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Polarization Observables and Sum Rules for Electromagnetic Reactions on the Deuteron

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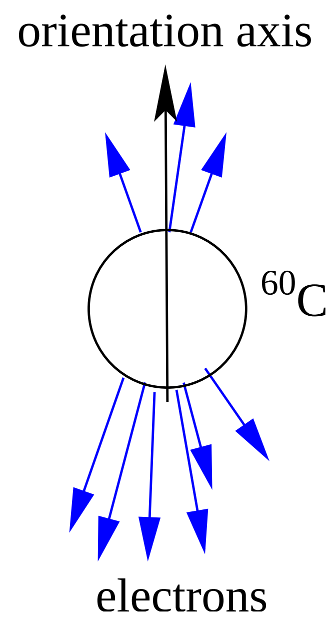
1. Introduction and Motivation

A few basic facts:

- The spin of a particle is an important property and opens a new DEGREE OF FREEDOM which plays an essential role in its dynamics.
- One should keep im mind that it is a RELATIVISTIC PROPERTY, i.e. the spin is of relativistic origin.
- The spin, integer or half-integer, determines the statistics in a system of identical particles, either Bose-Einstein or Fermi-Dirac statistics.
- The exploitation of SPIN DEGREES OF FREEDOM as manifest in POLARIZATION OBSERVABLES is a very important tool in various branches of physics.

A specific feature is the possibility to access small but interesting amplitudes which usually are buried under dominant contributions in an unpolarized experiment.

A very prominent example is parity violation in β-decay of oriented ^{60}Co (WU-experiment), where one measures the correlation between the orientation axis of ^{60}Co and the direction of the emitted electrons, which is a pseudoscalar quantity, generated by parity violation.



First I will consider some general features of polarization observables.

Then I will discuss their role in electromagnetic reactions taking the deuteron as an important example.

Finally I will review some general aspects of e.m. sum rules with special emphasis on the GERASIMOV-DRELL-HEARN sum rule.

2. Polarization Observables

Spin d.o.f become manifest in so-called "POLARIZATION OBSERVABLES".

They arise in reactions where the initial state is partially polarized and/or the final state is analyzed with respect to the polarization (polarimeter).

Consider, for example, a reaction:

$$a + b \rightarrow c + d + \ldots$$

Further assume that the initial system is prepared in a pure state $|\psi_{i,M_i^a,M_i^b}\rangle$ with specific spin projections M_i^a, M_i^b and that the detector selects another pure final state $|\psi_{f,M_f^c,M_f^d,\ldots}\rangle$.

Then the cross section is determined by the absolut square of the corresponding reaction matrix $\widehat{T}$

$$\sigma(a + b \rightarrow c + d + \ldots) \propto |\langle\psi_{f,M_f^c,M_f^d,\ldots}|\widehat{T}|\psi_{i,M_i^a,M_i^b}\rangle|^2 \,.$$

Spin degrees could then be investigated by varying the projections M_i^a, M_i^b and $M_f^c, M_f^d, \ldots$ of the initial and final states, and study how the cross section changes.

This ideal situation of pure states, however, is almost never met, because

(i) the initial preparation results in general in a statistical ensemble of a subset of different states (mixed state), and

(ii) similarly, the detector is sensitive to another statistical ensemble of states.

In other words, usually one has incomplete information about the initial and final states.

Consequence: STATISTICAL METHODS ARE NEEDED

$\longrightarrow$ Introduction of the DENSITY MATRIX for representing pure and mixed states on an equal footing.

Given an orthogonal ensemble of states $\{|\psi_i\rangle | i = 1, \ldots, n_0\}$ with probabilities p_i with $\sum_i p_i = 1$:

Definition of DENSITY MATRIX: $\boxed{\widehat{\rho} = \sum_i |\psi_i\rangle p_i \langle\psi_i|}$

The case of a pure ensemble represented by a pure state is contained for $p_i = \delta_{in_0}$ with $n_0 \in \{1, ..., n\}$.

SOME PROPERTIES OF THE DENSITY MATRIX:

(i) $\widehat{\rho}$ is hermitean: $\boxed{\widehat{\rho}^\dagger = \widehat{\rho}}$, and normalized $\boxed{\mathbf{Tr}\widehat{\rho} = 1}$.

(ii) Only for a pure state: $\boxed{\widehat{\rho}^2 = \widehat{\rho}}$, i.e. $\widehat{\rho}$ is *idempotent* or a projection operator onto the pure state.

(iii) The expectation value of an operator $\widehat{\Omega}$ is given by $\boxed{\langle\widehat{\Omega}\rangle_\rho = \mathbf{Tr}(\widehat{\Omega}\widehat{\rho})}$.

(iv) Equation of motion with $\widehat{H}$ as the hamiltonian of the system:

$$\boxed{\frac{\partial\widehat{\rho}}{\partial t} = \frac{i}{\hbar}\left[\widehat{\rho}, \widehat{H}\right]}$$

A few examples:

(i) **Density matrix for spin $1/2$:**

Two basis states: $\{|\frac{1}{2} \pm \frac{1}{2}\rangle\} \rightsquigarrow \rho^{(1/2)}$ represented by 2×2-matrix

$$\boxed{\rho^{(1/2)} = \tfrac{1}{2}(\mathbb{1} + \mathbf{P} \cdot \sigma)}$$

P_i probability to find spin component in direction of i-axis.
For a pure state $|\mathbf{P}| = 1$, while for a partially polarized particle one has $|\mathbf{P}| < 1$.

(ii) **Density matrix for real photon:**

Again two basis states, either linear or circular transverse photon polarizations: thus the density matrix $\rho^{(\gamma)}$ has the same structure as for a spin one-half particle

$$\boxed{\rho^{(\gamma)} = \tfrac{1}{2}(\mathbb{1} + \mathbf{P} \cdot \sigma)}$$

However, the meaning of $\mathbf{P}$ is different and depends on the chosen representation, either linear or circular polarization basis.

(iia) DENSITY MATRIX FOR VIRTUAL PHOTON (ELECTRON SCATTERING)

Electron scattering is in lowest order described by exchange of a virtual photon, thus has formal appearance of photo absorption of a virtual photon.

For longitudinally polarized electrons of degree h the normalized density matrix of the exchanged photon reads

$$(\widetilde{\rho}^{(\gamma^*)})_{\lambda'\lambda}=\frac{1}{\xi^2+\xi+2\eta}\begin{pmatrix} \xi^2 & \widetilde{\rho}_{01} & \widetilde{\rho}_{0-1} \\ \widetilde{\rho}_{01} & \frac{\xi}{2}+\eta-h\sqrt{\eta(\xi+\eta)} & -\frac{\xi}{2} \\ \widetilde{\rho}_{0-1} & -\frac{\xi}{2} & \frac{\xi}{2}+\eta+h\sqrt{\eta(\xi+\eta)} \end{pmatrix},$$

where $\widetilde{\rho}_{0\pm 1} = \pm\xi\sqrt{(\xi+\eta)/2} - h\xi\eta/2$, $\xi = 1-\omega^2/q^2$, and $\eta = \tan^2(\theta_e/2)$, (ω, q) energy and momentum transfer, and θ_e electron scattering angle.

