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RUB

Effective Field Theory and Nuclear Forces

Lecture 1: Introduction & first look into ChPT

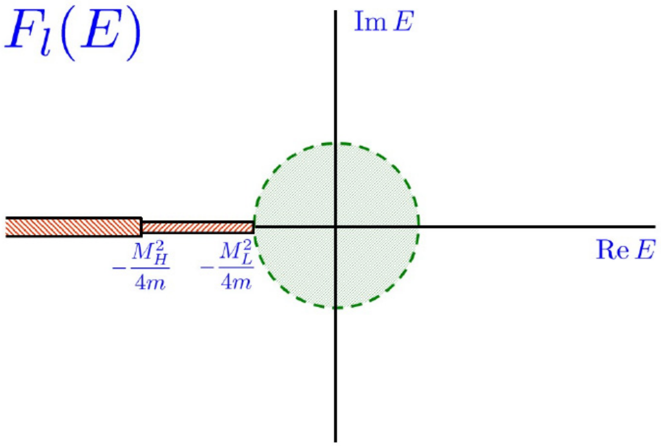
Lecture 2: Chiral EFT for two nucleons

- 2N beyond ERE: low-energy theorems
- Low-energy theorems for perturbative pions
- Exactly solvable (separable) toy model
- Toy model with local interactions
- Derivation of nuclear forces in chiral EFT

2N beyond ERE: Low-Energy Theorems

Both ERE & $\not{\pi}$-EFT provide an expansion of NN observables in powers of k/M_π , have the same validity range and incorporate the same physics

⇨ ERE ~ $\not{\pi}$-EFT (in the NN sector)



Two-range potential $V(r) = V_L(r) + V_S(r)$, $M_L^{-1} \gg M_H^{-1}$

- $\underline{F_l(k^2)}$ is meromorphic in $\underline{|k| < M_L/2}$
- $$\boxed{F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2}\cot\left[\delta_l(k) - \delta_l^L(k)\right]}$$

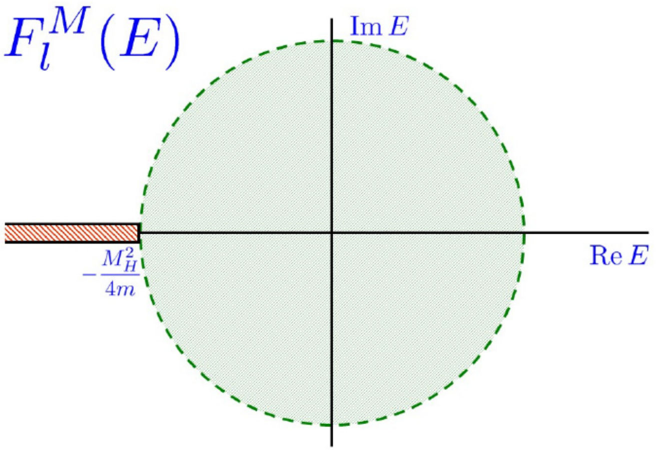
⇦ modified effective range function

Haeringen, Kok '82

$$\underbrace{f_l^L(k)}_{\text{Jost function for } V_L(r)} = \lim_{r\to 0}\left(\frac{l!}{(2l)!}(-2ikr)^l \underbrace{f_l^L(k,r)}_{\text{Jost solution for } V_L(r)}\right)$$

$$M_l^L(k) = Re\left[\frac{(-ik/2)^l}{l!}\lim_{r\to 0}\left(\frac{d^{2l+1}}{dr^{2l+1}}\frac{r^l f_l^L(k,r)}{f_l^L(k)}\right)\right]$$

Per construction, F_l^M reduced to F_l for $V_L = 0$ and is meromorphic in $|k| < M_H/2$



2N beyond ERE: Low-Energy Theorems

Example: proton-proton scattering

$$F_C(k^2) = C_0^2(\eta)\, k\, \cot[\delta(k) - \delta^C(k)] + 2k\,\eta\, h(\eta) = -\frac{1}{a^M} + \frac{1}{2} r^M k^2 + v_2^M k^4 + \ldots$$

where $\underbrace{\delta^C \equiv \arg\Gamma(1+i\eta)}_{\textit{Coulomb phase shift}}$, $\eta = \frac{m}{2k}\alpha$, $\underbrace{C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}}_{\textit{Sommerfeld factor}}$, $h(\eta) = \mathrm{Re}\Big[\underbrace{\Psi(i\eta)}_{\textit{Digamma function } \Psi(z) \equiv \Gamma'(z)/\Gamma(z)}\Big] - \ln(\eta)$

MERE and low-energy theorems

Long-range forces impose correlations between the ER coefficients (low-energy theorems)

Cohen, Hansen '99; Steele, Furnstahl '00

$$\boxed{F_l \;\equiv\; k^{2l+1}\cot\delta_l \;=\; -\frac{1}{a} \;+\; \frac{1}{2} r k^2 \;+\; v_2 k^4 \;+\; \ldots \;=\; \frac{A_l\, F_l^L \;-\; k^{4l+2}}{A_l \;+\; F_l^L}}$$

depend on F_l^M and quantities calculable from V_L

where $F_l^L = k^{2l+1}\cot\delta_l^L$, $A_l = (F_l^M + M_l^L)|f_l^L(k)|^2$

Compute $\delta_l^L(k)$, $f_l^L(k)$, $M_l^L(k)$ from V_L and use first n coefficients in the MERE as input

$$F_l^M(k^2) = -\frac{1}{a^M} \;+\; \frac{1}{2} r^M k^2 \;+\; v_2^M k^4 \;+\; v_3^M k^6 \;+\; v_4^M k^8 \;+\; \ldots$$

⟹ reproduce first n ERE coefficients and make predictions for all the higher ones (LETs)

2N with perturbative pions

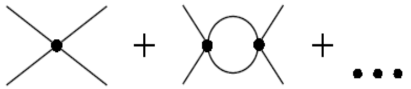
Chiral EFT for few nucleons: are pions perturbative?

It is straightforward to generalize the *KSW* power counting assuming that π-exchanges can be treated in perturbation theory, i.e.:

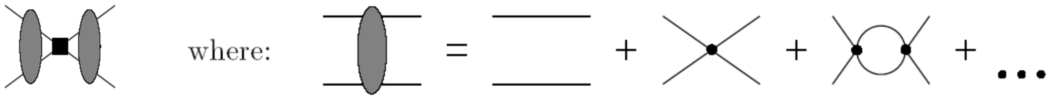
$$T(k) = T^{(-1)}(k) + T^{(0)}(k) + T^{(1)}(k) + \ldots$$

- EFT without pions

$T^{(-1)} =$



$T^{(0)} =$



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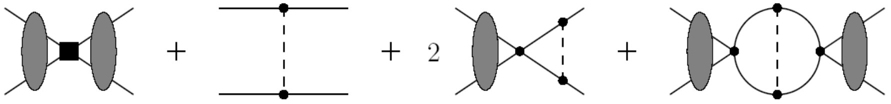
- EFT with perturbative pions

$T^{(-1)} =$

$T^{(0)} =$



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Testing 2N with perturbative π's: LETs

- **"Low-energy theorems"** *(Cohen & Hansen '99,'00; E.E. & Gegelia '09)*

  If pions are properly incorporated, one should be able to go beyond the effective range expansion, i.e. to predict the shape parameters.

