \mathcal{L}_F + U(\sigma)$$

Hence using EL-equation and considering mean values for the fields:

$$g_\sigma \sigma = \left(\frac{g_\sigma}{m_\sigma}\right)^2 \left[\frac{2}{\pi^2} \int_0^{k_F} k^2 dk \frac{m - g_\sigma \sigma}{[k^2 + (m - g_\sigma \sigma)^2]^{1/2}} - bm (g_\sigma \sigma)^2 - c (g_\sigma \sigma)^3\right]$$

$$g_\omega \omega_0 = \left(\frac{g_\omega}{m_\omega}\right)^2 \rho$$

$$m_\omega^2 \omega_k = 0$$

and finally the Dirac equation as mentioned before. Here, with scalar self coupling, only the scalar field equation is modified explicitly.

Energy density:

$$\begin{aligned}\epsilon &= \frac{1}{3}bm(g_\sigma\sigma)^3 + \frac{1}{4}c(g_\sigma\sigma)^4 \\ &+ \frac{1}{2}bm(m_\sigma\sigma)^2 + \frac{1}{2}m_\omega^2\omega_0^2 \\ &+ \frac{2}{\pi^2}\int_0^{k_F}(k^2+m^{*2})^{1/2}k^2dk\end{aligned}$$

Pressure:

$$\begin{aligned}P &= -\frac{1}{3}bm(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 \\ &- \frac{1}{2}bm(m_\sigma\sigma)^2 + \frac{1}{2}m_\omega^2\omega_0^2 \\ &+ \frac{2}{\pi^2}\frac{1}{3}\int_0^{k_F}\frac{k^2}{(k^2+m^{*2})^{1/2}}k^2dk\end{aligned}$$

Two additional parameters b and c allow us to find compressibility K and m^* at the saturation density.

Isospin Force:

To distinguish n and p- interaction with ρ-meson exchange is introduced. Interaction part of this Lagrangian:

$$\mathcal{L}_{int} = -g_\rho \vec{\rho}_\nu . \vec{I}^\nu$$

where the vector (in isospin space) meson current:

$$\vec{I}^\nu = \frac{1}{2}\overline{\psi}\gamma^\nu \vec{\tau}\psi + \vec{\rho}_\mu \times \vec{\rho}^{\nu\mu} + 2g_\rho(\vec{\rho}^\nu \times \vec{\rho}^\mu) \times \vec{\rho}_\mu$$

Then in the EL-equation, the extra term is

$$\frac{\partial \mathcal{L}_{int}}{\partial \overline{\psi}} = \frac{g_\rho}{2}\gamma_\nu \vec{\rho}^\nu . \vec{\tau}\psi$$

Dirac eqn. becomes:

$$\left[\gamma_\mu \left(k^\mu - g_\omega \omega^\mu - \frac{1}{2} g_\rho \tau_3 \rho_3^\mu\right) - m^*\right]\psi(k) = 0$$

Other new equations:

As usual

$$g_\rho \rho_3^k = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 < \overline{\psi}\gamma^k \tau_3 \psi >= 0$$

$$g_\rho \rho_3^0 = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 < \overline{\psi}\gamma^0\tau_3\psi >= \left(\frac{g_\rho}{m_\rho}\right)^2 \frac{1}{2}(\rho_p - \rho_n)$$

Here $\pm 1/2$ are the isospin eigen values for p and n. In this case also three vector part ρ_3^k does not contribute because of same reason. Further, ρ_1 and ρ_2, which can be expressed in terms of ρ^+ and ρ^- do not contribute for obvious reason.