This leads to following interpretation of the virtual photon density matrix:

(i) Fraction of longitudinal photons: $P_{\text{long}} = \frac{\xi^2}{\xi^2+\xi+2\eta}$.

(ii) Fraction of transverse photons: $P_{\text{trans}} = \frac{\xi+2\eta}{\xi^2+\xi+2\eta}$.

(ii) Their ratio $P_{\text{long}}/P_{\text{trans}} = \frac{\xi^2}{\xi+2\eta}$ approaches $\xi = 1-\omega^2/q^2$ for $\theta_e \to 0$.

Furthermore, the transverse part of $\widetilde{\rho}^{(\gamma^*)}$ is

$$\rho^{(\gamma^*)}(t) = \frac{\eta}{\xi^2+\xi+2\eta} \begin{pmatrix} 1+\frac{\xi}{2\eta}+h\sqrt{1+\frac{\xi}{\eta}} & -\frac{\xi}{2\eta} \\ -\frac{\xi}{2\eta} & 1+\frac{\xi}{2\eta}-h\sqrt{1+\frac{\xi}{\eta}} \end{pmatrix}$$

$$= \frac{\eta+\frac{\xi}{2}}{\xi^2+\xi+2\eta}(\mathbb{1}_2 - \varepsilon_l\sigma_x + h\varepsilon_c\sigma_z)$$

Therefore,

(i) for unpolarized electrons the transverse part of the virtual photon density matrix describes linear polarization in the scattering plane of degree $\varepsilon_l = \xi/(\xi+2\eta)$.

(ii) Longitudinally polarized electrons create in addition circular polarization of degree $h\varepsilon_c$ with $\varepsilon_c = 2\sqrt{\eta(\xi+\eta)}/(\xi+2\eta)$.

(iii) Density matrix for spin 1

Three basis states: $\{|1,\,0,\pm 1\rangle\} \rightsquigarrow \rho^{(1)}_{M'M}$ is a 3×3-matrix in spin-1 space. $\rho^{(1)}$ can be decomposed into 9 independent matrices, the unit matrix and 8 traceless hermitean matrices.

They can be chosen as irreducible tensors under rotations, i.e. three vector components $\widehat{\Omega}^{[1]}_M(1)$ and five tensor components $\widehat{\Omega}^{[2]}_M(1)$, defined by their reduced matrix elements

$$\langle 1\|\widehat{\Omega}^{[J]}(1)\|1\rangle = \sqrt{3(2J+1)}\,, \quad J = 1, 2\,.$$

$\rightsquigarrow$ $\rho^{(1)}$ is governed by eight independent parameters: 3 for vector polarization (P_{1M}) and 5 for tensor (P_{2M}).

$$\boxed{\rho^{(1)} = \frac{1}{3}\Big(\mathbb{1}_3 + \sum_{J=1,2}\sum_M (-)^M \widehat{\Omega}^{[J]}_M(1) P_{IM}\Big)}$$

For diagonal density matrix $\rho^{(1)}_{M'M} = \delta_{M'M} p_M$, where p_M probability for finding the particle in a state $|1M\rangle$ with respect to the z-axis as orientation axis.

$$P_{1M} = \delta_{M0}\sqrt{\frac{3}{2}}(p_1 - p_{-1}) = \delta_{M0} P_1^0$$

$$P_{2M} = \delta_{M0}\sqrt{\frac{1}{2}}(p_1 + p_{-1} - 2p_0) = \delta_{M0}\sqrt{\frac{1}{2}}(1 - 3p_0) = \delta_{M0} P_2^0 \,.$$

For arbitrary direction of orientation axis, the spin-1 density matrix is characterized by the spherical angles (θ, ϕ) of the orientation axis and two parameters, P_1^0 for vector and P_2^0 for tensor polarization. In this case

$$P_{JM}(\theta, \phi) = P_J^0\, D^J_{0M}(0, -\theta, -\phi) = P_J^0 e^{iM\phi} d^J_{M0}(\theta) \quad (I = 1, 2)$$

and

$$\rho^{(1)}_{m'm} = \frac{(-)^{1-m}}{\sqrt{3}} \sum_{J=0}^{2} P_J^0 \sum_{M=-J}^{J} \sqrt{2J+1} \begin{pmatrix} 1 & 1 & J \\ m & -m' & -M \end{pmatrix} e^{iM\phi} d^J_{M0}(\theta)$$

where for convenience I have introduced $P_0^0 = 1$.

(iv) Generalization to arbitrary spin I

$$\rho^{(I)} = \frac{1}{\widehat{I}^2}\left(\mathbb{1}_{2I+1} + \widehat{I}\sum_{J=1}^{2I}\sum_{M=-J}^{J}(-)^M \widehat{\Omega}^{[J]}_{-M}(I) P_{JM}\right)$$

with operators $\widehat{\Omega}^{[J]}(I)$ (with $\langle I||\widehat{\Omega}^{[J]}(I)||I\rangle = \widehat{I}\widehat{J}$, $\widehat{I} = \sqrt{2I+1}$)

Assume again an orientation axis at (θ, ϕ), for which $\rho^{(I)}$ becomes diagonal ($\rho^{(I)}_{m'm} = \delta_{m'm} p_m$) with $2I$ orientation parameters P^0_J ($P^0_0 = 1$).

$$\rightsquigarrow \quad \boxed{\rho^{(I)}_{m'm} = \frac{(-)^{I-m}}{\widehat{I}}\sum_{J=0}^{2I} P^0_J \sum_{M=-J}^{J} \widehat{J}\begin{pmatrix} I & I & J \\ m & -m' & -M \end{pmatrix} e^{iM\phi} d^J_{M0}(\theta)}$$

The orientation parameters P^0_J are related to the diagonal elements p_m by

$$P^0_J = \widehat{I}\widehat{J}\sum_m (-)^{I-m}\begin{pmatrix} I & I & J \\ m & -m & 0 \end{pmatrix} p_m$$

General type of polarization observable

The set of operators $\{\widehat{\Omega}^{[J]}_M(I);\ J=0,\ldots,2I+1\}$, introduced for the representation of the density matrix, serves also as a convenient basis for any polarization observable. (note $\widehat{\Omega}^{[0]}_0(I)=\mathbb{1}_{2I+1}$ and $\widehat{\Omega}^{[1]}_M(\frac{1}{2})=\sigma_M$)

Consider reaction $a+b\to c+d$ with spins $I_a,\ldots,I_d$ and reaction matrix $\widehat{T}$.