$^1S_0$ at NLO: $k\cot\delta = -a^{-1} + \frac{1}{2}rk^2 + v_2k^4 + v_3k^6 + v_4k^8 + \ldots$

$$\frac{g_A^2 m}{16\pi F_\pi^2}\left(-\frac{16}{3a^2M_\pi^4}+\frac{32}{5aM_\pi^3}-\frac{2}{M_\pi^2}\right)$$

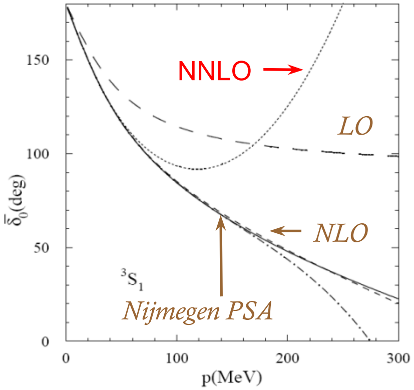
$$\frac{g_A^2 m}{16\pi F_\pi^2}\left(\frac{16}{a^2M_\pi^6}-\frac{128}{7aM_\pi^5}+\frac{16}{3M_\pi^4}\right)$$

	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)
theory	-3.3	17.8	-108.	-0.95	4.6	-25.
data	-0.5	3.8	-17.	0.04	0.7	-4.0

spin-singlet *spin-triplet*

- **Higher-order KSW calculation** *(Mehen & Stewart '00)*

  NNLO results obtained by Mehen & Stewart show no signs of convergence in spin-triplet channels



it seems necessary to treat pions non-perturbatively at momenta $p \sim M_\pi$

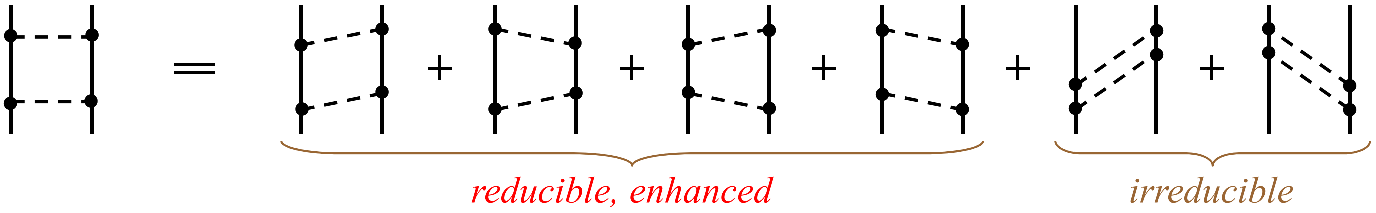
see, however, Beane, Kaplan, Vuorinen, arXiv:0812.3938 for an alternative scenario

Two nucleons: chiral EFT à la Weinberg

Weinberg '90,'91

Perturbation theory fails due to enhancement caused by reducible (i.e. infrared divergent in the limit $m_N \to \infty$) diagrams.

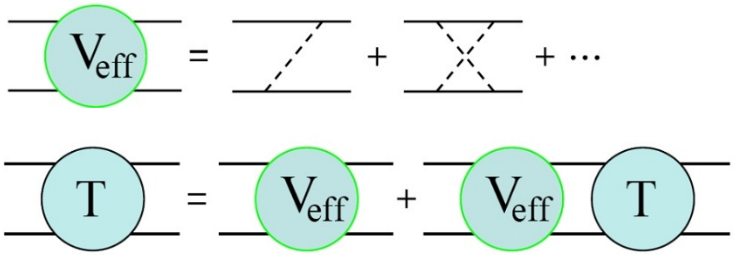
Switch to time-ordered theory: $\mathrm{Amp} = \langle NN|H^I|NN\rangle + \sum_\Psi \frac{\langle NN|H^I|\Psi\rangle\langle\Psi|H^I|NN\rangle}{E_{NN}-E_\Psi+i\epsilon} + \ldots$



$$\frac{1}{E_{NN}-E_\Psi} = \frac{m_N}{\vec{p}^{\,2}-\vec{q}^{\,2}} \sim \frac{m_N}{Q^2} \gg \frac{1}{Q} \qquad \frac{1}{E_{NN}-E_\Psi} \sim \frac{1}{M_\pi} \sim \frac{1}{Q}$$

Weinberg's approach

- Irreducible contributions can be calculated using ChPT
- Reducible contributions enhanced and should be summed up to infinite order



Two nucleons: chiral EFT à la Weinberg

$V_{\rm cont}$, V_π grow with increasing momenta ⟹ LS equation must be regularized & renormalized

$$T(\vec p, \vec k\,) = \Big[V_{\rm cont}(\vec p, \vec k\,) + V_\pi(\vec p, \vec k\,) \Big] + \int \frac{d^3 q}{(2\pi)^3} \Big[V_{\rm cont}(\vec p, \vec q\,) + V_\pi(\vec p, \vec q\,) \Big] \frac{m}{k^2 - q^2 + i\epsilon} T(\vec q, \vec k\,)$$

Regularization of the LS equation

- DR difficult to implement numerically due to appearance of power-law divergences
  Phillips et al.'00
- Cutoff (employed in most applications)
  — needs to be chosen $\Lambda \gg M_\pi$ to avoid large artifacts (i.e. large $1/\Lambda^n$-terms)
  — Λ can be employed at the level of $\mathcal{L}_{\rm eff}$ in order to preserve all relevant symmetries
  Slavnov '71; Djukanovic et al. '05,'07; also Donoghue, Holstein, Borasoy '98,'99

Renormalization à la Lepage

Ordonez et al.'96; Park et al.'99; E.E. et al.'00,'04,'05; Entem, Machleidt '02,'03

Choose $\Lambda \sim M_{\rm hard}$ & tune the strengths of $C_i(\Lambda)$ to fit low-energy observables.

- generally, can only be done numerically; requires solving nonlinear equations for $C_i(\Lambda)$,
- residual Λ dependence in observables survives,
- self-consistency checks via „Lepage plots“

Toy model

E.E., J. Gegelia, EPJ A41 (2009) 341

Two-range ($m_l \ll m_s \sim m$) spin-less separable model:

$$\boxed{V(p,\, p') = v_l\, F_l(p)\, F_l(p') + v_s\, F_s(p)\, F_s(p')}$$

with the formfactors $F_l(p) \equiv \dfrac{\sqrt{p^2+m_s^2}}{p^2+m_l^2}$ and $F_s(p) \equiv \dfrac{1}{\sqrt{p^2+m_s^2}}$

Lippmann-Schwinger equation (S-wave)

$$T(p',\, p;\, k) = V(p',\, p) + 4\pi \int \frac{l^2 dl}{(2\pi)^3} V(p',\, l) \frac{m}{k^2 - l^2 + i\epsilon} T(l,\, p;\, k)$$

can be solved analytically for interactions of a separable kind.