Energy density:

Energy eigen value:

$$\varepsilon_{I_3}(k) = E(k) + g_\omega \omega^0 + g_\rho I_3 \rho_3^0$$

where

$$E(k) = (k^2 + m^{*2})^{1/2}$$

Since $I_3|p >= +\frac{1}{2}|p >$ and $I_3|n >= -\frac{1}{2}|n >$, we have energy density

$$\begin{aligned}
\epsilon &= \frac{1}{3}bm(g_\sigma\sigma)^3 + \frac{1}{4}c(g_\sigma\sigma)^4 + \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\omega\omega_0^2 + \frac{1}{2}m_\rho\rho_{0_3}^2 \\
&+ \frac{1}{\pi^2}\int_0^{k_p} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega\omega_0 + \frac{1}{2}g_\rho\rho_3^0\right] \\
&+ \frac{1}{\pi^2}\int_0^{k_n} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega\omega_0 - \frac{1}{2}g_\rho\rho_3^0\right]
\end{aligned}$$

Pressure:

$$
\begin{aligned}
P &= -\frac{1}{3}bm(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 - \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\omega\omega_0^2 + \frac{1}{2}m_\rho\rho_{0_3}^2 \\
&+ \frac{1}{3}\frac{1}{\pi^2}\int_0^{k_p} k^2 dk \frac{k^2}{(k^2 + m^{*2})^{1/2}} \\
&+ \frac{1}{3}\frac{1}{\pi^2}\int_0^{k_n} k^2 dk \frac{k^2}{(k^2 + m^{*2})^{1/2}}
\end{aligned}
$$

Symmetry Energy:

The part of the energy contributes in the symmetry energy:

$$
\begin{aligned}
\epsilon_s &= \frac{1}{2} m_\rho \rho_{0_3}^2 + \frac{1}{\pi^2}\int_0^{k_p} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega \omega_0 + \frac{1}{2} g_\rho \rho_3^0 \right] \\
&+ \frac{1}{\pi^2}\int_0^{k_n} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega \omega_0 - \frac{1}{2} g_\rho \rho_3^0 \right]
\end{aligned}
$$

Define

$$
t = \frac{\rho_n - \rho_p}{\rho} \text{ and } \rho = \rho_n + \rho_p
$$

$\Longrightarrow$

$$
\begin{aligned}
\rho_n &= \frac{\rho}{2}(1+t)^{1/3} = \frac{k_F^3}{3\pi^2}(1+t) = \frac{k_n^3}{3\pi^2} \\
\rho_p &= \frac{\rho}{2}(1-t)^{1/3} = \frac{k_F^3}{3\pi^2}(1-t) = \frac{k_p^3}{3\pi^2}
\end{aligned}
$$

Then the symmetry energy per nucleon

$$
\begin{aligned}
\varepsilon_s &= \frac{E_s}{A} = \frac{\epsilon}{\rho} = \frac{1}{8}\left(\frac{g_\rho}{m_\rho}\right)^2 \rho t^2 + \frac{1}{\rho}\frac{1}{\pi^2}\int_0^{k_p} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega \omega_0 + \frac{1}{2} g_\rho \rho_3^0 \right] \\
&+ \frac{1}{\rho}\frac{1}{\pi^2}\int_0^{k_n} k^2 dk \left[(k^2 + m^{*2}(\sigma))^{1/2} + g_\omega \omega_0 - \frac{1}{2} g_\rho \rho_3^0 \right]
\end{aligned}
$$

The symmetry coefficient is defined as:

$$a_{sym} = \frac{1}{2}\left[\frac{\partial^2(\epsilon_s/\rho)}{\partial t^2}\right]_{t=0}$$

and is given by

$$a_{sym} = \frac{k_F^2}{6(k_F^2 + m^{*2})^{1/2}} \approx 32.5\text{MeV}$$

To obtain this result- while differentiating wrt t, we use

$$\frac{\partial}{\partial t}F(k_i) = \frac{\partial}{\partial k_i}F(k_i)\frac{\partial k_i}{\partial t}$$

where $i = n$ or p and F is a function of either k_p or k_n.

Algebraic Determination of Coupling Constants:

We consider σ-ω field model with self coupling.

Four unknown parameters: g_σ/m_σ, g_ω/m_ω, b and c.

Known quantities at saturation: ρ, B/A, K and m^*.