A convenient set of all possible polarization observables is given by

$$\mathcal{O}(J_a,M_a;J_b,M_b;J_c,M_c;J_d,M_d)= \mathbf{Tr}\Big(\widehat{T}^\dagger\widehat{\Omega}^{[J_c]}_{M_c}(I_c)\,\widehat{\Omega}^{[J_d]}_{M_d}(I_d)\,\widehat{T}\,\widehat{\Omega}^{[J_a]}_{M_a}(I_a)\,\widehat{\Omega}^{[J_b]}_{M_b}(I_b)\Big).$$

The trace refers to all spin degrees of initial and final particles.

For example, polarization of outgoing $s=1/2$ particle c is determined by

$$P_M(c)\propto \mathbf{Tr}\Big(\widehat{T}^\dagger\,\widehat{\Omega}^{[1_c]}_M(\frac{1}{2})\,\widehat{T}\Big)=\mathbf{Tr}\Big(\widehat{T}^\dagger\,\sigma_M(c)\,\widehat{T}\Big)\,.$$

Example (I): Deuteron photo disintegration

Differential cross section for partially polarized photon and oriented deuteron:

$$
\begin{aligned}
\frac{d\sigma(\theta)}{d\Omega} &= \mathbf{Tr}((\widehat{T}^{\gamma})^{\dagger}\widehat{T}^{\gamma}\,\rho^{(\gamma)}\rho^{(d)}) \\
&= \frac{d\sigma_0}{d\Omega}\Big(1 + P_l^{\gamma}\Sigma^l(\theta)\cos 2\phi \\
&\quad + \sum_{J=1,2} P_J^d \{ \sum_{M\le 0} (T_{JM}(\theta)\cos(M(\phi_d-\phi)-\delta_{J1}\frac{\pi}{2}) \\
&\quad + P_c^{\gamma} T_{JM}^c(\theta)\sin(M(\phi_d-\phi)+\delta_{J1}\frac{\pi}{2}))d^J_{M0}(\theta_d) \\
&\quad + P_l^{\gamma}\sum_{M=-J}^{J} T_{JM}^l(\theta)\cos(M(\phi_d-\phi)+2\phi-\delta_{J1}\frac{\pi}{2})d^J_{M0}(\theta_d)\}\Big)
\end{aligned}
$$

Defines various photon, target and photon-target asymmetries Σ^l, T_{JM}, T_{JM}^c, and T_{JM}^l.

Of particular interest is the vector target asymmetry for circularly polarized photons T_{10}^c, because it determines the GDH sum rule.

(i) PHOTON ASYMMETRY Σ^l for unoriented deuteron ($P_1^d = P_2^d = 0$) but linearly polarized photons perpendicular and parallel to reaction plane

$$\frac{d\sigma_\perp}{d\Omega} = \frac{d\sigma_0}{d\Omega}(1 - P_l^\gamma \Sigma^l(\theta)) \quad \text{for} \quad \phi = \pi/2 \quad \text{(perpendicular polarization)}$$

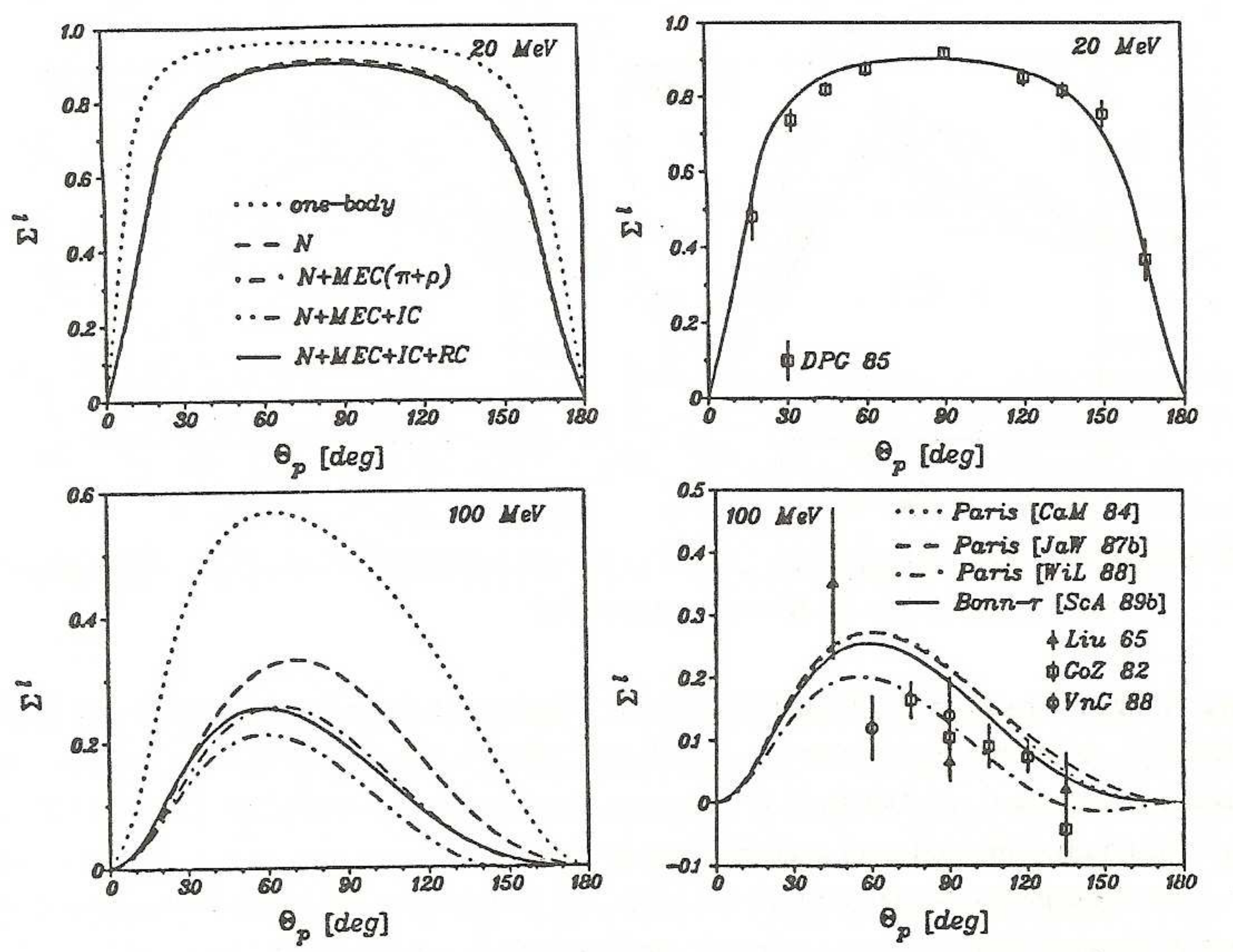
$$\frac{d\sigma_\parallel}{d\Omega} = \frac{d\sigma_0}{d\Omega}(1 + P_l^\gamma \Sigma^l(\theta)) \quad \text{for} \quad \phi = 0 \quad \text{(parallel polarization)}$$

Thus measuring the cross sections for parallel and perpendicular linear polarization allows one to determine the photon asymmetry

$$\boxed{\Sigma^l(\theta) = \frac{1}{P_l^\gamma} \frac{d\sigma_\parallel - d\sigma_\perp}{d\sigma_\parallel + d\sigma_\perp}}$$

Σ^l is sensitive to the interference of various multipoles.