I require a “natural” scattering length $a = \alpha_{s,l}/m_{s,l}$ with $|\alpha_{s,l}| \sim 1$

$$\Longrightarrow \quad v_l = -\frac{8\pi m_l^3 \alpha_l}{m\,(\alpha_l m_s^2 + m_l^2 \alpha_l - 2 m_s^2)} \quad \text{and} \quad v_s = -\frac{4\pi m_s \alpha_s}{m\,(\alpha_s - 1)}$$

(strong long-range and weak short-range interactions at momenta $k \sim m_l$)

Toy model

E.E., J. Gegelia, EPJ A41 (2009) 341

“Chiral” expansion of the coefficients in the ERE (S-wave):

$$a, r = \frac{1}{m_l}\left[\gamma_{a,r}^{(0)} + \gamma_{a,r}^{(1)}\frac{m_l}{m_s} + \gamma_{a,r}^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right]$$

$$v_i = \frac{1}{m_l^{2i-1}}\left[\gamma_{v_i}^{(0)} + \gamma_{v_i}^{(1)}\frac{m_l}{m_s} + \gamma_{v_i}^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right]$$

depend on the details of the interaction

Explicit calculation for the considered model yields:

- Scattering length: $\gamma_a^{(0)} = \alpha_l\,, \quad \gamma_a^{(1)} = (\alpha_l - 1)^2\alpha_s\,, \quad \gamma_a^{(2)} = (\alpha_l - 1)^2\alpha_l\alpha_s^2\,, \quad \ldots$
- Effective range: $\gamma_r^{(0)} = \frac{3\alpha_l - 4}{\alpha_l}\,, \quad \gamma_r^{(1)} = \frac{2\,(\alpha_l - 1)\,(3\alpha_l - 4)\,\alpha_s}{\alpha_l^2}\,,$

$$\gamma_r^{(2)} = \frac{(\alpha_l - 1)\,(3\alpha_l - 4)\,(5\alpha_l - 3)\,\alpha_s^2 + (2 - \alpha_l)\,\alpha_l^2}{\alpha_l^3}\,, \quad \ldots$$

Low-energy theorems à la KSW

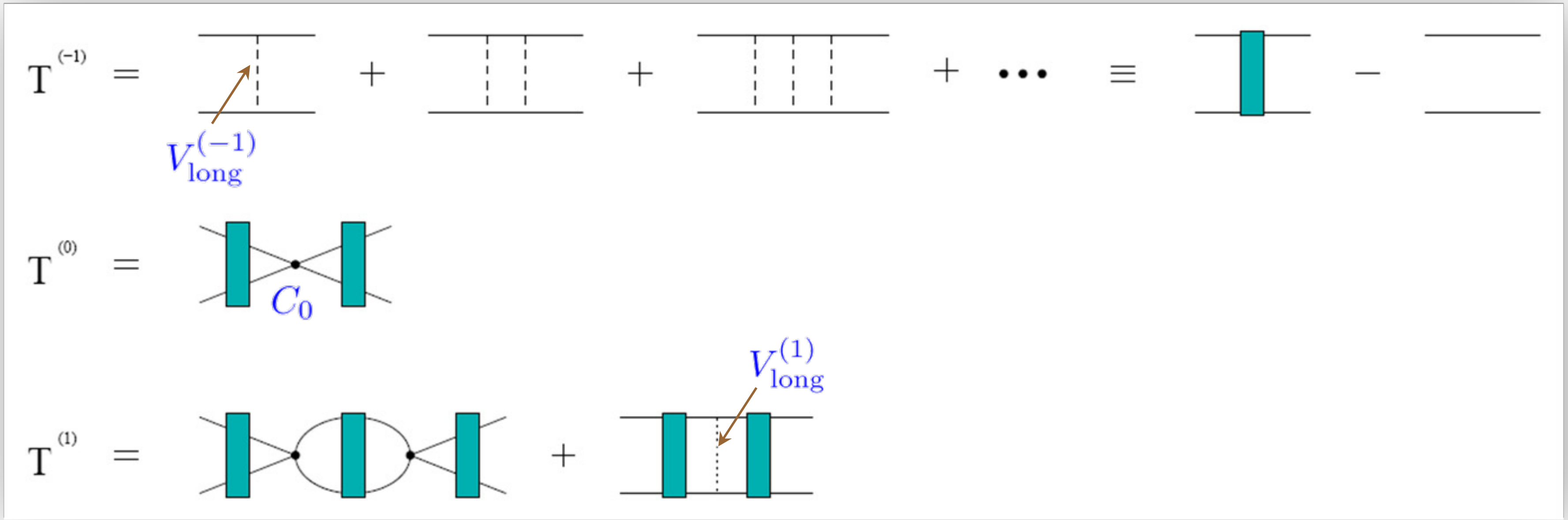
Effective theory: $V_{\rm eff}(p,\,p') = v_l\, F_l(p)\, F_l(p') \;+\; \Big[C_0 + C_2(p^2 + {p'}^2) + \ldots\Big]$

KSW-like approach: use subtractive renormalization (⟹ power counting at the level of diagrams) and keep track of the soft scales $Q = \{k,\, m_l,\, \mu\}$

Example of subtractive renormalization

$$I^{\rm reg} \equiv \int_0^\Lambda \frac{l^2 dl}{k^2 - l^2 + i\epsilon} = -\Lambda - i\frac{\pi}{2}k + \mathcal{O}(\Lambda^{-1}) \qquad\qquad I^{\rm subtr} \equiv \lim_{\Lambda\to\infty}\left[I^{\rm reg} + \int_\mu^\Lambda dl\right] = -\mu - i\frac{\pi}{2}k$$

Low-momentum expansion for the amplitude up to NNLO



Low-energy theorems à la KSW

Effective range function up to NNLO

$$k\cot\delta = -\frac{4\pi}{m}\Re\left[\underbrace{\frac{1}{T^{(-1)}}}_{\sim Q} - \underbrace{\frac{T^{(0)}}{[T^{(-1)}]^2}}_{\sim Q^2} + \underbrace{\frac{[T^{(0)}]^2}{[T^{(-1)}]^3} - \frac{T^{(1)}}{[T^{(-1)}]^2}}_{\sim Q^3}\right]$$

- **LO:** (leading long-range) $k\cot\delta = \underbrace{-\frac{m_l}{\alpha_l} + \frac{(3\alpha_l-4)}{2m_l\alpha_l}k^2 + \frac{(\alpha_l-2)}{2m_l^3\alpha_l}k^4}_{\gamma_{a,r}^{(0)} \text{ and } \gamma_{v_i}^{(0)} \text{ correctly reproduced for } \forall i}.$

- **NLO:** use $\gamma_a^{(1)}$ as input to fix $C_0^{(0)}$ and predict $\gamma_r^{(1)}$ and $\gamma_{v_i}^{(1)}$ for $\forall i$

  For example, the predicted effective range: $r = \frac{1}{m_l}\left[\frac{3\alpha_l-4}{\alpha_l} + \frac{2(\alpha_l-1)(3\alpha_l-4)\alpha_s}{\alpha_l^2 m_s}m_l\right].$

