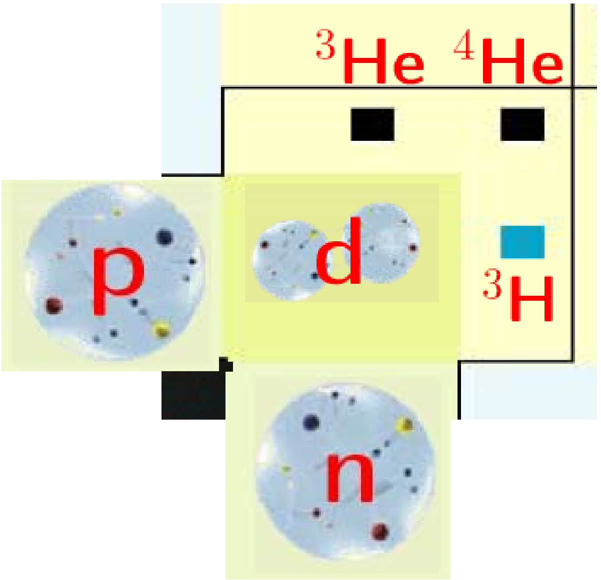
Few-Body Calculations in Three Dimensions

Ch. Elster

Supported by: U.S. DOE, OSC, NERSC

n+d & p+d scattering



Faddeev Approach

Los Alamos – Iowa

Mainz – Bonn - Dubna

Bochum – Cracow

Lisbon - Hannover

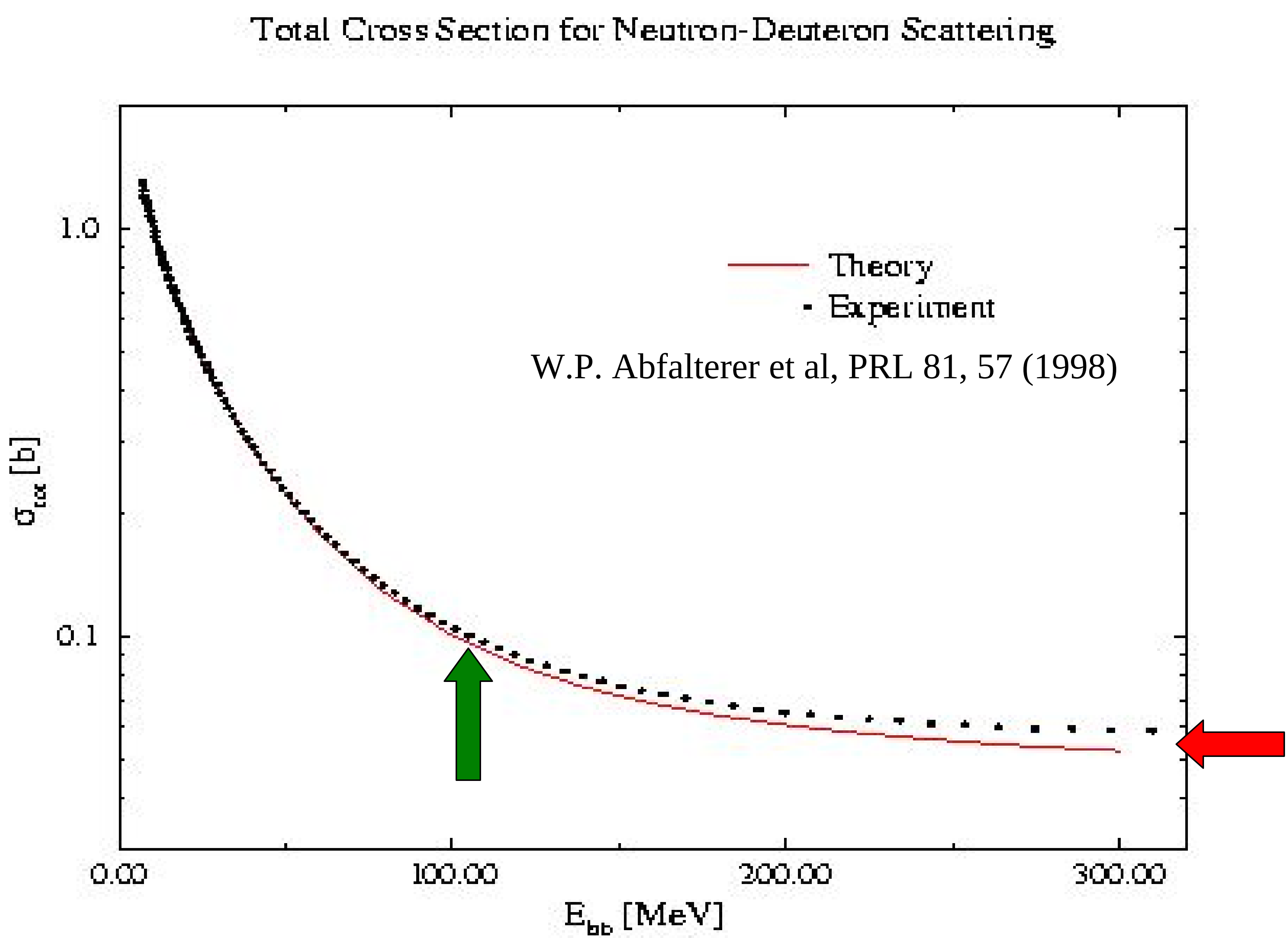
Variational Approach & HH

Pisa

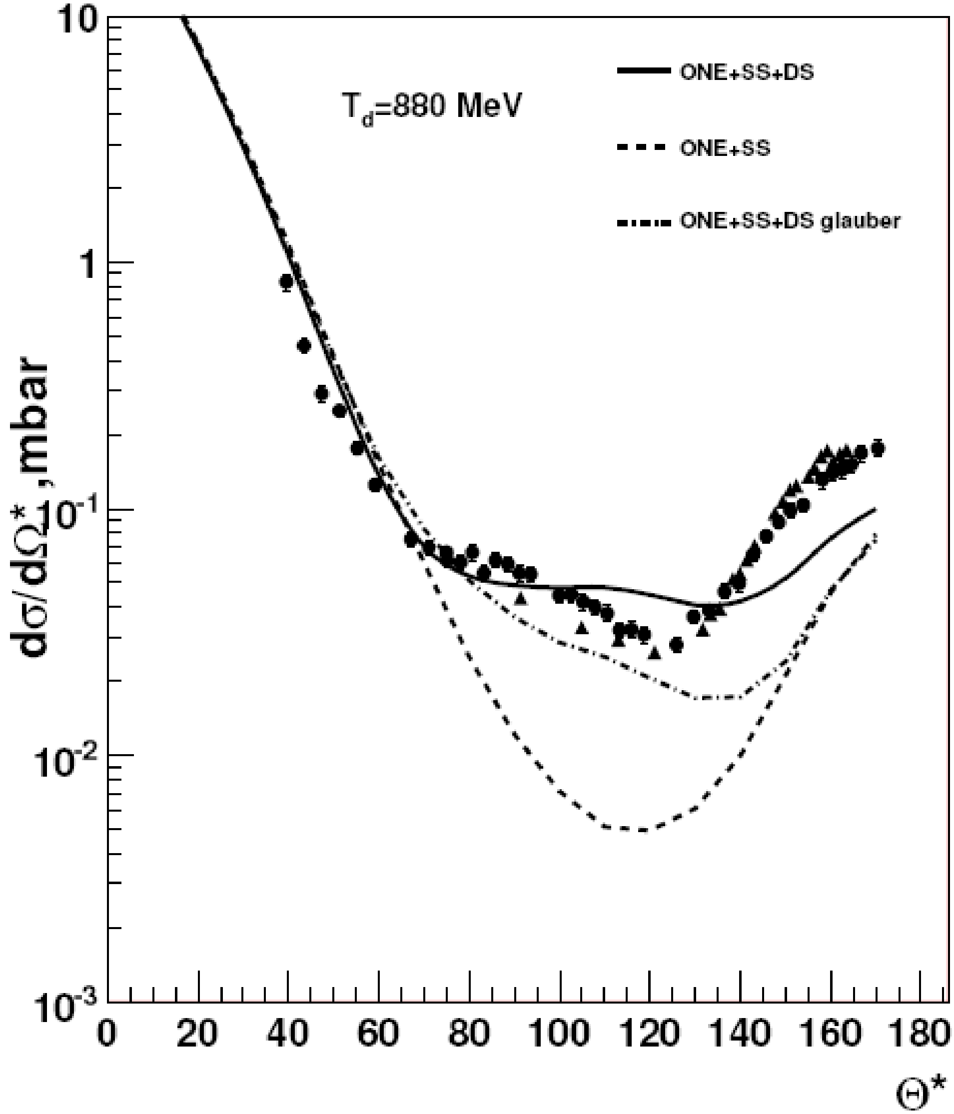
Challenges in 3N Physics

- **Test of nuclear forces in the simplest nuclear environment** (over a large energy range!)
  - Two-body forces
  - Genuine three-body forces

- **Reaction mechanisms**
  - Examples: deuteron breakup, (p,n) charge exchange, exclusive breakup (specific configurations) …
  - Higher Energy: Lorentz vs. Galilean Invariance
  - Check commonly used approximations (e.g. Glauber approach)


Total Cross Section for Neutron-Deuteron Scattering
Theory
Experiment
W.P. Abfalterer et al, PRL 81, 57 (1998)
1.0
0.1
σtot [b]
0.00
100.00
200.00
300.00
Elab [MeV]

Example: N.B. Ladygina: arXiv:0906.1910



Direct + Rescattering diagrams explicitly calculated.