PHOTON ASYMMETRY Σ^l FOR DEUTERON PHOTO DISINTEGRATION



At 20 and 100 MeV with contributions from meson exchange currents (MEC), isobar configurations (IC), relativistic effects (RC) (left panels), and comparison to experiments (right panels).

(ii) TARGET ASYMMETRIES T_{JM} for unpolarized photons ($P_l^\gamma = P_c^\gamma = 0$) but oriented deuterons:

By proper choices of deuteron orientation axis perpendicular to the photon momentum ($\theta_d = 90^\circ$), one can extract the asymmetries.

$$T_{11} = \frac{1}{\sqrt{2}P_1^d}\frac{d\sigma^\uparrow - d\sigma^\downarrow}{d\sigma_0}$$

$$T_{20} = \frac{1}{P_2^d}\left(2 - \frac{d\sigma^\odot + \frac{1}{2}(d\sigma^\uparrow + d\sigma^\downarrow)}{d\sigma_0}\right)$$

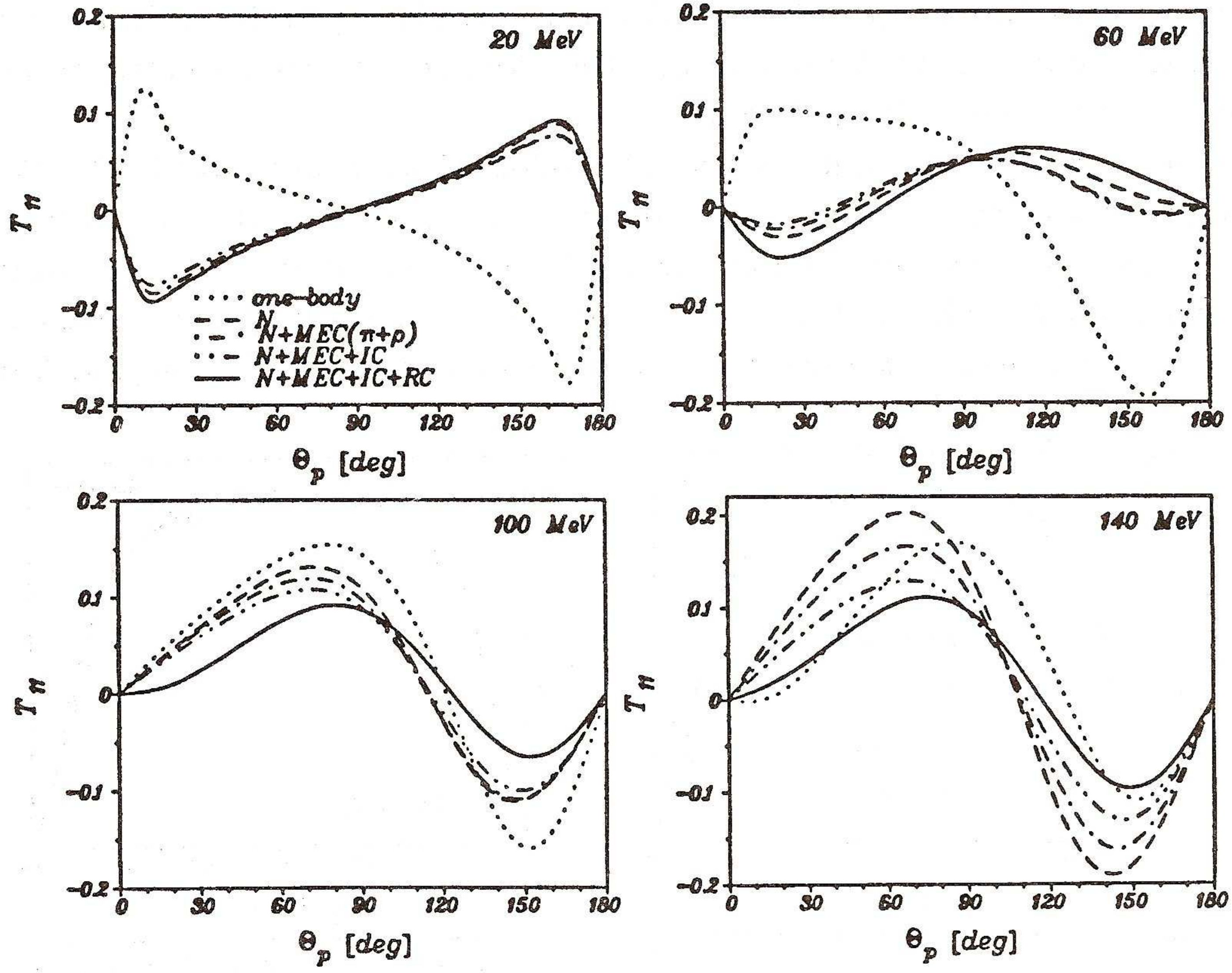
$$T_{22} = \frac{\sqrt{2}}{\sqrt{3}P_2^d}\frac{d\sigma^\odot - \frac{1}{2}(d\sigma^\uparrow + d\sigma^\downarrow)}{d\sigma_0}$$

with different orientation angle ϕ_d:

$$\uparrow: \ \phi_d = 90^\circ \quad \odot: \ \phi_d = 0^\circ \quad \downarrow: \ \phi_d = -90^\circ$$

perpendicular to photon momentum in the reaction plane ($\odot$) and perpendicular to the reaction plane ($\uparrow$, $\downarrow$).

Vector target asymmetry T_{11} for deuteron photo disintegration

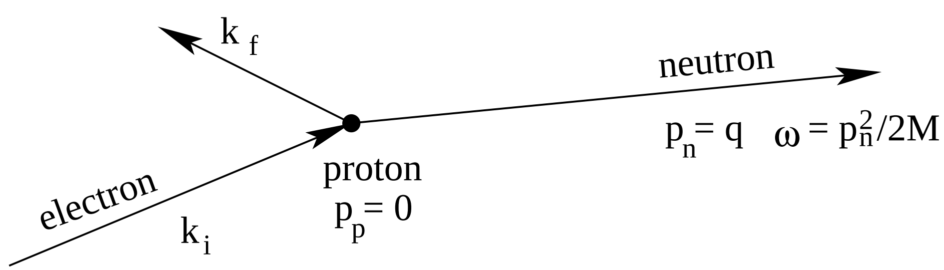


At various energies with contributions from meson exchange currents (MEC), isobar configurations (IC), relativistic effects (RC).