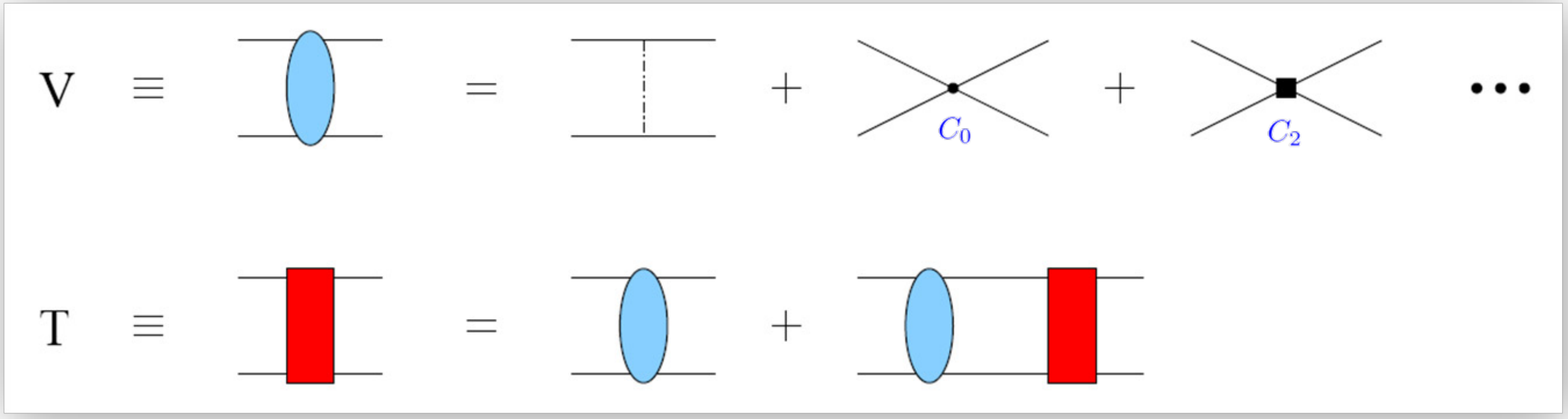
- **NNLO:** use $\gamma_a^{(2)}$ as input to fix $C_0^{(1)}$ and predict $\gamma_r^{(2)}$ and $\gamma_{v_i}^{(2)}$ for $\forall i$

$$r = \frac{1}{m_l}\left[\frac{3\alpha_l-4}{\alpha_l} + \frac{2(\alpha_l-1)(3\alpha_l-4)\alpha_s}{\alpha_l^2 m_s}m_l + \frac{(\alpha_l-1)(3\alpha_l-4)(5\alpha_l-3)\alpha_s^2 + (2-\alpha_l)\alpha_l^2}{\alpha_l^3 m_s^2}m_l^2 + \mathcal{O}(Q^4)\right]$$

Low-energy theorems à la Weinberg

It is difficult to apply the above renormalization scheme to OPEP (non-separable) ⟹ cutoff regularization and the Weinberg-Lepage scheme:

Expansion for the amplitude in Weinberg's approach



- **LO:** same as before (only long-range force), ⟹ $\gamma^{(0)}_{a,r}$ and $\gamma^{(0)}_{v_i}$ correctly reproduced for $\forall i$
- **NLO:** $V_{\text{eff}}(p,\, p') \;=\; V_{\text{long}}(p,\, p') + C_0$

  Solve the LS equation for a given value of Λ and adjust the LEC $C_0(\Lambda)$ to reproduce the scattering length

Low-energy theorems à la Weinberg

$$a = \frac{\pi m_s \left\{C_0 m \left[2\alpha_l \left(m_s \left(\Lambda - s m_l\right) + 2m_l^2 \ln(m_s/2\Lambda)\right) + \pi m_l m_s\right] + 4\pi^2 \alpha_l m_s\right\}}{m_l \left[2\pi m_s^2 \left(C_0 m \Lambda + 2\pi^2\right) - C_0 m m_l \alpha_l \left[s m_s - 2 m_l \ln(m_s/2\Lambda)\right]^2\right]}$$

$$\left(2\sqrt{m_s^2 - m_l^2}/m_s\right) \mathrm{arccot}\left(m_l/\sqrt{m_s^2 - m_l^2}\right)$$

$$\overset{(!)}{=} \underbrace{\frac{m_l \left(2\alpha_l - 1\right) \alpha_s - \alpha_l m_s}{m_l \left(m_l \alpha_l \alpha_s - m_s\right)}}_{\textit{scatt. length in the underlying model}} \Longrightarrow C_0(\Lambda) = \ldots$$

Prediction for the effective range:

$$r = \frac{1}{m_l}\left[\underbrace{\frac{3\alpha_l - 4}{\alpha_l}}_{\gamma_r^{(0)}} + \underbrace{\frac{2\left(\alpha_l - 1\right)\left(3\alpha_l - 4\right)\alpha_s}{\alpha_l^2}}_{\gamma_r^{(1)}} \frac{m_l}{m_s} + \right.$$

$$\left. + \underbrace{\left(\frac{4\left(\alpha_l - 2\right)\alpha_s}{\pi \alpha_l}\left(\ln\frac{m_s}{2\Lambda} + 1\right) + \frac{\left(\alpha_l - 1\right)\left(3\alpha_l - 4\right)\left(5\alpha_l - 3\right)\alpha_s^2 + \left(2 - \alpha_l\right)\alpha_l^2}{\alpha_l^3}\right)}_{\gamma_r^{(2)}} \frac{m_l^2}{m_s^2} + \mathcal{O}\left(m_l^3\right)\right]$$

The first nontrivial LET for $\gamma_r^{(1)}$ correctly reproduced provided one chooses $\Lambda \sim m_s$. Same conclusions for the shape parameters v_i.

Misconception: Infinite cutoff limit

It is possible to take the limit $\Lambda \to \infty$ for T-matrix while keeping the scattering length correctly reproduced. Notice that the infinite cutoff limit does not commute with the "chiral expansion", i.e. with the Taylor expansion of r in powers of m_l:

$$\lim_{\Lambda\to\infty} T_{m_l}\Big[m_l\, r(m_l, m_s, \Lambda)\Big] \neq T_{m_l}\Big[\lim_{\Lambda\to\infty} m_l\, r(m_l, m_s, \Lambda)\Big]$$

Taylor expansion

- finite cutoff-removed result for the effective range:

$$r_\infty = \frac{m_l^3\alpha_s + m_l^2(\alpha_l - 2)\, m_s + m_l(2\alpha_l - 3)\, m_s^2\alpha_s + (4 - 3\alpha_l)\, m_s^3}{m_l m_s^2\left(m_l(2\alpha_l - 1)\,\alpha_s - \alpha_l m_s\right)}$$