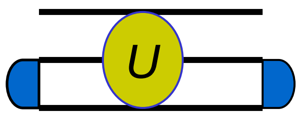
Relativistic formulation.

Does not appear converged

Three-Body Scattering - General

- Transition operator for elastic scattering

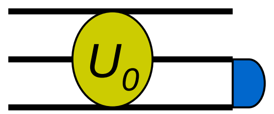
$$U = PG_0^{-1} + PT$$



- Transition operator for breakup scattering

$$U_0 = (1 + P)T$$

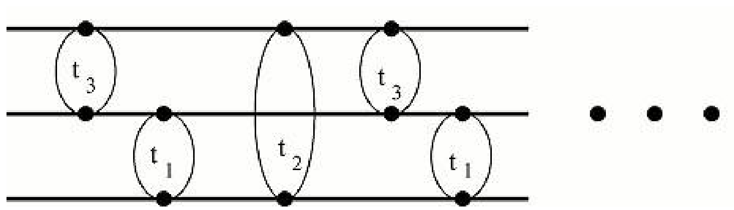
$$T = tP + tG_0PT$$



- Faddeev equation (Multiple Scattering Series)

$$T = tP + tG_0PtP + \cdots$$

1st Order in tP



$t = v + vg_0t$ =: NN t-matrix

$P = P_{12}P_{23} + P_{13}P_{23} \equiv$ Permutation Operator

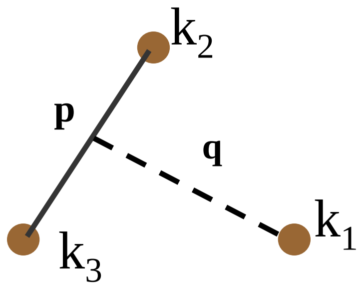
3-Body Transition Amplitude

$$T\left|\mathrm{q}_0\varphi_d\right\rangle = tP\left|\mathrm{q}_0\varphi_d\right\rangle + tG_0PT\left|\mathrm{q}_0\varphi_d\right\rangle$$

$$\mathrm{p} = \frac{1}{2}\left(\mathrm{k}_2 - \mathrm{k}_3\right)$$

$$\mathrm{q} = \frac{2}{3}\left(\mathrm{k}_1 - \frac{1}{2}(\mathrm{k}_2 + \mathrm{k}_3)\right)$$

non-relativistic relation



The Faddeev Equation in momentum space by using Jacobi Variables

$$\left\langle \mathrm{pq}\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle = \varphi_d\left(\mathrm{q}+\tfrac{1}{2}\mathrm{q}_0\right)\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}_0, E-\tfrac{3}{4m}q^2\right)$$

$$+\int d^3\mathrm{q}''\frac{\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}'', E-\tfrac{3}{4m}q^2\right)}{E-\tfrac{1}{m}\left(q^2+q''^2+\mathrm{q}\cdot\mathrm{q}''\right)+i\varepsilon}\frac{\left\langle \mathrm{q}+\tfrac{1}{2}\mathrm{q}'',\mathrm{q}''\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle}{E-\tfrac{3}{4m}q''^2-E_d+i\varepsilon}$$

$\hat{t}_s \equiv$ symmetrized 2-body t-matrix

Computational Challenge:

3N and 4N systems:

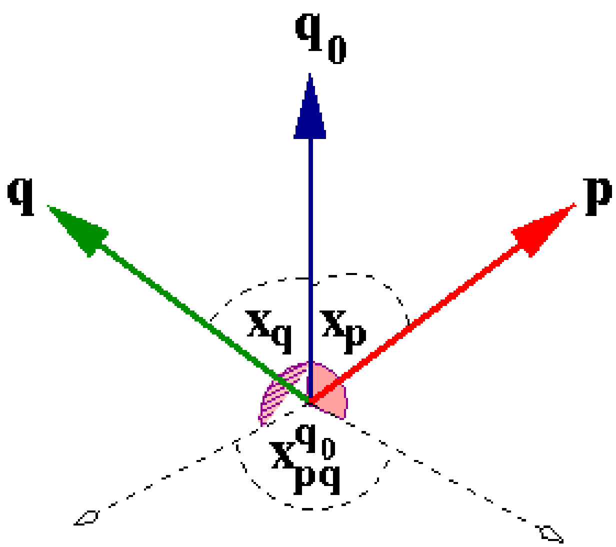
- standard treatment based on pw projected momentum space successful (3N scattering up to ≈250 MeV) but rather tedious
- 2N: j_{max}=5,   3N: J_{max}=25/2 → 200 `channels'
- Computational maximum :
- 2N: j_{max}=7,   3N: J_{max}=31/2

⇒ **Suggested approach:**

⇒ **NO partial wave decomposition of basis states**

Variables for 3D Calculation

3 distinct vectors in the problem: $\mathbf{q_0}$ $\mathbf{q}$ $\mathbf{p}$



5 independent variables:

$$p = |\mathrm{p}| \ , \ q = |\mathrm{q}|$$

$$x_p = \hat{\mathrm{p}} \cdot \hat{\mathrm{q}}_0 \ , x_q = \hat{\mathrm{q}} \cdot \hat{\mathrm{q}}_0$$

$$x_{pq}^{q_0} = (\mathrm{q}_0 \times \mathrm{q}) \cdot (\mathrm{q}_0 \times \mathrm{p})$$

q system : **z || q**

q_0 system : **z || $\mathbf{q_0}$**

Variables invariant under rotation:

freedom to choose coordinate system for numerical calculation

3D Integral Equation in 5 Variables

Solved by Padé Summation

$$
\begin{aligned}
&\langle p, x_p, x_{pq}^{q_0}, x_q, q|\hat{T}|q_0\varphi_d\rangle \\
= \;& \varphi_d\left(\sqrt{q^2+\frac{1}{4}q_0^2+qq_0x_q}\right) \\
\times \;& \hat{t}_s\left(p, \sqrt{\frac{1}{4}q^2+q_0^2+qq_0x_q}, \frac{\frac{1}{2}qy_{pq}+q_0x_p}{\sqrt{\frac{1}{4}q^2+q_0^2+qq_0x_q}}; E-\frac{3}{4m}q^2\right) \\
+ \;& \int_0^\infty dq'' q''^2 \int_{-1}^{+1} dx'' \int_0^{2\pi} d\varphi'' \frac{1}{E-\frac{1}{m}(q^2+qq''x''+q''^2)+i\varepsilon} \\
\times \;& \hat{t}_s\left(p, \sqrt{\frac{1}{4}q^2+q''^2+qq''x''}, \frac{\frac{1}{2}qy_{pq}+q''y_{pq''}}{\sqrt{\frac{1}{4}q^2+q''^2+qq''x''}}; E-\frac{3}{4m}q^2\right) \\
\times \;& \frac{\left\langle \sqrt{q^2+\frac{1}{4}q''^2+qq''x''}, \frac{qx_q+\frac{1}{2}q''y_{q_0q''}}{\sqrt{q^2+\frac{1}{4}q''^2+qq''x''}}, \frac{\frac{qx''+\frac{1}{2}q''}{\sqrt{q^2+\frac{1}{4}q''^2+qq''x''}}-x_{\pi_p}x_{\pi_q}}{\sqrt{1-x_{\pi_p}^2}\sqrt{1-x_{\pi_q}^2}}, y_{q_0q''}, q''|\hat{T}|q_0\varphi_d\right\rangle}{E-\frac{3}{4m}q''^2-E_d+i\varepsilon}
\end{aligned}
$$

Moving Singularities:

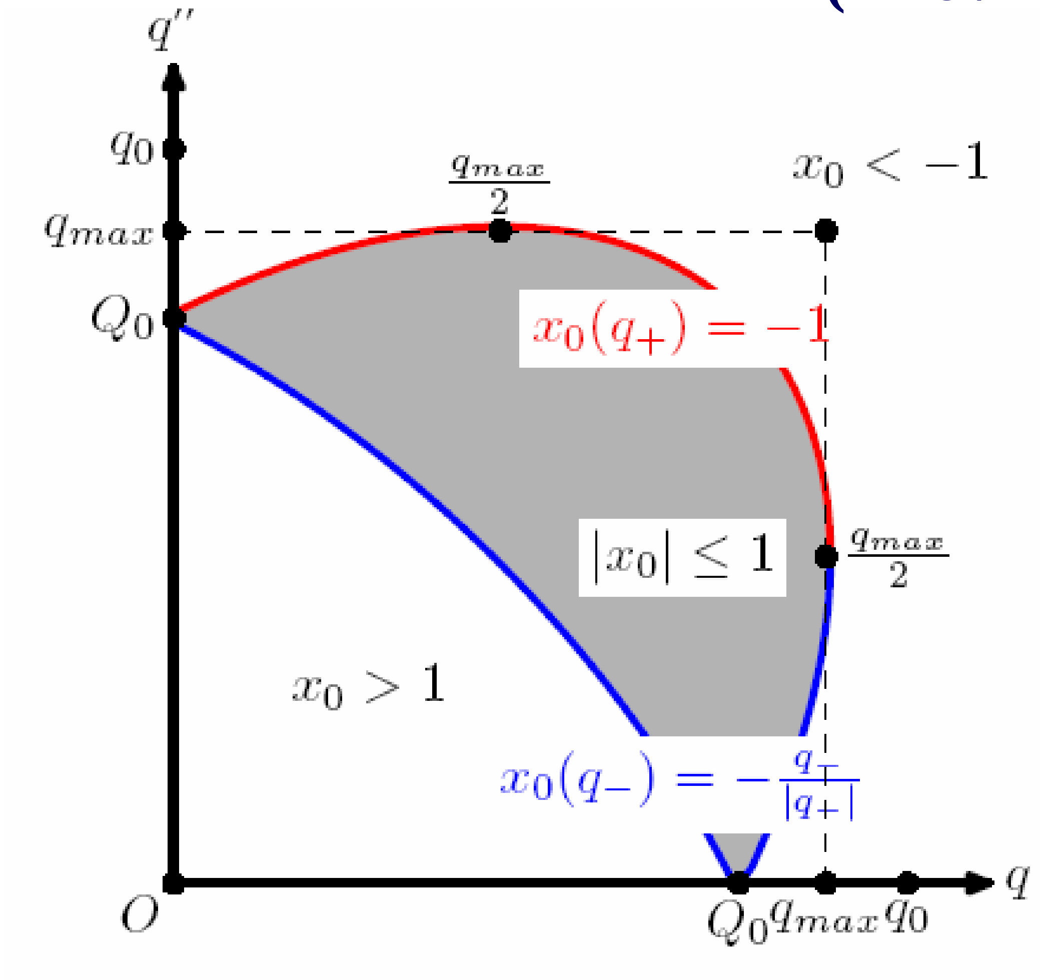
Position depends on

q, q”, x”

Fixed deuteron pole

Singularities of 3N Propagator

("moving" singularities)



PhD thesis H. Liu, OU

New approach for treating propagator singularities :

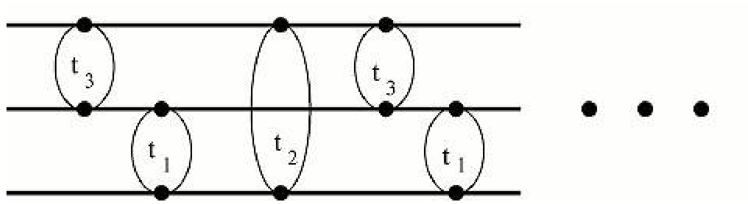
Elster, Glöckle, Witala, FBS **45**, 1 (2009)