Now

$$\begin{aligned}
\frac{d\epsilon}{d\rho} &= \frac{d\sigma}{d\rho}\frac{\partial\epsilon}{\partial\sigma} \\
&= \frac{d\sigma}{d\rho}\left(m_\sigma^2\sigma + \frac{dU}{d\sigma} - g_\sigma\frac{2}{\pi^2}\int_0^{k_F}\frac{m-g_\sigma\sigma}{[k^2+(m-g_\sigma\sigma)^2]^{1/2}}k^2dk\right) \\
&+ \ g_\omega\omega_0 + (k_F^2 + m^{*2})^{1/2}
\end{aligned}$$

The long expression within the parenthesis vanishes identically by virtue of the scalar field equation.

$$g_\sigma\sigma = \left(\frac{g_\sigma}{m_\sigma}\right)^2\left[\frac{2}{\pi^2}\int_0^{k_F}k^2dk\frac{m-g_\sigma\sigma}{[k^2+(m-g_\sigma\sigma)^2]^{1/2}} - bm(g_\sigma\sigma)^2 - c(g_\sigma\sigma)^3\right]$$

$$g_\omega\omega_0 = \left(\frac{g_\omega}{m_\omega}\right)^2\rho$$

$$m_\omega^2\omega_k = 0$$

Hence

$$\frac{d\epsilon}{d\rho} = g_\omega \omega_0 + (k_F^2 + m^{*2})^{1/2} = \mu \;\; (\text{chemical potential})$$

where all quantities, ω_0, k_F, m^*.. etc are evaluated at saturation density.

Now

$$g_\omega \omega_0 = \left(\frac{g_\omega}{m_\omega}\right)^2 \rho$$

and since ϵ/ρ is minimum at saturation Red

$$\frac{d}{d\rho}\left(\frac{\epsilon}{\rho}\right) = 0 = \frac{1}{\rho}\left(\frac{d\epsilon}{d\rho} - \frac{\epsilon}{\rho}\right)$$

Hence at saturation

$$m + \frac{B}{A} \equiv \frac{\epsilon}{\rho} = \left(\frac{g_\omega}{m_\omega}\right)^2 \rho + (k_F^2 + m^{*2})^{1/2}$$

Further

$$\frac{d^2}{d\rho^2}\left(\frac{\epsilon}{\rho}\right) = \frac{1}{\rho}\frac{d^2\epsilon}{d\rho^2} = \frac{1}{\rho}\frac{d\mu}{d\rho}$$

Therefore

$$K = 9\rho \frac{d\mu}{d\rho}$$

where

$$\frac{d\mu}{d\rho} = \left(\frac{g_\omega}{m_\omega}\right)^2 + \frac{1}{E(k_F)}\left(k_F \frac{dk_F}{d\rho} - m^* g_\sigma \frac{d\sigma}{d\rho}\right)$$

Hence

$$\begin{aligned} K &= \left(\frac{g_\omega}{m_\omega}\right)^2 \frac{6k_F^3}{\pi^2} + \frac{3k_F^2}{E(k_F)} \\ &- \frac{\frac{6k_F^2}{\pi^2}\left(\frac{m^*}{E(k_F)}\right)^2 \left(\frac{g_\sigma}{m_\sigma}\right)^2}{1 + \left(\frac{g_\sigma}{m_\sigma}\right)^2 \left[\frac{d^2U}{d(g_\sigma\sigma)^2} + \frac{2}{\pi^2}\int_0^{k_F} \frac{k^4}{E^3(k)} dk\right]} \end{aligned}$$

$g_\sigma \sigma = m - m^*$ is known from empirical data ($\sim 0.74m - 0.82m$). So the unknown quantities in the above equation are g_σ/m_σ, b and c. Two additional equations: Scalar field equation and the energy equation with the same unknowns. Energy equation is given by

$$\epsilon = \rho\left[m + \frac{B}{A}\right]$$

Hence all the parameters can be obtained. If we consider isospin force into account, the fifth unknown parameter is given by

$$\left(\frac{g_\rho}{m_\rho}\right)^2 = \frac{8}{\rho}\left[a_{sym} - \frac{k_F^2}{6(k_F^2 + m^{*2})^{1/2}}\right]$$

Reference:

- N.K. Glendenning, Compact Stars, Nuclear Physics, Particle Physics and General Relativity, Second Edison, Springer (2000).
- S. Chakrabarty, D. Bandopadhyay and S. Pal, Phys. Rev. Lett. 78 (1997) 2898.
- D. Bandopadhyay, S. Chakrabarty and S. Pal, Phys. Rev. Lett. 79 (1997) 2176.
- D. Bandopadhyay, S. Pal and S. Chakrabarty, Jour. Phys. G24 (1998) 1647.
- D. Bandopadhyay, S. Chakrabarty, P. Dey and S. Pal, Phys. Rev. D58 (1998) 121301 (Rapid Communication).
- S. Pal, D. Bandopadhyaya and S. Chakrabarty, Jour. Phys. G25 (1999) L117.