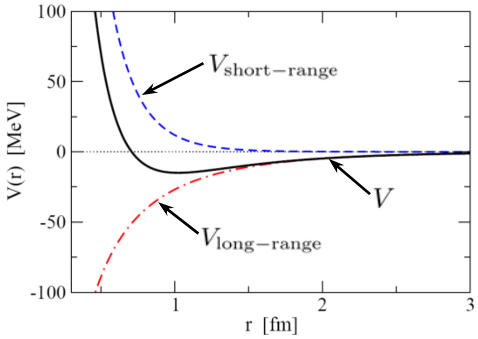
$$= \frac{1}{m_l}\left[\underbrace{\frac{3\alpha_l - 4}{\alpha_l}}_{\gamma_r^{(0)}} + \underbrace{\frac{4(\alpha_l - 1)^2\alpha_s}{\alpha_l^2}\frac{m_l}{m_s}}_{\neq\,\gamma_r^{(1)}} + \frac{\alpha_l^3(8\alpha_s^2 - 1) + \alpha_l^2(2 - 20\alpha_s^2) + 16\alpha_l\alpha_s^2 - 4\alpha_s^2}{\alpha_l^3}\frac{m_l^2}{m_s^2} + \mathcal{O}(m_l^3)\right]$$

- the first non-trivial LET is broken after taking the limit $\Lambda \to \infty$

  Similarly, the LETs for the shape parameters are also broken in the infinite-Λ limit.

Toy model with a local potential

$$V(\vec q) = \underbrace{\frac{\alpha_l}{\vec q^{\,2}+M_l^2}}_{M_l\,=\,200\text{ MeV}} + \underbrace{\frac{\alpha_h}{\vec q^{\,2}+M_h^2}}_{M_h\,=\,750\text{ MeV}} \quad\rightarrow\quad V(\vec r) = \underbrace{\frac{\alpha_l}{4\pi r}e^{-M_l r}}_{\textit{long-range}} + \underbrace{\frac{\alpha_h}{4\pi r}e^{-M_h r}}_{\textit{short-range}}$$



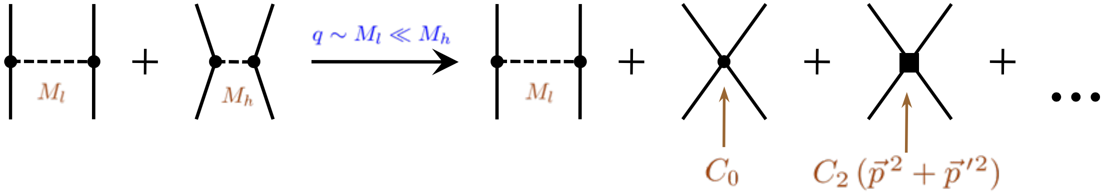
$$\left.\begin{aligned}\alpha_l &= -1.50\\ \alpha_h &= 10.81\end{aligned}\right\}$$ ⇨ S-wave bound state with: $E_B = 2.2229$ MeV

Effective theory

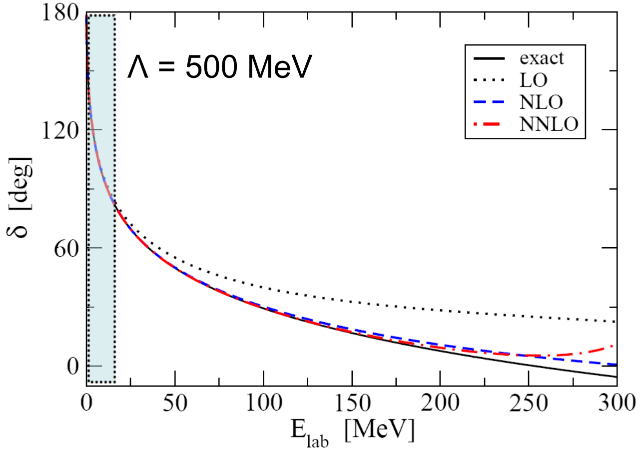
At low energy, $q \sim M_l \ll M_h$, the precise structure of $V_{\rm short-range}$ is irrelevant

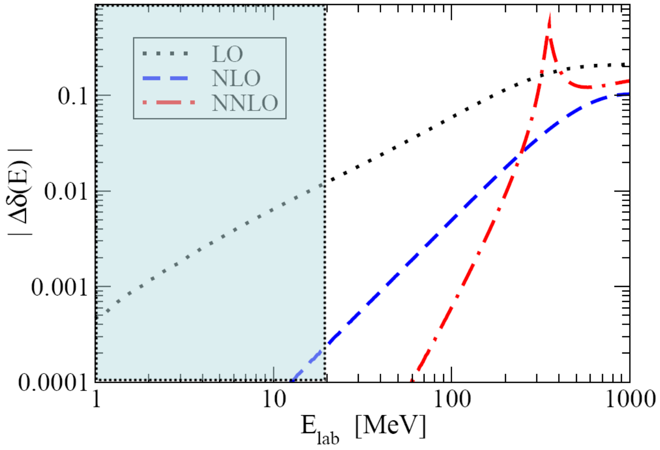
⇨ mimic $V_{\rm short-range}$ by a generic set of point-like interactions

$$\boxed{V \rightarrow V_{\rm eff} = V_{\rm long-range} + \left[C_0 + C_2\left(\vec p^{\,2} + \vec p^{\,\prime 2}\right) + C_4 \vec p^{\,2}\vec p^{\,\prime 2} + \ldots\right]\exp\left(-\frac{\vec p^{\,2}+\vec p^{\,\prime 2}}{\Lambda^2}\right)}$$



Toy model with a local potential





Error at order ν : $\Delta\delta(k) \sim (k/\bar{\Lambda})^{2\nu}$, $\bar{\Lambda} \sim 400 \text{ MeV}$ ⟸ agrees with $\bar{\Lambda} \sim M_h/2$

Results for the bound state: $E_B = \underbrace{2.1594}_{LO} + \underbrace{0.0638}_{NLO} - \underbrace{0.0003}_{NNLO} = 2.2229 \text{ MeV}$

Lessons learned:

- Incorporate the correct long-range force.
- Add local correction terms to V_{eff} . Respect symmetries.
- Introduce an ultraviolet cutoff Λ of the order of the natural hard scale.
- Fix unknown constants from some date and make predictions.

⟹ At low energy model independent and systematically improvable approach!