Faddeev multiple scattering series

$$T = tP + tG_0PT$$

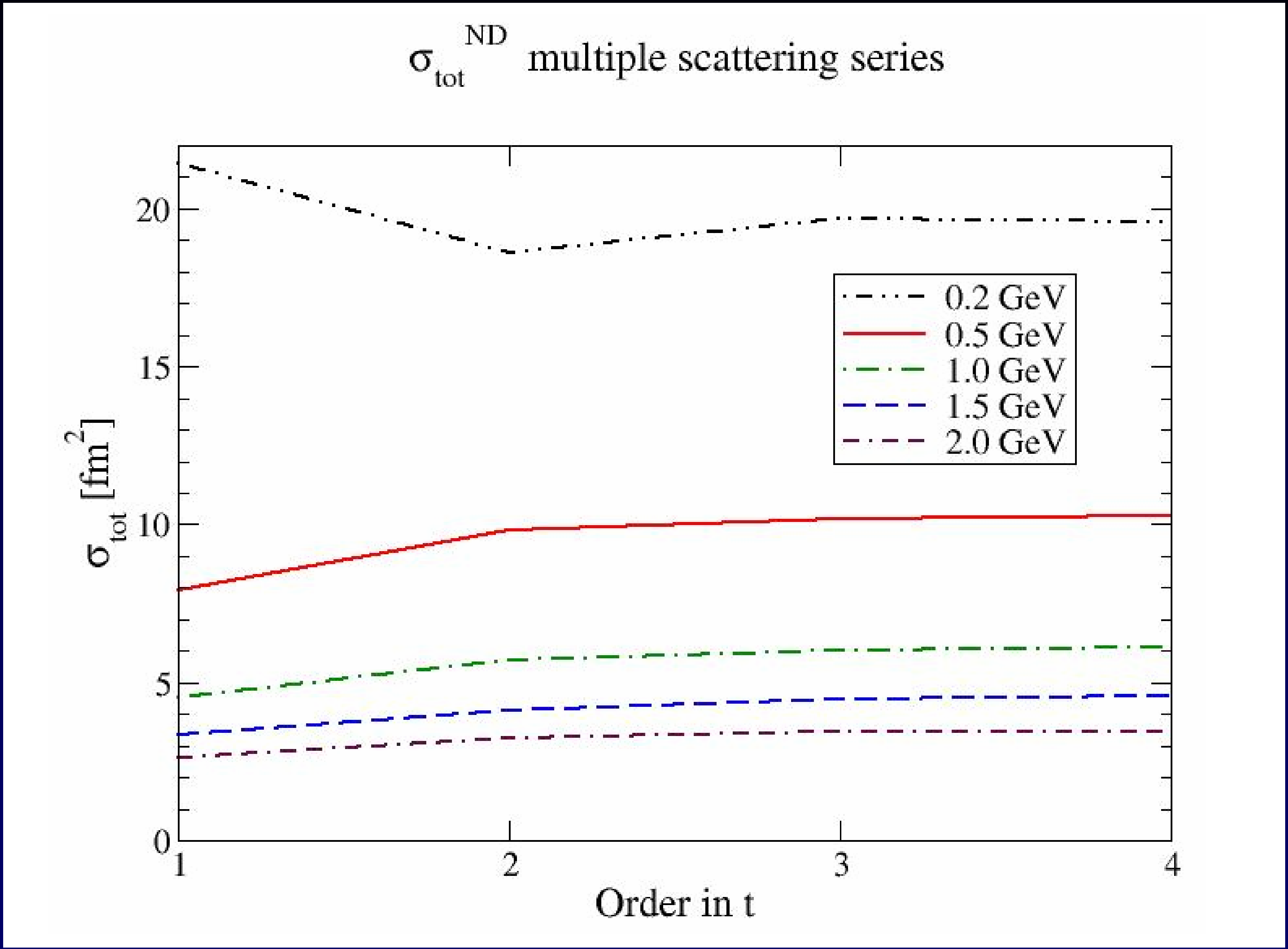
$$T = tP + tG_0PtP + \cdots$$

1st Order or IA

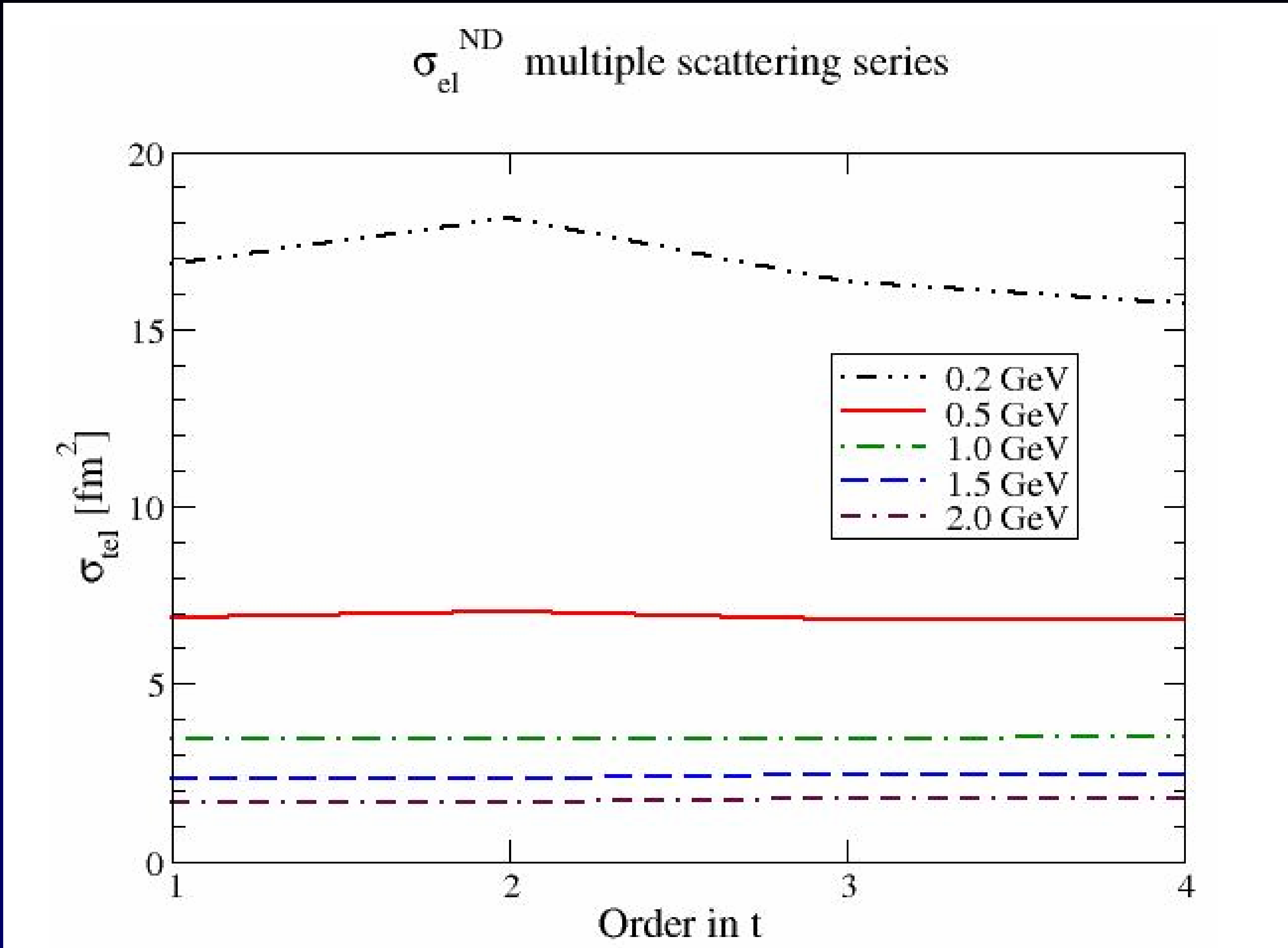


Convergence of the MS as function of energy?

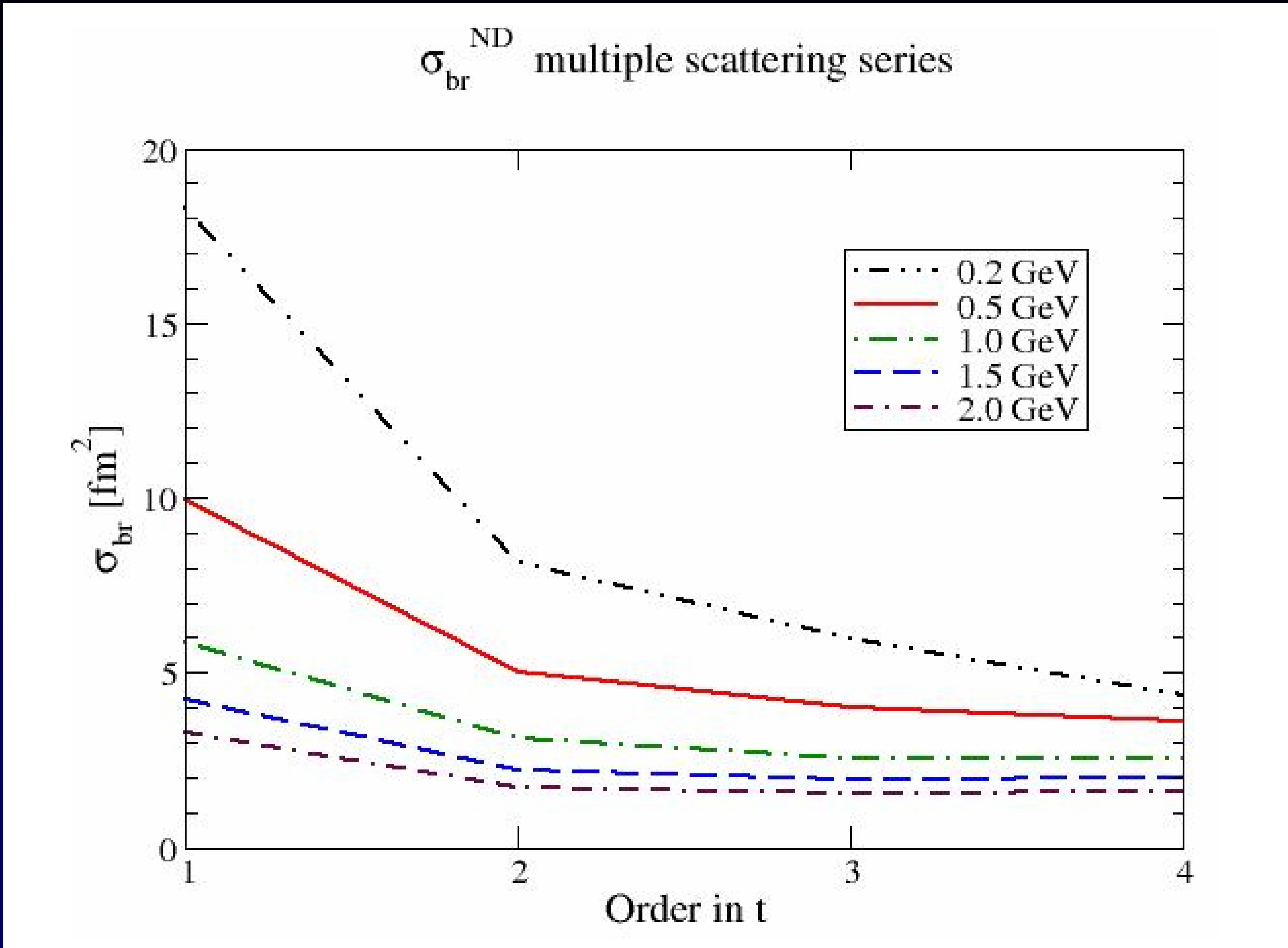
Consider total cross sections



σ_tot^ND multiple scattering series
σ_tot [fm^2]
Order in t
0.2 GeV
0.5 GeV
1.0 GeV
1.5 GeV
2.0 GeV
0
5
10
15
20
1
2
3
4



σ_el^ND multiple scattering series
σ_tel [fm^2]
Order in t
0.2 GeV
0.5 GeV
1.0 GeV
1.5 GeV
2.0 GeV
0
5
10
15
20
1
2
3
4



σ_{br}^{ND} multiple scattering series
σ_{br} [fm^2]
Order in t
0.2 GeV
0.5 GeV
1.0 GeV
1.5 GeV
2.0 GeV

Unitarity Relation

$$\langle \phi | U | \phi' \rangle^* - \langle \phi' | U | \phi \rangle = \int d^3 q \langle \phi | U | \phi' \rangle^* \, 2\pi \, i \, \vartheta\left(E - E_q\right) \langle \phi_q | U | \phi \rangle$$
$$+ \tfrac{1}{3} \int d^3 p d^3 q \langle \phi_0 | U_0 | \phi' \rangle \, 2\pi \, i \, \vartheta\left(E - E_{pq}\right) \langle \phi_0 | U_0 | \phi \rangle$$

$$\Rightarrow \quad -(2\pi)^3 \frac{4m}{3q_0} \operatorname{Im} \langle q_0, 1, \varphi_d | U | q_0 \varphi_d \rangle = \sigma_{tot} = \sigma_{el} + \sigma_{br}$$

$$U = P G_0^{-1} + P t G_0 U$$

n+d Cross Sections in the GeV Regime

- Original suggestion for calculation
  - R.J. Glauber 1955
- Specfic calculations:
  - **D.R. Harrington, Phys. Rev. 135, B358 (1964)**
  - **V. Franco, R.J. Glauber, Phys. Rev. 142, 1195 (1966)**
  - **F.W. Byron, C.J. Joachain, E.H. Mund, Phys. Rev. D8, 2622 (1973)**
- Calculations are

non-relativistic

neglect spin degrees of freedom

~~use an Eikonal form of the two-body NN interaction~~

Unique Opportunity

- Compare the Glauber approximation for 3-body scattering with an exact Faddeev calculation
- Use the same two-body input for both calculations
  - Two-body NN interaction of Malfliet-Tjon type
  - Solve for the deuteron wave function
  - Solve for the two-body t-matrix and scattering amplitude

$$f_s^c(p,\theta_{c.m.}) = -\frac{m}{2}(2\pi)^2\, t_s(p,p,x_c;E_p)$$

$$f_s^l(k_l,\theta_{lab}) = 2\sqrt{\cos\theta_l}\; f_s^c(p,\theta_c(\theta_l))$$

- Use the same kinematics – here non-relativistic

Glauber 3N amplitude

$$F_{3N}^{Gl}(\vec{q}) = 2\,S\left(\tfrac{1}{2}\vec{q}\right)\,f(\vec{q}) + \frac{i}{2\pi\,k}\int d^2q'\,S(\vec{q}\,')\,f\left(\tfrac{1}{2}\vec{q}+q'\right)f\left(\tfrac{1}{2}\vec{q}-q'\right)$$