- Theoretical Nuclear and Sub-Nuclear Physics, J.D. Walecka, Oxford University Press (1995).
- M.H. Johnson and E. Teller, Phys. Rev. 98, 783 (1955).
- J.D. Walecka, Ann. Phys. 83, 491 (1974).
- B.D. Serot, Phys. Lett. B86, 146 (1979).

Outer Crust Matter:

Outer crust:Mainly dense crystal of fully ionized metallic iron. Iron nuclei are arranged in regular lattice surrounded by electron gas. It is assumed that each nucleus surrounded by electron gas form a charge-neutral cell. The cells are arranged in a regular manner. These are called Wigner-Seitz (WS) cells. The statistical treatment to obtain EOS for outer crust matter is the Thomas-Fermi (TF) method- the semi-classical approach for many electron system.

It is assumed that the electron gas within WS cells are fully degenerate and chemical potential or the Fermi energy is constant throughout the cell otherwise, the electrons will accumulate at minimum Fermi energy / chemical potential region.

So in the NR picture

$$E_F = \mu_e = \frac{p_F^2}{2m_e} - e\phi(r) = \text{constant}$$

where $\phi(r)$ is the electrostatic potential, satisfies the Poisson's equation.:

$$\nabla^2\phi = 4\pi n(r) - 4\pi Ze\delta^3(r - r_n)$$

Second term on the rhs $\longrightarrow$ nuclear contribution. We want $\phi(r)$ in electron gas, outside the nuclei $\longrightarrow$ the nuclear contribution is neglected.

With $\hbar = 1$, we have

$$n_e(r) = \frac{p_F^3(r)}{3\pi^2} = \frac{1}{3\pi^2}[2m_e(\mu_e + e\phi(r))]^{3/2}$$

Then in spherical polar coordinate with spherical symmetry, the Poisson's equation:

$$\frac{1}{r}\frac{d^2}{dr^2}(r\phi(r)) = \frac{4e}{3\pi}[2m_e(\mu_e + e\phi(r))]^{3/2}$$

$\longrightarrow$ TF equation. To solve it numerically, we use the boundary conditions:
On the nuclear surface:

$$\mathrm{Lim}_{r \to r_n} r\phi(r) = Ze$$

the constant electrostatic potential by the nuclear charge. At the WS cell boundary

$$\frac{d\phi}{dr} = 0 \ \text{ for } \ r = r_s$$

where $r_s \longrightarrow$ radius of the WS cell. Overall charge neutrality inside WS-cell $\Longrightarrow$ electric field vanishes at the surface.

For a convenient form: Define $r = \mu x$, $x \longrightarrow$ dimensionless and $\mu \longrightarrow$ unknown constant and

$$\mu_e + e\phi(r) = Ze^2\frac{\psi(r)}{r}$$

The TF equation reduces to

$$\frac{d^2\psi}{dx^2} = \frac{\psi^{3/2}}{x^{1/2}}$$

with

$$\mu = \left(\frac{9\pi^2}{128Z}\right)^{1/3} a_0, \quad \text{with} \quad a_0 = \frac{1}{me^2} \quad \text{Bohr radius}$$

Boundary conditions: $\psi(x = x_n) = 1$, where $x_n = r_n/\mu$ and

$$\frac{d\psi}{dx} = \frac{\psi}{x} \quad \text{for} \quad x = x_s, \quad \text{with} \quad x_s = \frac{r_s}{\mu}$$