Example (II): Deuteron electro disintegration and neutron electric form factor

- For e-n scattering the unpolarized cross section is dominated by G_M, while G_E gives a tiny contribution only: $\longrightarrow$ would be difficult to measure.
- For polarized electron and initial polarized neutron the target asymmetry $A_{n,x}$ is linear in the ratio G_E/G_M and thus more sensitive to G_E.
- The same holds for an unpolarized initial neutron and the final neutron polarization component P_x.

In the absence of neutron targets of sufficient density, light nuclei are used as effective neutron targets, and one studies, e.g., inelastic electron scattering $d(e, e'n)p$ in quasi-free kinematics, where the emitted neutron takes up energy and momentum transfer such that the recoiling proton remains at rest in the laboratory:



- For quasi-free kinematics the cross section is roughly given by the cross section on the neutron, thus allowing the determination of G_M.
- However, since the electric form factor G_E of the neutron is much smaller than its magnetic one, its contribution is buried under competing binding and final state interaction effects.

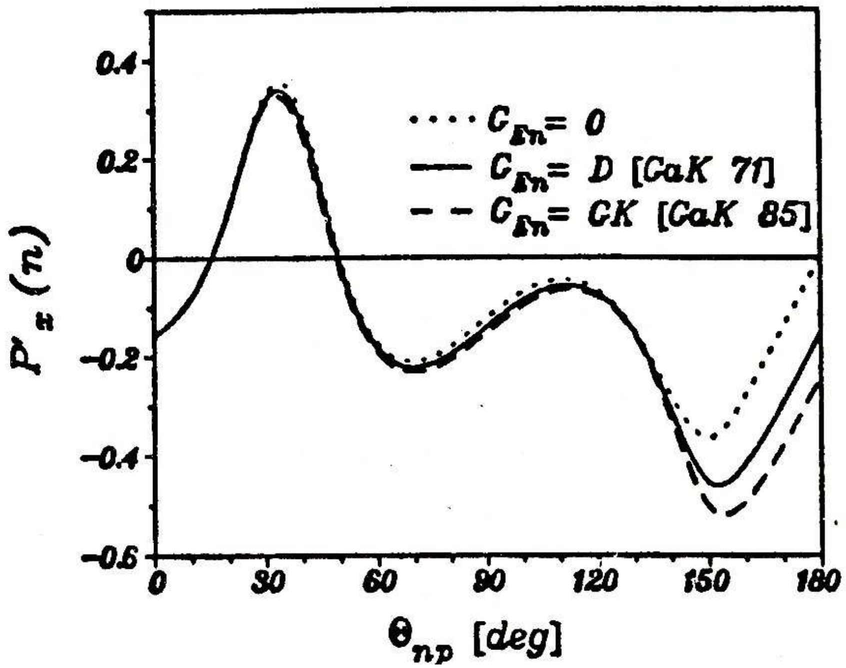
One needs a vector polarized deuteron target for having an initially polarized neutron, because

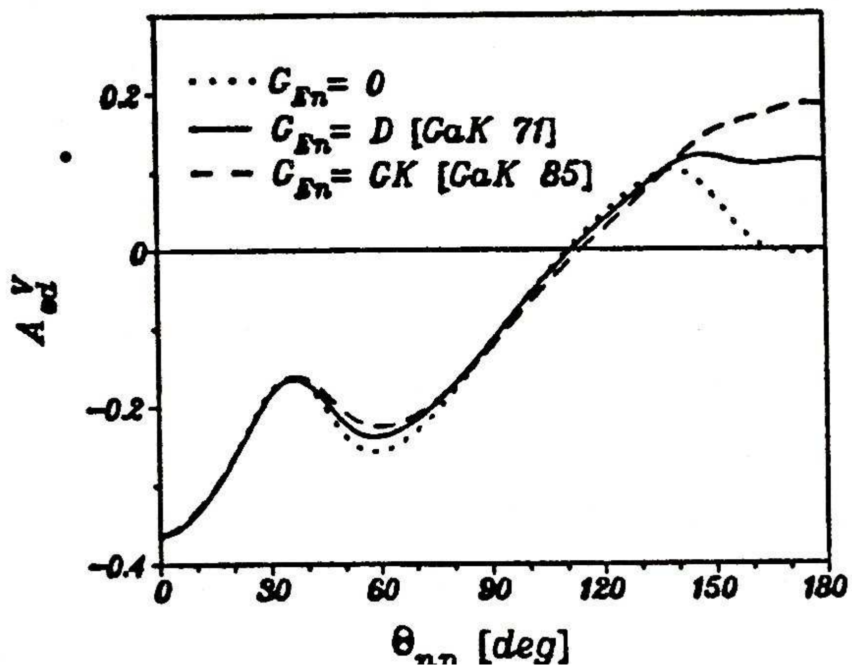
(i) the bound neutron is almost completly polarized along the orientation axis, i.e. $P_n = (p_1 - p_{-1})(1 - \frac{3}{2}P_D)$, with P_D for the D-wave probability.

(ii) For quai-free kinematics the deuteron target vector asymmetry is proportional to G_E/G_M.

Instead of using an initially polarized neutron, one can measure the polarization of the outgoing neutron in $d(\overrightarrow{e}, e' \overrightarrow{n})p$ which again is proportional to G_E/G_M.

TWO EXAMPLES FOR QUASI-FREE KINEMATICS ($\mathbf{q}^2 = 12$ fm^{-2})





Polarization component P'_x
of final neutron.
Dotted $G_{En} = 0$

Vector asymmetry A^V_{ed}
at $\theta_d = 90°, \phi_d = 0$.
Dotted $G_{En} = 0$

Different models for G_E (dipol and GARI-KRÜMPELMANN).

3. Electromagnetic Sum Rules

The interest in sum rules arises from the fact, that by summing an observable over the whole spectrum of excited states, one often arrives at a simple interpretation in terms of ground state properties without explicit knowledge of the complicated excitation spectrum.

A well known example is the Thomas-Reiche-Kuhn sum rule in atomic as well as in nuclear physics. Asuming unretarded $E1$ contributions and using completeness one finds for the integrated total photoabsorption cross section

$$\int dE\, \sigma_{tot}^{E1}(E) = 2\pi^2 e^2 \frac{NZ}{AM}(1+\kappa)\,,$$

where $\kappa = (MA/NZ)\langle 0|[D_z[V, D_z]]|0\rangle$ is measure of exchange effects in the NN-interaction and electromagnetic current.

This sum rule was later generalized by GELL-MANN, GOLDBERGER, THIRRING applying dispersion relations.