Further reading

Breakdown of NN EFT with perturbative pions

- *Cohen, Hansen, Phys. Rev. C59 (99) 13; Phys. Rev. C59 (99) 3047; arXiv:nucl-th/9908049*
- *Fleming, Mehen, Stewart, Nucl. Phys. A677 (00) 313*

How to renormalize the Schrödinger equation

- *Lepage, "How to renormalize the Schrödinger equation", arXiv:nucl-th/9706029*
- *Lepage, "Tutorial: renormalizing the Schrödinger equation", talk at the INT Program 00-2 "Effective Field Theories and Effective Interactions", see:*

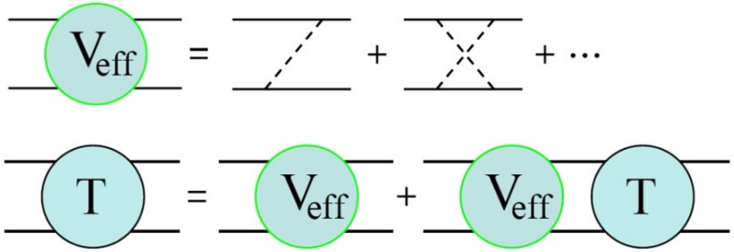
  http://www.int.washington.edu/talks/WorkShops/int_00_2/People/Lepage_TUT/ht/01.html
- *E.E., Gegelia, Eur. Phys. J. A41 (09) 341*

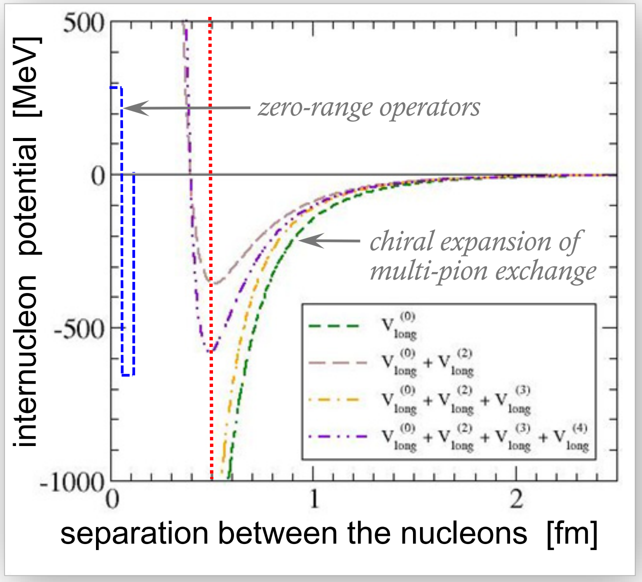
Nuclear chiral EFT à la Weinberg

Weinberg '90,'91

Weinberg's approach

- Irreducible contributions can be calculated using ChPT
- Reducible contributions enhanced and should be summed up to infinite order





Structure of chiral nuclear forces

$$V_{\rm eff} = \sum_{\nu} \Big[\underbrace{V^{(\nu)}_{\rm short-range}}_{\textit{parametrized}} + \underbrace{V^{(\nu)}_{\rm long-range}}_{\chi\textit{-symm. constrained}} \Big]$$

— how to derive nuclear forces from $\mathcal{L}_{\rm eff}$?

Derivation of nuclear forces

Nuclear forces are defined as irreducible (i.e. non-iterative) contributions to the amplitude and can be derived using various methods.

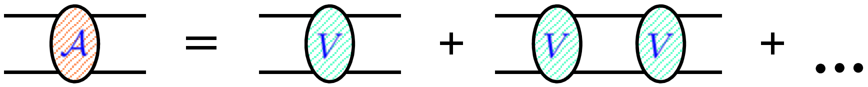
S-matrix-based method

Robilotta, da Rocha ’97; Kaiser et al. ’97,’01,...; Higa et al. ’03,’04; ...

Idea: the potential is derived through (perturbative) matching to the scattering amplitude.

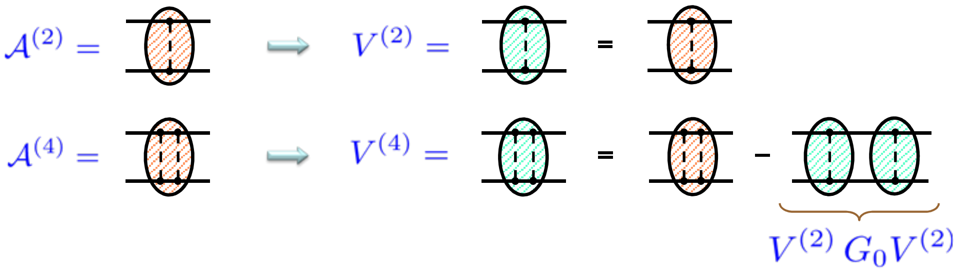
$$\underbrace{\mathcal{A}}_{\text{calculated using standard methods}} = \underbrace{V + V\,G_0\,V + V\,G_0\,V\,G_0\,V + \ldots}_{\text{matching to } \mathcal{A} \text{ allows to define } V}$$

calculate in ChPT ⟹



⟸ *define V by matching to $\mathcal{A}$*

For example:

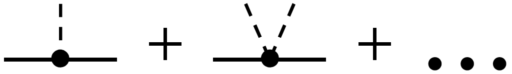


Derivation of nuclear forces

Old-fashioned time-ordered perturbation theory

Weinberg '90,'91; Ordonez et al. '92,'94; van Kolck '94

Consider mesons interacting with non-relativistic nucleons: $H = H_0 + H_I \,, \quad H_I =$



Schrödinger equation:

nucleonic states $|N\rangle, \, |NN\rangle, \, \ldots$

$$\begin{pmatrix} \eta H \eta & \eta H \lambda \\ \lambda H \eta & \lambda H \lambda \end{pmatrix} \begin{pmatrix} |\phi\rangle \\ |\psi\rangle \end{pmatrix} = E \begin{pmatrix} |\phi\rangle \\ |\psi\rangle \end{pmatrix}$$

projectors

states with mesons $|N\pi\rangle, \, |N\pi\pi\rangle, \, \ldots$

can not solve (infinite-dimensional eq.)

Effective Schrödinger equation for $|\phi\rangle$:

$$|\psi\rangle = \frac{1}{E - \lambda H \lambda} H |\phi\rangle \quad \Longrightarrow \quad \left(H_0 + V_{\rm eff}^{\rm t-o}(E) \right) |\phi\rangle = E |\phi\rangle$$

where $V_{\rm eff}^{\rm t-o}(E) = \eta H_I \eta + \eta H_I \lambda \frac{1}{E - \lambda H \lambda} \lambda H_I \eta$

$$= \eta H_I \eta + \eta H_I \frac{\lambda}{E - H_0} H_I \eta + \eta H_I \frac{\lambda}{E - H_0} H_I \frac{\lambda}{E - H_0} H_I \eta + \ldots$$

- $V_{\rm eff}^{\rm t-o}$ depends on E
- $|\phi\rangle$ not orthonormal: $\langle \phi_i | \phi_j \rangle = \langle \Psi_i | \Psi_j \rangle - \langle \psi_i | \psi_j \rangle = \delta_{ij} - \langle \phi_i | H_I \left(\frac{1}{E - \lambda H \lambda} \right)^2 H_I | \phi_j \rangle$

Derivation of nuclear forces

Method of unitary transformation

Taketani, Mashida, Ohnuma'52, Okubo '54, E.E., Glöckle, Meißner '98,'00,'05

Find a unitary operator U such that: $\tilde{H} \equiv U^\dagger \begin{pmatrix} \eta H \eta & \eta H \lambda \\ \lambda H \eta & \lambda H \lambda \end{pmatrix} U = \begin{pmatrix} \eta \tilde{H} \eta & 0 \\ 0 & \lambda \tilde{H} \lambda \end{pmatrix}$

- no dependence on energy (per construction),
- unitary transformation preserves the norm of $|\phi\rangle$

How to compute U ?