We can write

$$\frac{p_F^2(x)}{2m_e} = \mu_e + e\phi(x) = Ze^2\frac{\psi(x)}{\mu x}$$

Hence the cell averaged electron pressure

$$\begin{aligned} P(x_s) &= \frac{1}{\pi^2}\frac{1}{3}\int_0^{p_F(x_s)} \frac{p^2}{m_e}p^2 dp \\ &= \frac{1}{15\pi^2 m_e}\left(\frac{2m_e Ze2}{\mu}\right)^{5/2}\left(\frac{\psi(x_s)}{x_s}\right)^{5/2} \end{aligned}$$

By some rearrangement

$$P(x_s) = \frac{Z^2 e^2}{10\pi\mu^4}\left(\frac{\psi(x_s)}{x_s}\right)^{5/2}$$

Whereas, the energy density, coming from the nucleon parts at rest:

$$\epsilon_0 = \frac{3Am_B}{4\pi\mu^3 x_s^3} = \frac{32}{3\pi^3}\left(\frac{m_p}{a_0^3}\right)\frac{AZ}{x_s^3}$$

Hence we get the EOS.

To obtain x_s (or r_s), one has to solve TF equation with guess values for $\psi'(x = x_n)$.
For $\psi'(x_n) > -1.5889$, solution diverges.
For $\psi'(x_n) \approx -1.5889 \longrightarrow$ surface condition is achieved asymptotically.
Asymptotic solution $\Longrightarrow$ zero pressure case $\Longrightarrow$ free atom has infinite radius $\Longrightarrow$
Defect of TF-model, which is the semi-classical version of Hartree calculation.

Thomas-Fermi-Dirac Approximation (TFD):

Hartree term:

$$e\phi(r) = V(r) + \sum_{j=1}^{Z} \int \phi_j^*(r') \frac{e^2}{|r - r'|} \phi_j(r') d^3 r'$$

Here, ϕ in the lhs is the electrostatic potential.

Hartree-Fock term:

$$U_{ex}(r)\phi_i(r) = \sum_{j=1}^{Z} \phi_j(r)\delta(\sigma_j, \sigma_i) \int \phi_j^*(r') \frac{e^2}{|r - r'|} \phi_i(r') d^3 r'$$

With plane wave states:

$$\phi_i(r) = \frac{1}{V^{1/2}} \exp(ip.r) \quad \text{with} \quad \hbar = 1$$

and

$$\phi_j(r) = \frac{1}{V^{1/2}} \exp(ip'r)$$

Now

$$\frac{1}{V} \sum_{j=1}^{Z} \delta(\sigma_i, \sigma_j) \rightarrow \frac{1}{(2\pi)^3} \int d^3 p' \Theta(p_F - p')$$

Hence

$$U_{ex}\phi_i(r) = \frac{e^2}{(2\pi)^3}\frac{1}{V^{1/2}}\exp(ip.r)\int d^3p'\Theta(p_F - p')d^3x\frac{\exp[-i(p-p').x]}{x}$$

where $x = r' - r$.

x-integral give:

$$I_x = \frac{4\pi}{\mid p - p' \mid^2}$$

Then

$$U_{ex}(r) = \frac{e^2}{2\pi^2}\int \Theta(p_F - p')\frac{1}{\mid p - p' \mid^2}d^3p'$$

p'-integral:

Let $\theta \longrightarrow \angle p, p' \Longrightarrow \mid p - p' \mid^2 = p^2 + p'^2 - 2pp'X$, where $X = \cos(\theta)$.

θ or X-integral gives:

$$I_\theta(p, p') = \frac{\pi}{p}\int_0^{p_F} p'dp' \ln\left|\frac{p + p'}{p - p'}\right|$$

Integration over $p' \longrightarrow$

$$U_{ex} = \frac{e^2}{2\pi}\left[\frac{(p_F^2 - p^2)}{p}\ln\left|\frac{p_F + p}{p_F - p}\right| + 2p_F\right]$$

Hence energy per particle

$$\varepsilon(p) = \frac{p^2}{2m_e} - e\phi(r) - \frac{e^2}{2\pi}\left[\frac{(p_F^2 - p^2)}{p}\ln\left|\frac{p_F + p}{p_F - p}\right| + 2p_F\right]$$

where $p_F \equiv p_F(r)$.