Another example is the energy weighted sum rule related to the electric polarizability α_E

$$\int \frac{dE}{E^2}\,\sigma_{tot}^{E1}(E) \propto \alpha_E\,.$$

Here I would like to concentrate on the GERASIMOV-DRELL-HEARN sum rule (GDH):

$$\int_0^\infty \frac{dk}{k}\Big(\underbrace{\sigma^P(k)-\sigma^A(k)}_{\text{spin asymmetry of total absorption cross section}}\Big) = 4\,\pi^2\kappa^2\frac{e^2}{M_t^2}\,I\,,$$

with I as ground state spin, M_t as mass and κ as anomalous magnetic moment. Thus the GDH sum rule links a ground state property, κ, to the whole internal excitation spectrum, i.e., to the total integrated and energy weighted spin asymmetry of the absorption cross section.

GDH sum rule belongs to a larger class of photoabsorption sum rules based on dispersion relations of the forward elastic COMPTON scattering amplitude. They relate the polarization observables of the total photoabsorption cross section to ground state properties.

The general form of the total absorption cross section with beam and target polarization (target spin I) is (H.A., Phys.Rev. 171(1968)1212).

$$\sigma_{\rm tot}(k,\rho^\gamma,\rho^t)=\frac{1}{2}\sum_{J=0}^{2I} P_J^t\,[(1+(-)^J)\,\sigma_J^{11}(k)+(1-(-)^J)\,P_c^\gamma\,\sigma_J^{11}(k)\,P_J(\cos\theta_t)$$

$$+(1+(-)^J)P_l^\gamma\,\sigma_J^{-11}(k)\,d_{20}^J(\theta_t)\,\cos(2\phi_t)]\,,$$

where P_l^γ and P_c^γ denote the degree of linear and circular photon polarization, respectively, and P_J^t and (θ_t,ϕ_t) the target polarization parameters.

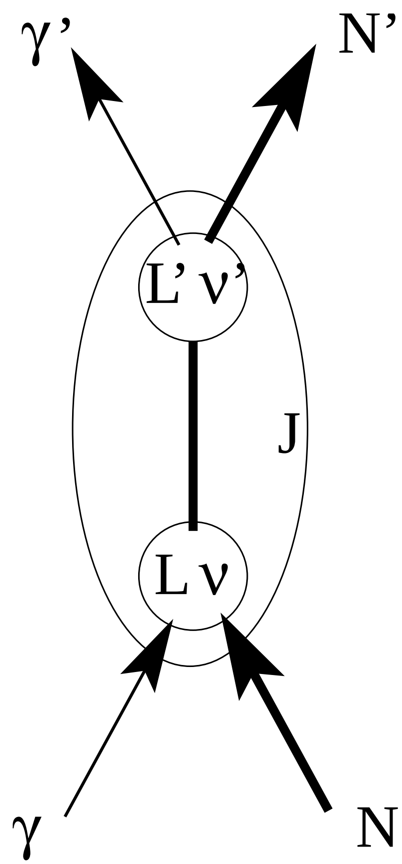
The $\sigma_J^{\lambda'\lambda}$ $(\lambda',\lambda=\pm 1)$ are related to the forward COMPTON scattering amplitude via the optical theorem

$$\sigma_J^{\lambda'\lambda}(k)=\frac{4\pi}{k}\,\mathcal{I}m\,T^J_{\lambda'\lambda}(k)\,,$$

where $T^J_{\lambda'\lambda}$ is a partial wave contribution to the elastic forward scattering amplitude

$$T_{\lambda' M', \lambda M}(k) = (-)^{I-M}\, \hat{I} \sum_{J=0}^{2I} \hat{J} \begin{pmatrix} I & J & I \\ -M' & \lambda - \lambda' & M \end{pmatrix} T^J_{\lambda'\lambda}(k)\,.$$

It can be expressed in terms of generalized e.m. polarizabilities $P_J^{L'L\lambda'\lambda}(k)$



$$T^J_{\lambda'\lambda}(k) = \frac{\hat{J}}{\hat{I}} \sum_{L'L} (-)^{L'+L} \begin{pmatrix} L & L' & J \\ \lambda & -\lambda' & \lambda' - \lambda \end{pmatrix} P_J^{L'L\lambda'\lambda}(k)\,.$$

with

$$P_J^{L'L\lambda'\lambda}(k) = \sum_{\nu',\nu=0,1} \lambda'^{\nu'}\, \lambda^{\nu} P_J(M^{\nu'}L', M^{\nu}L; k)\,,$$

where $M^0 = E$ (electric) and $M^1 = M$ (magnetic) multipole.

The $T^J_{\lambda'\lambda}$ are also related to the expansion of the scattering amplitude in terms of the complete set of operators $\widehat{\Omega}^{[J]}$ introduced for the general density matrix:

$$T_{\lambda' M',\lambda M}(k) = \sum_{J=0}^{2I} (-)^{-\lambda'+\lambda} \langle IM' | \widehat{\Omega}^{[J]}_{\lambda-\lambda'} | IM \rangle T^J_{\lambda'\lambda}(k)\,,$$

Specifically one has

$$\sigma^{11}_J = \frac{\hat{J}}{\hat{I}} \sum_M (-)^{I-M} \begin{pmatrix} I & J & I \\ -M & 0 & M \end{pmatrix} \sigma_{1M}\,,$$

where σ_{1M} denotes the absorption cross section of a photon with helicity $\lambda = 1$ at a target with definite spin projection M on the photon momentum.

Corresponding expressions hold for σ^{-11}_J with respect to the absorption of linearly polarized photons.

In detail one finds for $J = 0, 1, 2$

$$\sigma_0^{11} = \frac{1}{\hat{I}^2} \sum_M \sigma_{1M}\ , \text{ unpolarized total cross section,}$$

$$\sigma_1^{11} = \frac{\sqrt{3}}{\hat{I}^2\,\sqrt{I(I+1)}} \sum_M M\,\sigma_{1M}\ ,$$

$$\sigma_2^{11} = \frac{\sqrt{5}}{\hat{I}^2\,\sqrt{I(I+1)}} \sum_M \frac{(3M^2 - I(I+1))}{(\widehat{I-1})\,(\widehat{I+1})}\,\sigma_{1M}\ ,$$

And for the spin asymmetry ($\sigma^{P/A} = \sigma_{1,\pm I}$)

$$\sigma^P - \sigma^A = \hat{I} \sum_J \hat{J}\,(1 - (-)^J) \begin{pmatrix} I & I & J \\ I & -I & 0 \end{pmatrix} \sigma_J^{11}\ .$$