With Deuteron Form Factor:

$$S(\vec{q}) = \int d^3r\left|\varphi_d(r)\right|^2 e^{i\vec{q}\cdot\vec{s}} = \int d^3r\left|\varphi_d(r)\right|^2 e^{i\vec{q}\cdot\vec{r}}$$

using $\vec{r} = \vec{s} + \hat{k}(\hat{k}\cdot\vec{r})$

3N Total Cross Section – 1st Order

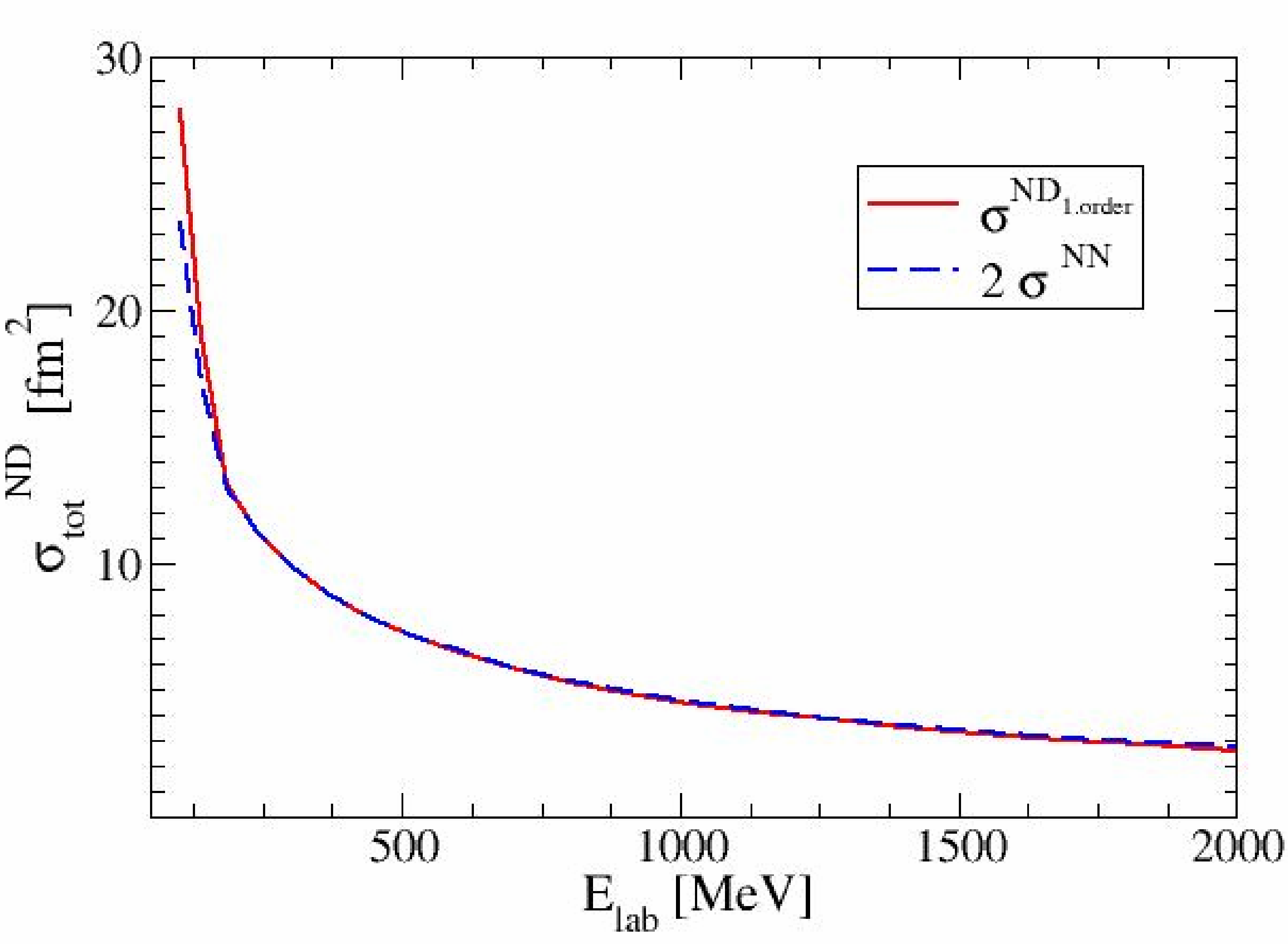
$$F_{3N}^{1st}\left(\vec{q}\right)=2\,S\left(\tfrac{1}{2}q\right)\,f_s\left(\vec{q}\right)$$

$$\sigma_{tot}^{Gl(1st)}=\frac{4\pi}{k_l}\,\Im m\,F_{3N}^{1st}\left(0\right)$$

$$\sigma_{tot}^{ND,1st}=-\left(2\pi\right)^3\frac{4m}{3q_0}\,\Im\mathrm{m}\left\langle q_0,1,\varphi_d\left|U^{1st}\right|q_0\varphi_d\right\rangle$$

Reminder: $U = PG_0^{-1} + PT$

3N Total Cross Section – 1st Order



Glauber 3N amplitude:

$$F_{3N}^{Gl}(\vec{q}) = 2\,S\left(\tfrac{1}{2}\vec{q}\right) f(\vec{q}) + \frac{i}{2\pi k}\int d^2q'\,S(\vec{q}\,')\,f\left(\tfrac{1}{2}\vec{q}+q'\right) f\left(\tfrac{1}{2}\vec{q}-q'\right)$$

Correction to total Cross Section explicitly

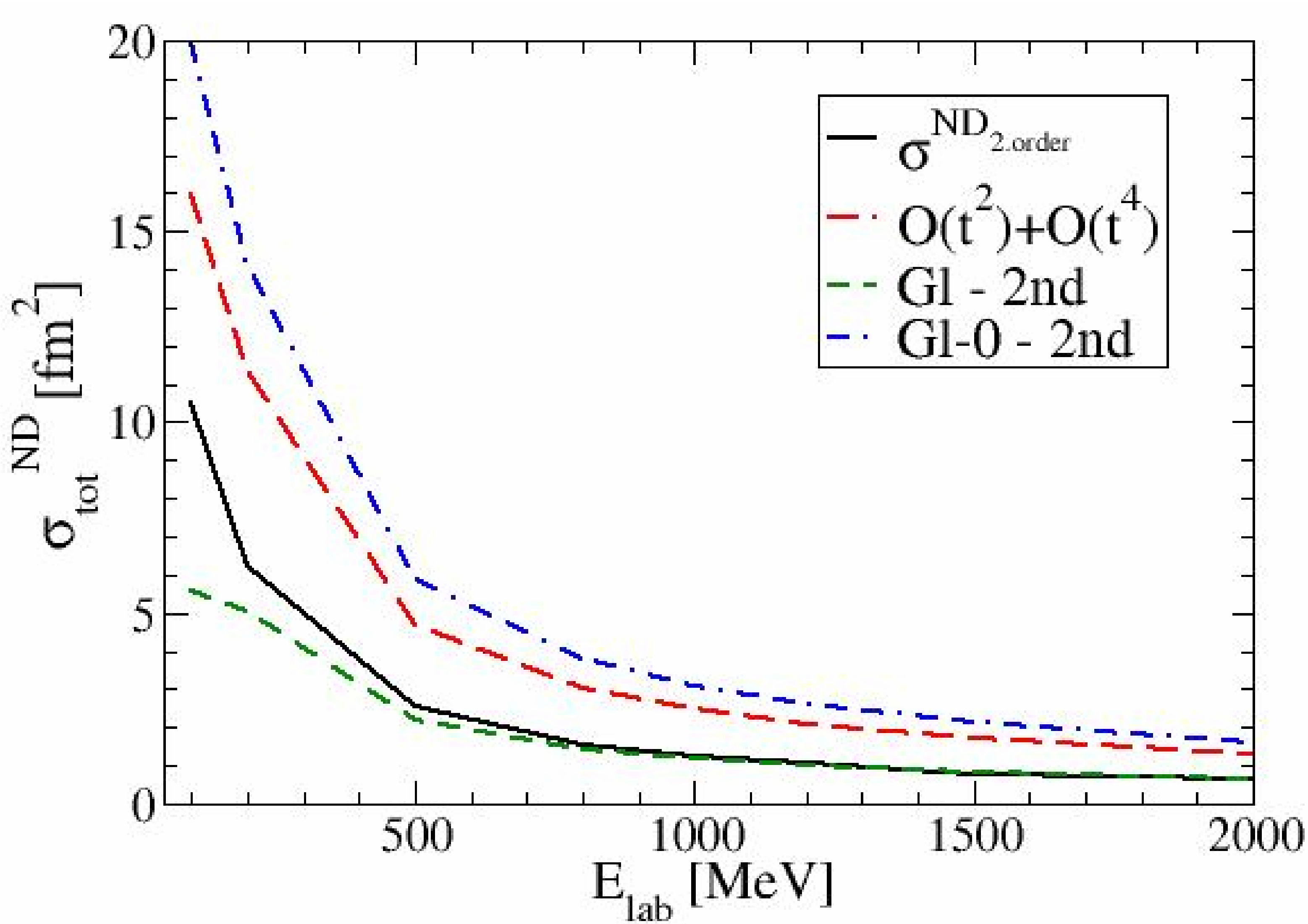
$$\delta\sigma = -\frac{4\pi}{p^2}\int_0^{q_{max}} dq\,'q\,'S(q\,')\,\Re e\left[f^2(p, x = 1 - \frac{q\,'^2}{2p^2}; E_p)\right]$$