Hence the Fermi energy / chemical potential

$$\mu_e = \varepsilon(p_F) = \frac{p_F^2}{2m_e} - e\phi(r) - \frac{e^2}{\pi}p_F(r)$$

Hence

$$p_F(r) = \frac{m_e e^2}{\pi} + \left[\left(\frac{2m_e e^2}{\pi}\right)^2 + 2m_e(\mu_e + e\phi(r))\right]^{1/2}$$

$\Longrightarrow$ TFD equation:

$$\frac{1}{r}\frac{d^2}{dr^2}(r\phi(r)) = \frac{4e}{3\pi}\left[\frac{m_e e^2}{\pi} + \left\{\left(\frac{m_e e^2}{\pi}\right)^2 + 2m_e(\mu_e + e\phi(r))\right\}^{1/2}\right]^3$$

Define:

$$2m_e e^4 + (\mu_e + e\phi(r)) = Ze^2 \frac{\omega(r)}{r} \quad \text{and} \quad \alpha = \left(\frac{6\pi}{Z^2}\right)^{1/3}$$

$\Longrightarrow$ TFD equation:

$$\frac{d^2\omega}{dx^2} = x\left[\alpha + \frac{\omega^{1/2}}{x^{1/2}}\right]^3$$

where $r = \mu x$ and μ is given in TF formalism.

Boundary conditions:

$$\text{(a) } \omega(x = x_n) = 1 \quad \text{and} \quad \frac{d\omega}{dx} = \frac{\omega}{x} \quad \text{for} \quad x = x_s$$

$\Longrightarrow$ Pressure:

$$P = f(\xi) P_{free}$$

where

$$f(\xi) = \frac{1}{3}\left[\alpha + \left(\frac{\omega(x_s)}{x_s}\right)^{1/2}\right]^3 \left[1 - \frac{\frac{5}{4}}{\alpha + \left(\frac{\omega(x_s)}{x_s}\right)^{1/2}}\right]^{3/5}$$

$$P_{free} = \frac{4\pi^2}{5m_e}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{Z\rho}{Am_p}\right)^{5/2}$$

and

$$\xi = \frac{a_0}{Z^{2/3}}\left(\frac{3Z}{4\pi a_0 x_s}\right)^{1/3} \approx \frac{0.701}{x_s}$$

Now $P_{TFD} << P_{TF} \Longrightarrow$ TFD model gives soft EOS compared to TF.

Again $f(\xi) \rightarrow 1$ as density increases. With the increase in density r_s or x_s decreases. For $\xi = 1$,

$$r_s = \left(\frac{9\pi^2}{128}\right)^{1/3}\frac{0.701a_0}{Z^{1/3}}$$

$\Longrightarrow$

$$\rho_{\xi=1} = 3Am_p 4\pi r_s^2 \approx 10AZ \;\; \text{gm cm}^{-3}$$

For Fe^{56}, $\rho_{xi=1} \approx 10^4$gm cm^{-3} $\Longrightarrow$ above $\rho_{xi=1}$, the electron gas may be assumed to be a free degenerate gas. Below this density TFD correction has to be done. On the other hand, at very low density, the electron cell effect has to be taken into account. $\Longrightarrow$ TF or TFD calculations are therefore not valid for laboratory metallic iron.

Relativistic version

Almost the same algebraic procedure is followed for the relativistic version of TFD equation. Here the single particle energy: $\varepsilon = (p^2 + m_e^2)^{1/2}$, with $c = 1$.

Here

$$\mu_e = [(3\pi^2 n_e)^{2/3} + m_e^2]^{1/2} - eV(r) - m_e = \text{constant}$$

where

$$n_e = \left[\frac{2m_e(\mu_e + eV(r))}{3\pi^2}\right]^{3/2} \left[1 + \frac{(eV(r) + \mu_e)}{2m_e}\right]^{3/2}$$

With the same substitution $\Longrightarrow$ TFD equation

$$\frac{d^2\phi}{dx^2} = \frac{\phi^{3/2}}{x^{1/2}} \left[1 + \frac{Z}{Z_{cr}} \frac{\phi}{x}\right]^{3/2}$$

where

$$r = \mu x, \quad Z_{cr} = \left(\frac{3\pi}{4e^2}\right)^{1/2} \quad \text{and} \quad \mu = \frac{(3\pi)^{2/3}}{me^2 2^{7/3} Z^{1/3}}$$

References:

- S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983) pp. 42 and 188.