Then sum rules are obtained by use of dispersion relations in conjunction with crossing symmetry

$$(T^J_{\lambda'\lambda}(-k))^* = (-)^J\, T^J_{\lambda'\lambda}(k)\,.$$

Assuming for J = even a once-subtracted dispersion relation for $T^J_{\lambda'\lambda}(k)$:

$$\mathcal{R}e\left(T^J_{\lambda'\lambda}(k) - T^J_{\lambda'\lambda}(0)\right) = \frac{2k^2}{\pi}\,\mathcal{P}\int_0^\infty \frac{dk'}{k'}\,\frac{\mathcal{I}m\,T^J_{\lambda'\lambda}(k')}{k'^2-k^2}$$

$$= \frac{k^2}{2\pi^2}\,\mathcal{P}\int_0^\infty dk'\,\frac{\sigma_J^{\lambda'\lambda}(k')}{k'^2-k^2}\,,$$

while for J = odd an unsubtracted dispersion relation may be used:

$$\mathcal{R}e\,T^J_{\lambda'\lambda}(k) = \frac{2k}{\pi}\,\mathcal{P}\int_0^\infty dk'\,\frac{\mathcal{I}m\,T^J_{\lambda'\lambda}(k')}{k'^2-k^2}$$

$$= \frac{k}{2\pi^2}\,\mathcal{P}\int_0^\infty dk'\,k'\,\frac{\sigma_J^{\lambda'\lambda}(k')}{k'^2-k^2}\,.$$

With the help of a power series expansion for the partial amplitude

$$T^J_{\lambda'\lambda}(k) = \begin{cases} \sum_{\nu=0}^{\infty} t_\nu^{\lambda'\lambda,\,J}\, k^\nu & \text{for } J \text{ even,} \\ \sum_{\nu=0}^{\infty} t_\nu^{\lambda'\lambda,\,J}\, k^{\nu+1} & \text{for } J \text{ odd,} \end{cases}$$

where the coefficients are given as ground state expectation values of certain operators, and an analogous one for the dispersion integral, one finds by comparison of the coefficients a general class of sum rules

$$t_\nu^{\lambda'\lambda,\,J} = \begin{cases} \frac{1}{2\pi^2}\,\int_0^\infty dk'\, \frac{\sigma_J^{\lambda'\lambda}(k')}{k'^{2\nu}} & \text{for } J \text{ even and } \nu = 1, 2, \dots, \\ \frac{1}{2\pi^2}\,\int_0^\infty dk'\, \frac{\sigma_J^{\lambda'\lambda}(k')}{k'^{2\nu+1}} & \text{for } J \text{ odd and } \nu = 0, 1, \dots, \end{cases}$$

one of which is the GDH, namely for $J = 1$ and $\nu = 0$.

This is achieved in conjunction with the low-energy expansion of the Compton amplitude

$$T_{\lambda M,\lambda M}(k) = -e^2\,\frac{Q^2}{M_t} + \lambda\,\kappa^2\,\frac{e^2}{M_t^2}\,\langle|S_z|\rangle_{IM}\,k + \mathcal{O}(k^2)\,.$$

This low energy expansion gives specifically for the partial amplitudes

$$T^J_{\lambda'\lambda}(k)=-\delta_{\lambda'\lambda}\,\delta_{J0}\,e^2\,\frac{Q^2}{M_t}+\mathcal{O}(k^2)\,,$$

for $J=$ even, and

$$T^J_{\lambda'\lambda}(k)=k\Big(\delta_{\lambda'\lambda}\,\delta_{J1}\,\lambda\,\kappa^2\,\frac{e^2}{M_t^2}\,\frac{\sqrt{I(I+1)}}{\sqrt{3}}+\mathcal{O}(k^2)\Big)\,,$$

for $J=$ odd.

The latter yields the GDH sum rule in the form

$$\begin{aligned}4\,\pi^2\,\frac{\kappa^2\,e^2}{M_t^2}\,I&=\frac{2\sqrt{3I}}{\sqrt{I+1}}\int_0^\infty\frac{dk'}{k'}\,\sigma_1^{11}(k')\\&=\int_0^\infty\frac{dk'}{k'}\Big(\sigma^P(k')-\sigma^A(k')\Big)\,.\end{aligned}$$

The latter results from the fact that the integrals over the higher order contributions σ_J^{11} for $J>1$ vanish.

Example: GDH sum rule for the deuteron

Deuteron is isoscalar and its anomalous magnetic moment is very small $\kappa_d = -0.143$, resulting in $\rightarrow$ $I_d^{GDH} = 0.65\,\mu$b.

This is in contrast to the nucleon GDH-values $I_p^{GDH} = 205\,\mu$b for proton and $I_n^{GDH} = 233\,\mu$b for neutron.

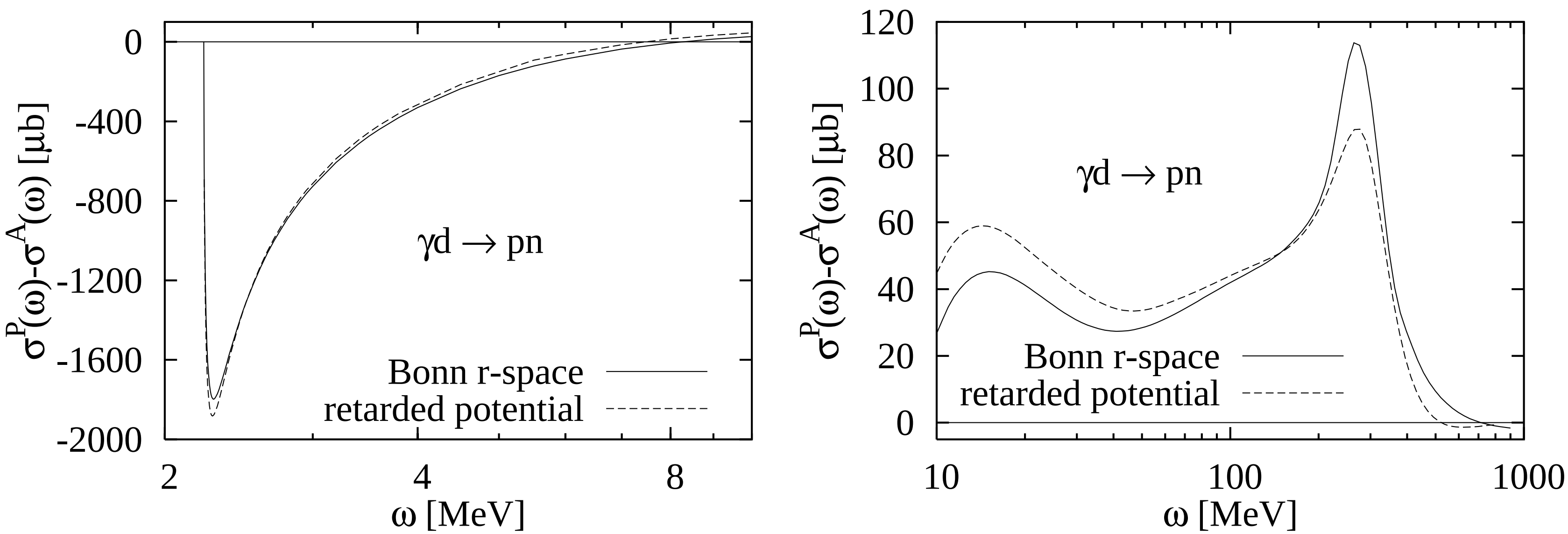
Photo absorptive processes on the deuteron:
(i) photodisintegration $\gamma + d \rightarrow n + p$,
(ii) meson production.