It is convenient to parameterize U in terms of the operator $A = \lambda A \eta$ *(Okubo '54):*

$$U = \begin{pmatrix} \eta(1 + A^\dagger A)^{-1/2} & -A^\dagger(1 + AA^\dagger)^{-1/2} \\ A(1 + A^\dagger A)^{-1/2} & \lambda(1 + AA^\dagger)^{-1/2} \end{pmatrix}$$

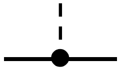
Require that $\eta \tilde{H} \lambda = \lambda \tilde{H} \eta = 0$ ⟹ $\boxed{\lambda \left(H - [A, \; H] - AHA \right) \eta = 0}$

The major problem is to solve the nonlinear decoupling equation.

Notice: similar methods widely used in particle & nuclear physics (Lee-Suzuki) and to deal with few- and many-body problems.

Derivation of nuclear forces

Example: expansion in powers of the coupling constant

$H_I =$  $\propto g$ ⇨ ansatz: $A = A^{(1)} + A^{(2)} + A^{(3)} + \ldots$

Recursive solution of the decoupling equation $\lambda \left(H - [A,\ H] - A H A \right) \eta = 0$

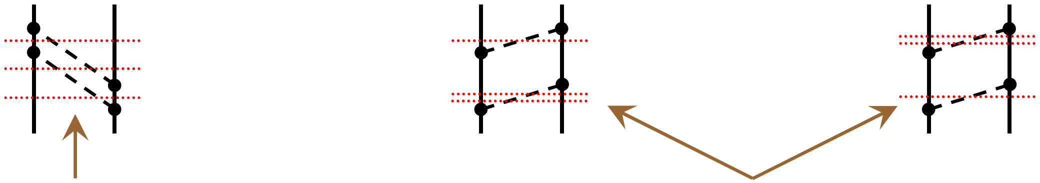
$$g^1: \quad \lambda(H_I - [A^{(1)},\ H_0])\eta = 0 \quad \Longrightarrow \quad A^{(1)} = -\lambda \frac{H_I}{E_\eta - E_\lambda} \eta$$

$$g^2: \quad \lambda(H_I\, A^{(1)} - [A^{(2)},\ H_0])\eta = 0 \quad \Longrightarrow \quad A^{(2)} = -\lambda \frac{H_I\, A^{(1)}}{E_\eta - E_\lambda} \eta$$

$$\ldots$$

In the static approximation, i.e. in the limit $m \to \infty$, one has: $E_\eta - E_\lambda \sim E_\pi$. One obtains:

$$\begin{aligned} V_{\rm eff} \;\; = \;\; & -\eta H_I \frac{\lambda}{E_\pi} H_I \eta \\ & \underbrace{-\eta H_I \frac{\lambda}{E_\pi} H_I \frac{\lambda}{E_\pi} H_I \frac{\lambda}{E_\pi} H_I \eta} + \underbrace{\frac{1}{2} \eta H_I \frac{\lambda}{E_\pi} H_I \eta H_I \frac{\lambda}{E_\pi^2} H_I \eta} + \underbrace{\frac{1}{2} \eta H_I \frac{\lambda}{E_\pi^2} H_I \eta H_I \frac{\lambda}{E_\pi} H_I \eta} + \ldots \end{aligned}$$



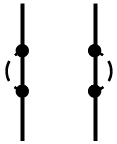
same as in old-fashioned perturbation theory

wave-function renormalization (missing in old-fashioned perturbation theory)

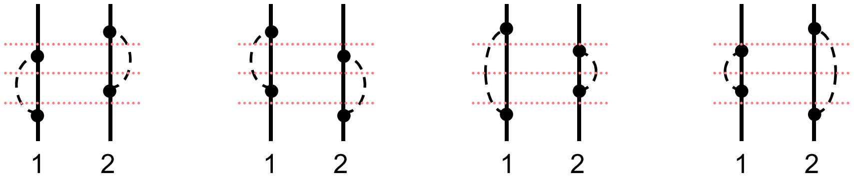
Derivation of nuclear forces

Consider self-energy insertions at 2 non-interacting nucleons:

Expect no contributions to the 2N Hamilton operator!



- **old-fashioned perturbation theory**



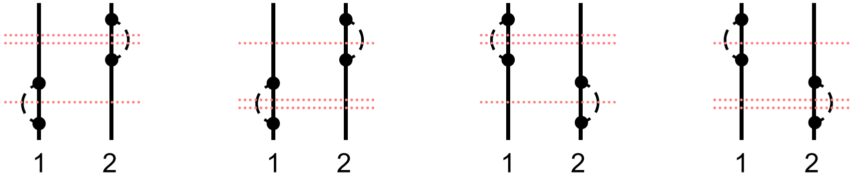
$$
\begin{aligned}
V_{\rm eff}^{\rm t-o} &= -\eta H_I \frac{\lambda}{E_\pi} H_I \frac{\lambda}{E_\pi} H_I \frac{\lambda}{E_\pi} H_I \eta \\
&= \mathcal{M} \left(-\frac{2}{\omega_1 \omega_2 (\omega_1 + \omega_2)} - \frac{1}{\omega_1^2 (\omega_1 + \omega_2)} - \frac{1}{\omega_2^2 (\omega_1 + \omega_2)} \right) \\
&= \mathcal{M} \left(-\frac{1}{\omega_1^2 \omega_2} - \frac{1}{\omega_1 \omega_2^2} \right)
\end{aligned}
$$

common isospin, spin & momentum structure (depends on the form of H_I)

What is wrong ??

- **method of unitary transformation**

Additional contributions
(wave-function renormalization)



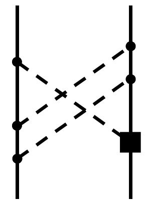
$$
V_{\rm eff} = V_{\rm eff}^{\rm t-o} + \frac{1}{2} \eta H_I \frac{\lambda}{E_\pi} H_I \eta H_I \frac{\lambda}{E_\pi^2} H_I \eta + \frac{1}{2} \eta H_I \frac{\lambda}{E_\pi^2} H_I \eta H_I \frac{\lambda}{E_\pi} H_I \eta = V_{\rm eff}^{\rm t-o} + \mathcal{M} \left(\frac{1}{\omega_1^2 \omega_2} + \frac{1}{\omega_1 \omega_2^2} \right) = 0
$$

Derivation of nuclear forces

Application to chiral Lagrangians *(E.E. et al., '98)*

expansion in g ⇒ chiral expansion

Power counting



$\sim \left(\frac{Q}{\Lambda}\right)^{\nu}$

Count powers of Q using dimensional analysis
Alternatively: count powers of Λ!