- E.H. Lieb and B. Simon, Phys. Rev. Lett. 31, (1973) 681.

- E.H. Lieb, J.P. Solovej and J. Yngvason, Phys. Rev. Lett. 69, (1992) 749.

- E.H. Lieb, Bull. Amer. Math. Soc., 22, (1990) 1;

- R.O. Mueller, A.R.P. Rau and L. Spruch, Phys. Rev. Lett. 26, (1971) 1136.

- A.R.P. Rau, R.O. Mueller and L. Spruch, Phys. Rev. A11, (1975) 1865.

- S.H. Hill, P.J. Grout and N.H. March, Jour. Phys. B18, (1985) 4665.

- R. Ruffini, ”Exploring the Universe”, a Festschrift in honour of Riccardo Giacconi, Advance Series in Astrophysics and Cosmology, World Scientific, Eds. H. Gursky, R. Rufini and L. Stella, Vol. 13, (2000) 383

- R. Ruffini, Int. Jour. of Mod. Phys. 5, (1996) 507.

- Y.S. Leung, Physics of Dense Matter, World Scientific, Singapore, (1984) pp. 35.

- Nandini Nag, Sutapa Ghosh and Somenath Chakrabarty, Ann. Phys. 324, (2009) 499.

- Nandini Nag, Sutapa Ghosh and Somenath Chakrabarty, Euro. J. Phys. A45, (2010) 99.

- Arpita Ghosh, Sutapa Ghosh and Somenath Chakrabarty, IJMPD (in press).

Oppenheimer-Volkoff (OV) Equation for Neutron Stars:

We use Gravitational or Geometrical units:
$G = c = k = 1$.

Hence the OV or the GR hydrostatic equilibrium equation along with the subsidiary mass equation:

$$\frac{dP}{dr} = -\frac{\rho m}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^2}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

OV equation is obtained from GR Einstein's equation with Schwarzschild metric, valid for a static, non-rotating system in vacuum.

How to solve the equations (numerically)?:

1. EOS $P(\rho)$ is known from the core to the crust.
2. Pick a value of central density ρ_c. The Pressure is known.
3. At the centre $m = 0$ (take an extremely small number for numerical calculation.)
4. Integrate the above equations out ward from $r = 0$.
5. Each time a new value for ρ and also a new value for $m(r)$ will be obtained, hence get $P(\rho)$.
6. At $r = R$, the radius of the star, $P = 0$.
7. At $r =$, $m(R) = M$, the mass of the star.
8. Hence we get $M(R)$ and density profile for a given ρ_c
9. Change the value of ρ_c and repeat (1-8).
10. We get $M(\rho_c)$.
11. The value of ρ_c be such that $dM/d\rho_c > 0$, otherwise the system becomes general relativistically unstable.

At the core of NS chemical equilibrium among the constituents: $\Longrightarrow$

$n \rightarrow p + e^- + \overline{\nu}_e$, $p + e^- \rightarrow n + \nu_e \Longrightarrow \mu_n = \mu_p + \mu_e$. Neutrinos are non-degenerate, leave the immediately after their formation.

Charge neutrality: $n_p = n_e$.

Self-consistent solution of these equations along with the equations discussed in $\sigma - \omega - \rho$-meson model will give EOS for the core material.

More complicated cases:

(i) If $\mu_e > m_\mu$, μ-mesons or muons will be created.
(ii) if $\mu_{n-p} > m_B$, B-is some baryon resonances- they have to be considered.
(iii) Presence of π-mesons and kaons are also important.
(iv) Super-fluidity of neutron matter.
(v) Superconductivity of proton matter.
(vi) Phase transition to any exotic matter.
(vii) Effect of magnetic field: EOS becomes softer.
(viii) Effect of magnetic field on various physical properties of dense neutron star matter.
(ix) Effect of magnetic field on phase transition to quark matter (!).
(x) Effect of magnetic field on neutron matter super-fluidity.