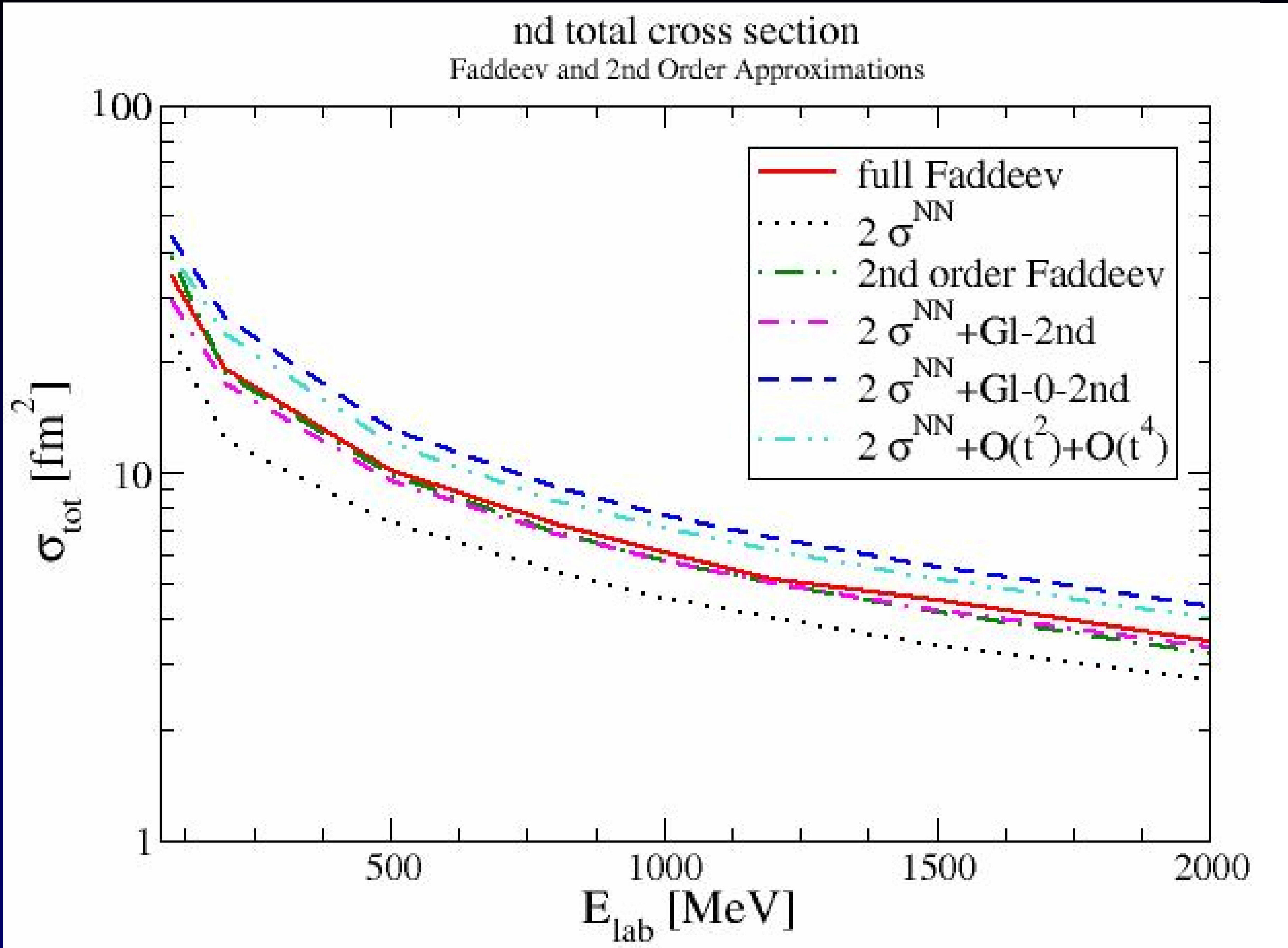
Further Approximation suggested by Franco-Glauber:

$$\delta\sigma = -\frac{4\pi}{p^2}\,\Re e\left[f^2(p, x = 1)\right]\int_0^{q_{max}} dq\,'q\,'S(q\,'),$$

Faddeev 2nd Order Correction: $\delta\sigma \propto t_s G_0 P t_s P$

2^{nd} order correction to 3N total cross section





nd total cross section
Faddeev and 2nd Order Approximations
full Faddeev
2 σ^NN
2nd order Faddeev
2 σ^NN+Gl-2nd
2 σ^NN+Gl-0-2nd
2 σ^NN+O(t^2)+O(t^4)
σ_tot [fm^2]
E_lab [MeV]
100
10
1
500
1000
1500
2000

3N Differential Cross Section – 1st Order

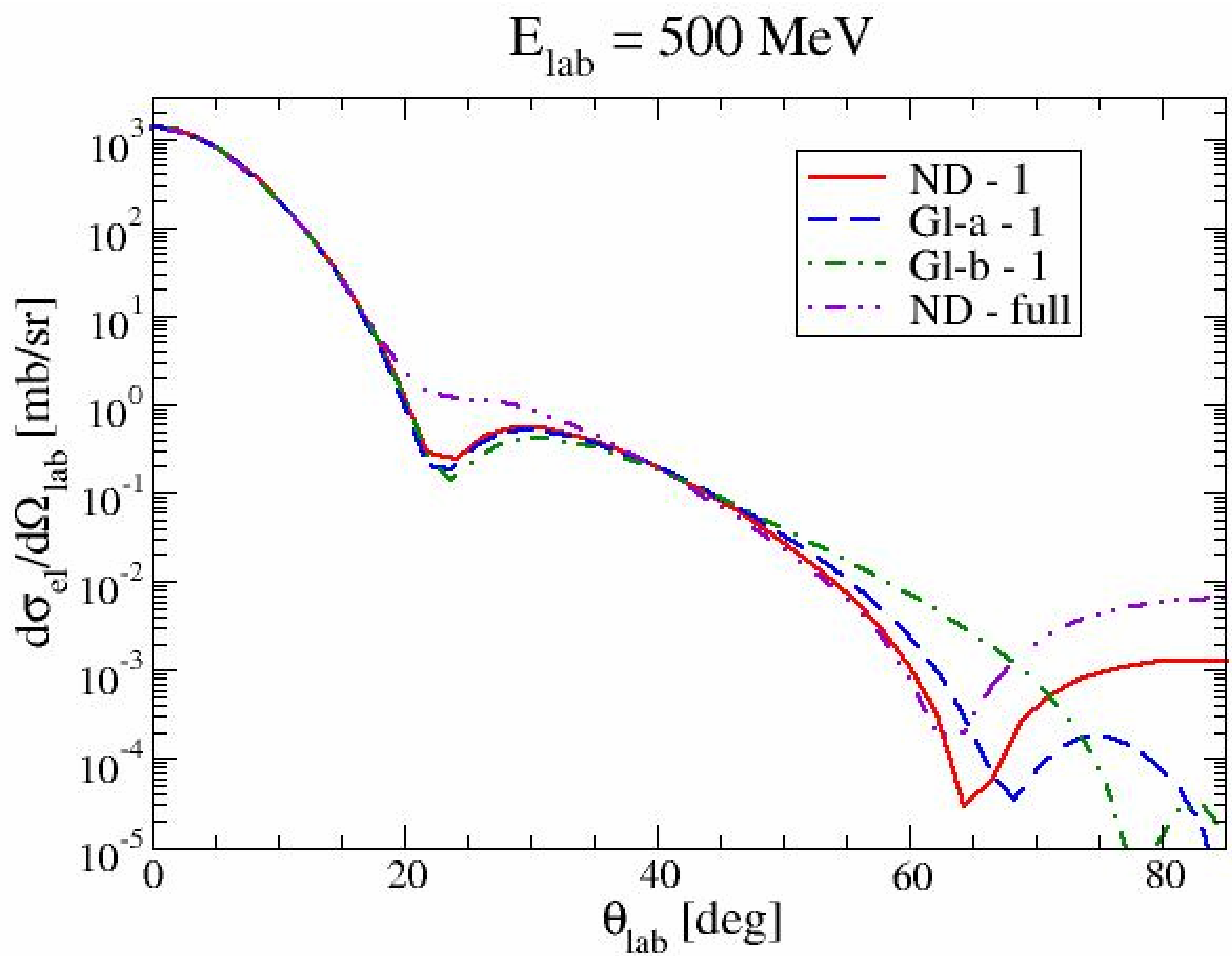
$$F_{3N}^{1st}\left(\vec{q}\right)=2\,S\left(\tfrac{1}{2}q\right)\,f\left(\vec{q}\right)$$

$$\frac{d\sigma_{3N}^{1st}}{d\Omega}=\left|F_{3N}^{1st}\left(\vec{q}\right)\right|^2$$