Neglecting interference effects, i.e. considering the deuteron as an incoherent n-p target results in an estimate of a large positive GDH contribution from meson production:

$$I_p^{GDH} + I_n^{GDH} = 438\,\mu\text{b}$$

It has to be canceled by the other absorptive reaction, namely photodisintegration.

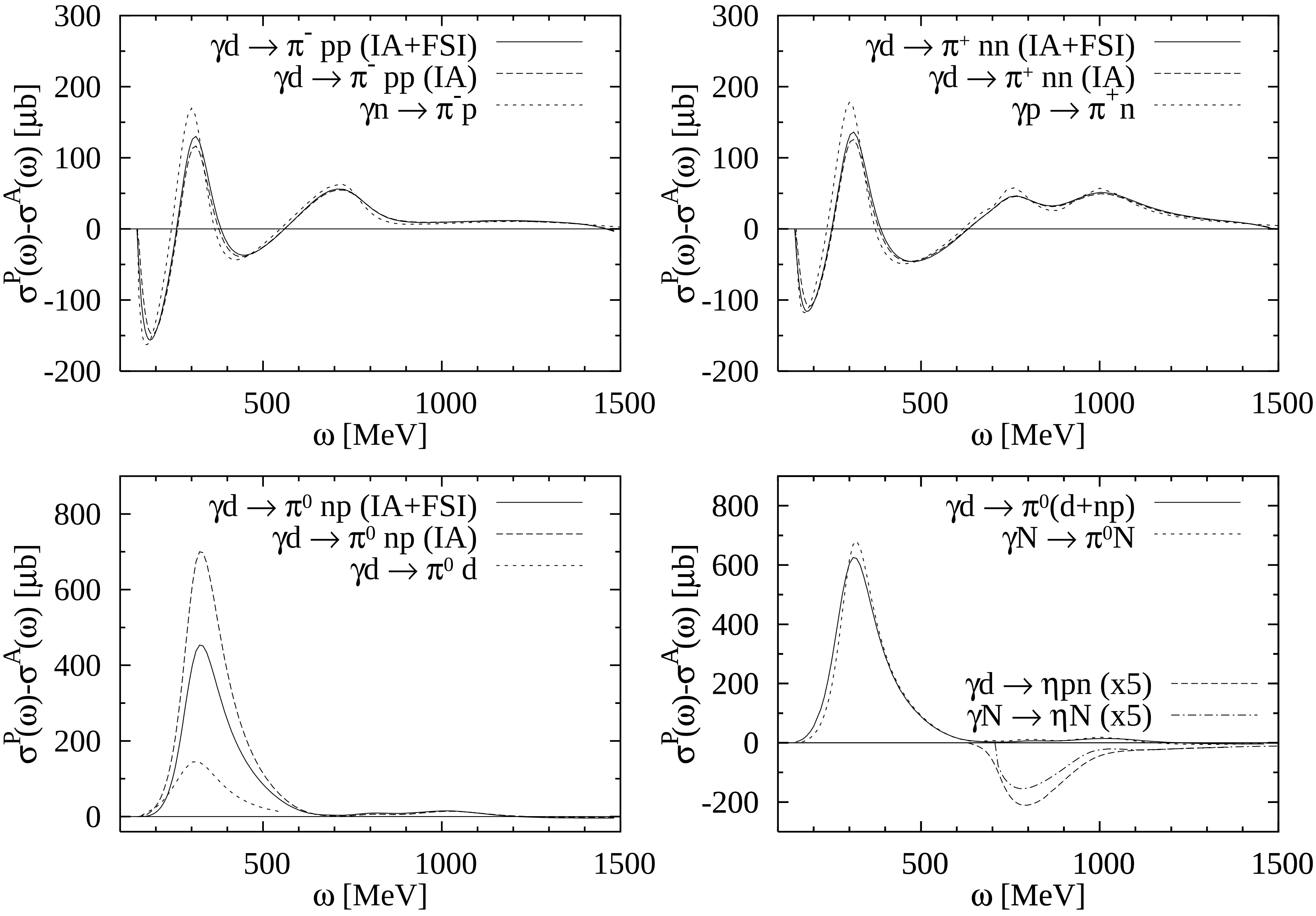
Spin asymmetry for deuteron photodisintegration



Near threshold spin asymmetry is dominated by isovector M1 transition to $^1S_0$-state → huge negative spin asymmetry of order [mb]. The positive asymmetry at higher energies is much less important for the GDH-integral because of the energy weighting.

As a result a large negative GDH contribution of $-381\,\mu$b is found. Furthermore, relativistic effects reduce quite sizable the spin asymmetry at low energies and thus are quite important.

Spin asymmetry for pion production on deuteron and nucleon



In addition, two-pion and eta meson production give substantial contributions to the GDH-integral.

Summary of contributions of various channels to finite GDH integral up to 0.9 GeV for $\gamma d \to np$ and 1.5 GeV for $\gamma d \to \pi(d + NN)$, $\gamma N \to \pi N$ and $\gamma N \to \pi\pi N$ in μb.

	$\int$ np	$\int \pi$	$\int \pi\pi$	$\int \eta$	$\Sigma \int$	GDH
n		138.95	55.06	-5.77	188.24	233.16
p		176.38	51.24	-8.77	218.85	204.78
d	-381.52	263.44	104.13	-13.95	-27.90	0.65

The present value of the GDH-integral is quite a success for the theory if one takes into account the large cancellations (small difference of large numbers). The deviation from the sum rule value might reflect the theoretical uncertainties and neglected channels as well, thus leaving room for additional contributions like, e.g, three meson production.

4. Conclusions and Outlook

The main conclusions are:

- Polarization observables provide us with much more detailed information on a reaction than just unpolarized total and differential cross sections.
- They serve as more stringent tests for theoretical models and are in general more sensitive to small but interesting effects like, e.g. parity violation.
- Sum rules provide links between ground state properties and the whole excitation spectrum.

Future experimental and theoretical efforts should be devoted to

- The development of more intense polarized beams and targets as well as highly efficient polarimeters.
- Experimental studies of various polarization observables of lightes nuclei (d and ^{3}He) over a large range of energy and momentum transfers.
- Theoretical predictions for preferentially a complete set of polarization observables also over a large range of energy and momentum transfers.