The only source of Λ are the coupling constants ⇒ $\boxed{\nu = -2 + \sum_i V_i \kappa_i}$

$$\mathcal{L}_i = c_i \, (N^\dagger(\ldots)N)^{\frac{n_i}{2}} \, \pi^{p_i} \, (\partial_\mu, M_\pi)^{d_i} \quad \Longrightarrow \quad [c_i] = (mass)^{-\kappa_i} \quad with \quad \kappa_i = d_i + \tfrac{3}{2} n_i + p_i - 4$$

Remember:

$\kappa_i < 0$ – *relevant (superrenorm.)*
$\kappa_i = 0$ – *marginal (renorm.)*
$\kappa_i > 0$ – *irrelevant (nonrenorm.)*

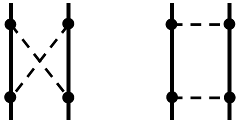
Examples:

$N^\dagger \boldsymbol{\tau} \vec{\sigma} N \cdot \vec{\nabla} \boldsymbol{\pi} \longrightarrow \kappa_i = 1$
$(N^\dagger N)\,(N^\dagger N) \longrightarrow \kappa_i = 2$

- expansion in coupling constant ($H_i \sim g^{n_i}$) ⟷ chiral expansion ($H_i \sim (Q/\Lambda)^{\kappa_i}$)
- perturbation theory works since all $\kappa_i > 0$ (as a consequence of χ–symmetry)

Derivation of nuclear forces

Example: chiral 2π-exchange potential proportional to $g_A{}^4$:



$$V_{2\pi}^{(2)}(q) = \; -\eta H_I \frac{\lambda}{E_\pi} H_I \frac{\lambda}{E_\pi} H_I \frac{\lambda}{E_\pi} H_I \eta \; + \; \frac{1}{2} \eta H_I \frac{\lambda}{E_\pi} H_I \eta H_I \frac{\lambda}{E_\pi^2} H_I \eta \; + \; \frac{1}{2} \eta H_I \frac{\lambda}{E_\pi^2} H_I \eta H_I \frac{\lambda}{E_\pi} H_I \eta$$

$$= -\frac{g_A^4}{2(2F_\pi)^4} \int \frac{d^3 l}{(2\pi)^3} \frac{\omega_+^2 + \omega_+ \omega_- + \omega_-^2}{\omega_+^3 \omega_-^3 (\omega_+ + \omega_-)} \left\{ \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \left(\vec{l}^{\,2} - \vec{q}^{\,2} \right)^2 + 6 (\vec{\sigma}_2 \cdot [\vec{q} \times \vec{l}\,])(\vec{\sigma}_1 \cdot [\vec{q} \times \vec{l}\,]) \right\}$$

$$\omega_\pm = \sqrt{(\vec{q} \pm \vec{l}\,) + 4 M_\pi^2}$$

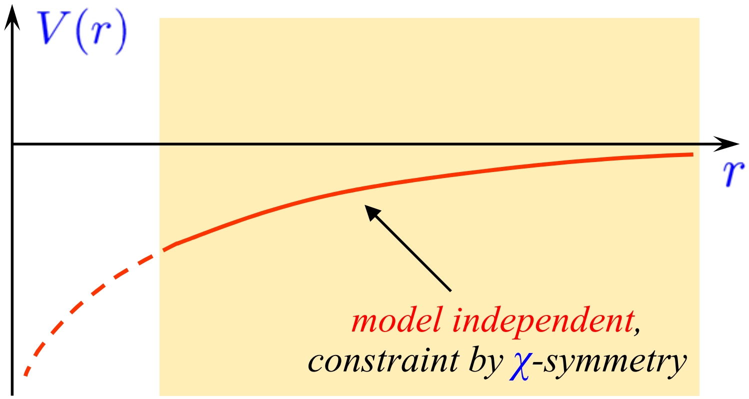
$$= -\frac{g_A^4}{384 \pi^2 F_\pi^4} \left[\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \left(20 M_\pi^2 + 23 q^2 + \frac{48 M_\pi^4}{4 M_\pi^2 + q^2} \right) - 18 \left(\vec{\sigma}_1 \cdot \vec{q}\, \vec{\sigma}_2 \cdot \vec{q} - q^2\, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \right] L(q) + \ldots$$

where the loop function is given by (in DR):

$$L(q) = \frac{1}{q} \sqrt{4 M_\pi^2 + q^2} \, \ln \frac{\sqrt{4 M_\pi^2 + q^2} + q}{2 M_\pi}$$

The integral has logarithmic and quadratic divergences can be absorbed into short-range terms:

$$V_{\rm cont} = (\alpha_1 + \alpha_2 \, q^2) \, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + \alpha_3 (\vec{\sigma}_1 \cdot \vec{q}) (\vec{\sigma}_2 \cdot \vec{q}) \\ + \alpha_4 (\vec{\sigma}_1 \cdot \vec{\sigma}_2) \, q^2$$



model independent,
constraint by χ-symmetry

Further reading

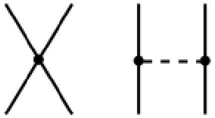
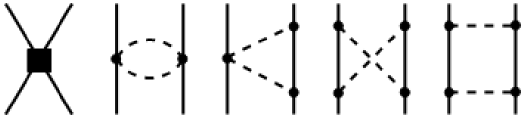
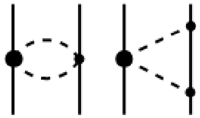
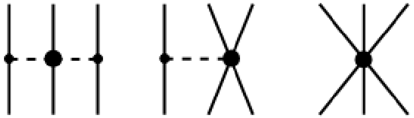
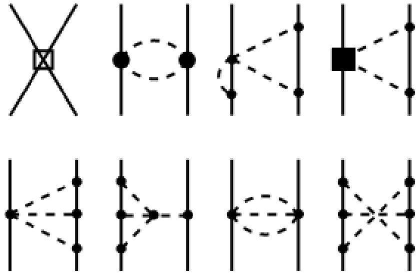
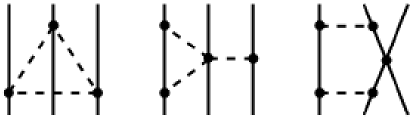
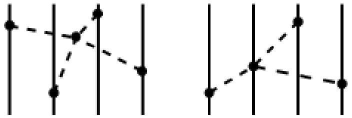
Nuclear potentials from field theory

- *Tamm, J. Phys. (USSR) 9 (45) 449; Dancoff, Phys. Rev. 78 (50) 382*
- *Okubo, Prog. Theor. Phys. (Japan) 12 (54) 603*
- *Fukuda, Sawada, Taketani, Prog. Theor. Phys. (Japan) 12 (54) 156*
- *Friar, Ann. Phys. 104 (77) 380*
- *Phillips, Reports on Progress in Physics XXII (59) 562 [review article]*

Derivation of the nuclear force from chiral EFT (selected papers)

- *Weinberg, Nucl. Phys. B363 (91) 3*
- *Ordonez, Ray, van Kolck, Phys. Rev. C53 (96) 2086*
- *Kaiser, Brockmann, Weise, Nucl. Phys. A625 (97) 758*
- *EE, Glöckle, Meißner, Nucl. Phys. A714 (03) 535*
- *EE, Prog. Part. Nucl. Phy.s. 57 (06) 654 [review article]; arXiv:1001.3229 [nucl-th] [lecture notes]*
- *EE, Hammer, Meißner, Rev. Mod. Phys. 81 (09) 1773 [review article]*

Nuclear forces from chiral EFT

	2N force	3N force	4N force
LO		—	—
NLO		—	—
N^{2}LO			—
N^{3}LO		 ...	 ...