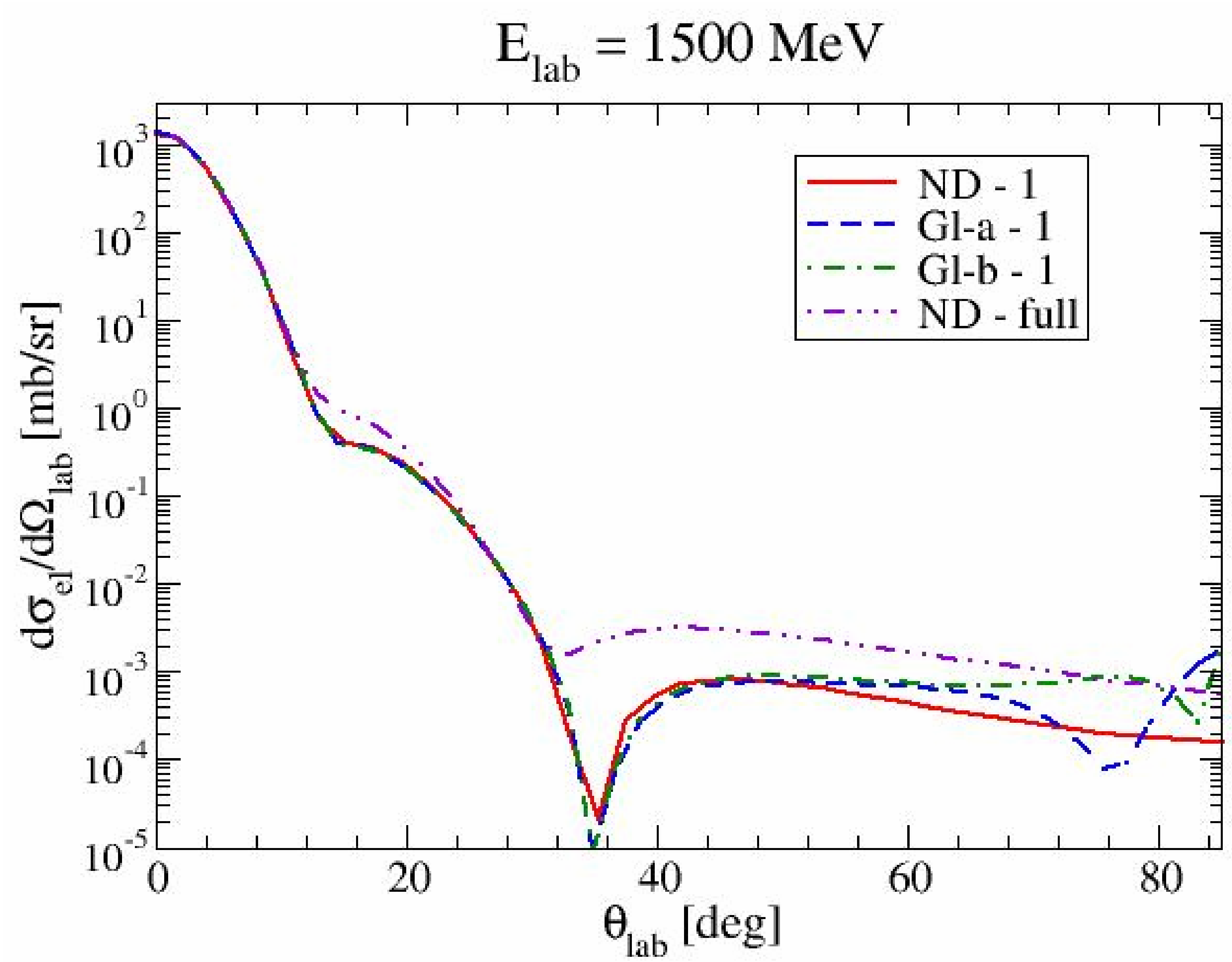
Position of **q** :

(a) In plane perpendicular to $k \equiv k_{lab}$ (Glauber)

(b) In plane perpendicular to k+k' (D.Harrington, C.Joachain)



E_lab = 500 MeV
ND - 1
Gl-a - 1
Gl-b - 1
ND - full
dσ_el/dΩ_lab [mb/sr]
θ_lab [deg]
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
20
40
60
80

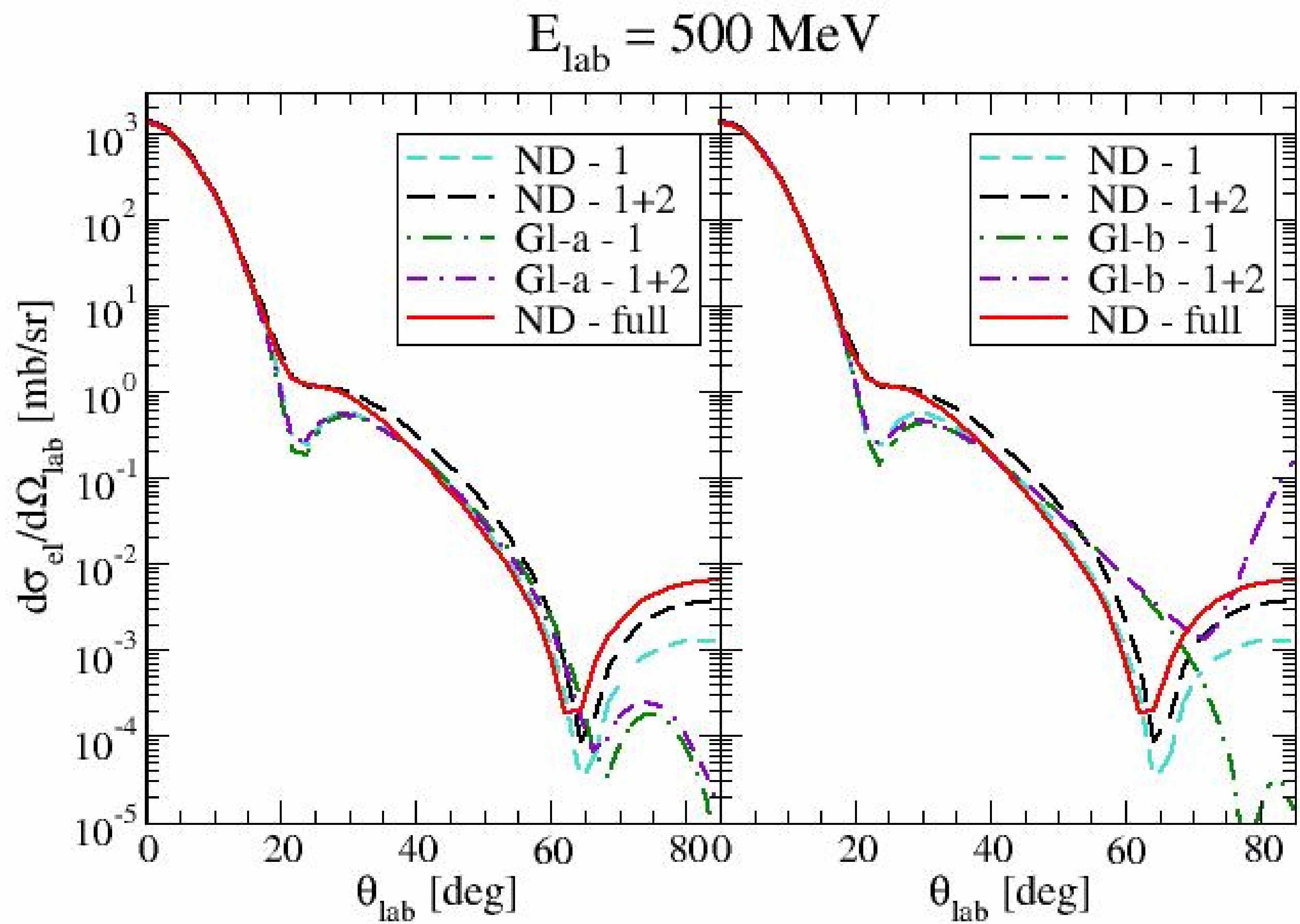


E_{lab} = 1500 MeV
ND - 1
Gl-a - 1
Gl-b - 1
ND - full
$d\sigma_{el}/d\Omega_{lab}$ [mb/sr]
θ_{lab} [deg]

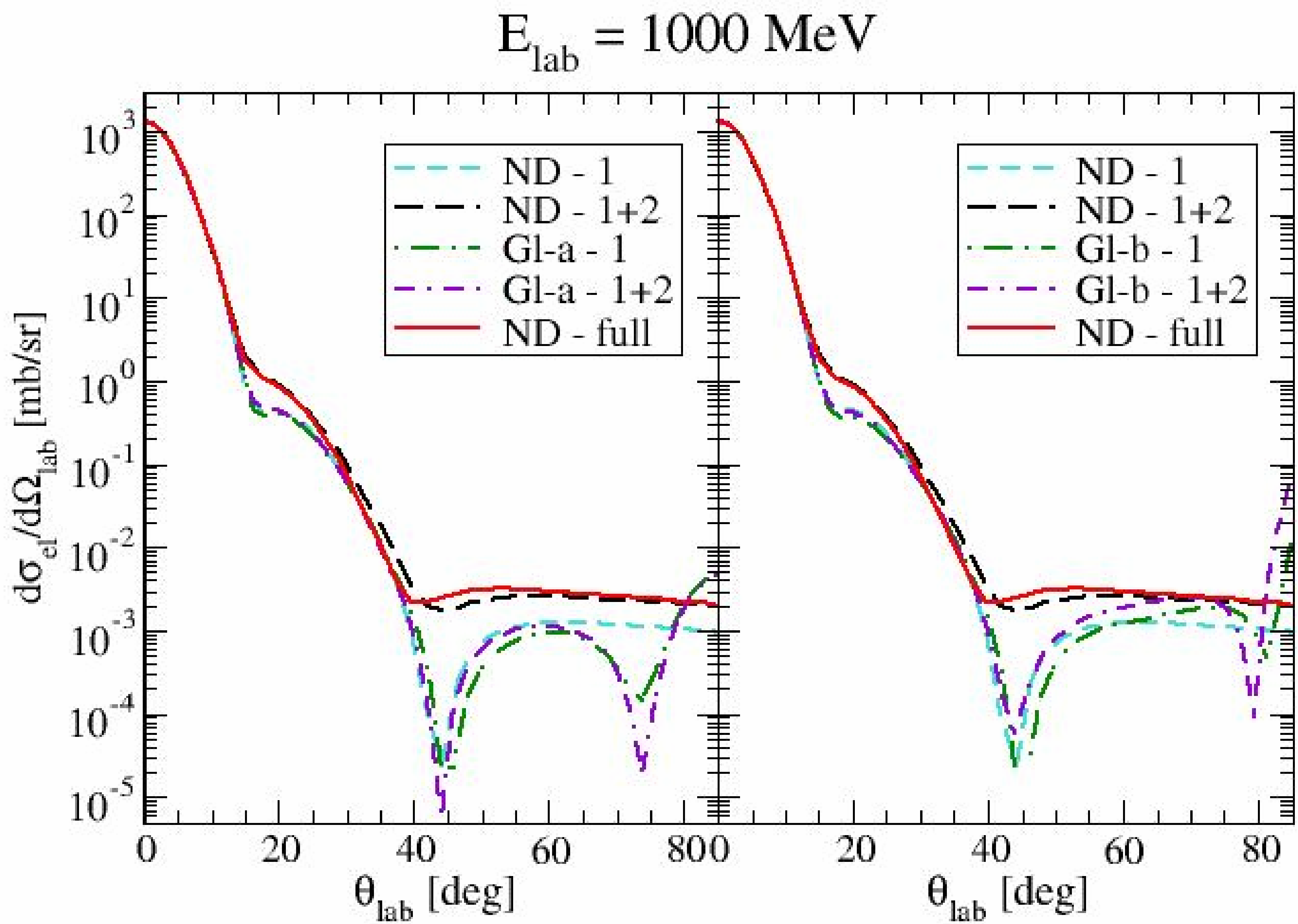
Glauber 3N amplitude

$$F_{3N}^{Gl}(\vec{q}) = 2\,S\left(\tfrac{1}{2}\vec{q}\right) f(\vec{q}) + \frac{i}{2\pi\,k}\int d^2q'\,S(\vec{q}\,')\,f\left(\tfrac{1}{2}\vec{q}+\vec{q}\,'\right) f\left(\tfrac{1}{2}\vec{q}-\vec{q}\,'\right)$$

$$\frac{d\sigma_{3N}^{Gl}}{d\Omega} = \left|F_{3N}(\vec{q})\right|^2$$



E_lab = 500 MeV
ND - 1
ND - 1+2
Gl-a - 1
Gl-a - 1+2
ND - full
Gl-b - 1
Gl-b - 1+2
dσ_el/dΩ_lab [mb/sr]
θ_lab [deg]
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
20
40
60
80



E_lab = 1000 MeV
dσ_el/dΩ_lab [mb/sr]
θ_lab [deg]
ND - 1
ND - 1+2
Gl-a - 1
Gl-a - 1+2
ND - full
Gl-b - 1
Gl-b - 1+2
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
20
40
60
80

Total Cross Sections
Total Cross Sections for Breakup Reactions

	Faddeev			Glauber		
E_{lab} [MeV]	σ_{tot}^{ND} [fm^2]	σ_{el}^{ND} [fm^2]	σ_{br}^{ND} [fm^2]	σ_{tot}^{Gl} [fm^2]	σ_{br}^{Gl}(a)[fm^2]	σ_{br}^{Gl}(b)[fm^2]
100	34.16	26.53	7.63	29.00	0.76	-
200	19.00	15.31	3.69	17.39	2.33	1.93
500	10.30	6.74	3.56	9.49	2.97	2.93
800	7.22	4.32	2.90	6.77	2.58	2.56
1000	6.00	3.46	2.54	5.74	2.33	2.31
1200	5.23	2.96	2.27	5.01	2.13	2.12
1500	4.37	2.41	1.96	4.22	1.89	1.88
2000	3.35	1.76	1.59	3.36	1.56	1.59

Integration over the

Fivefold differential cross section

Via optical theorem

Compare: Faddeev & Glauber

- **Total Cross Sections**
  - 1^{st} order terms agree from ≈ 250 MeV
  - 2^{nd} order terms agree from ≈ 700 MeV
  - At ≈ 700 MeV Faddeev and Glauber agree
- **Differential Cross Sections**
  - 1^{st} order calculations agree for forward angles
  - 2^{nd} order correction in Glauber does **not** fill in 1^{st} minimum
- **Total Cross Sections for Elastic Scattering**
  - 1^{st} order agrees with Faddeev at ≈ 1 GeV (2^{nd} order at 2 GeV)
- **Total Cross Sections for Breakup Reactions**
  - Faddeev & Glauber agree at 2 GeV

Published in PRC 78, 034002 (2008)

Similar study for p+^{11}Be by R. Crespo et al in PRC 76, 014620 (2007)

Platonova & Kukulin repeated the Glauber study for p+d with cd-bonn PRC 81, 014004 (2010)

Relativistic Three-Body Problem

- Context: Poincarė Invariant Quantum Mechanics
  - Poincarė invariance is exact symmetry, realized by a unitary representation of the Poincarė group on a few-particle Hilbert space
  - Instant form
  - Faddeev equations same operator form but different ingredients
- **Kinematics**
  - Lorentz transformations between frames
- **Dynamics**
  - Bakamjian-Thomas Scheme: Mass Operator $M=M_0+V$ replaces Hamiltonian $H=H_0+\mathbf{v}$
  - Connect Galilean two-body **v** with Poincarė two-body v
  - Construct V := $\sqrt{M^2+q^2}-\sqrt{M_0^2+q^2}$

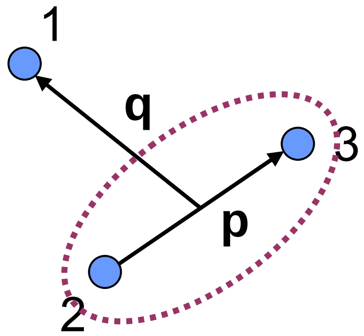
Kinematic Relativistic Ingredients:

- Lorentz transformation Lab $\rightarrow$ c.m. frame (3-body)
- Phase space factors in cross sections
- Poincarė-Jacobi momenta
- Permutations for identical particles

Kinematics: Poincaré-Jacobi momenta

- **Nonrelativistic (Galilei)**

$$\mathbf{p} = \frac{1}{2}(\mathbf{k}_2 - \mathbf{k}_3)$$
$$\mathbf{q} = \frac{2}{3}(\mathbf{k}_1 - \frac{1}{2}(\mathbf{k}_2 + \mathbf{k}_3))$$



- **Relativistic (Lorentz)**

$$\mathbf{p} = \frac{1}{2}(\mathbf{k}_2 - \mathbf{k}_3) + \frac{\mathbf{k}_2 + \mathbf{k}_3}{2m_{23}}\left(\frac{(\mathbf{k}_2 - \mathbf{k}_3)\cdot(\mathbf{k}_2 + \mathbf{k}_3)}{(E_2 + E_3) + m_{23}} - (E_2 - E_3)\right)$$
$$\mathbf{q} = \mathbf{k}_1 + \frac{\mathbf{K}}{M}\left(\frac{\mathbf{k}_1 \cdot \mathbf{K}}{E + M} - E_1\right)$$

in c.m. frame K = 0

$$|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle = \left|\frac{\partial(K\mathbf{pq})}{\partial(\mathbf{k}_2\mathbf{k}_3)}\right|^{1/2} |K\mathbf{pq}\rangle \neq 1$$

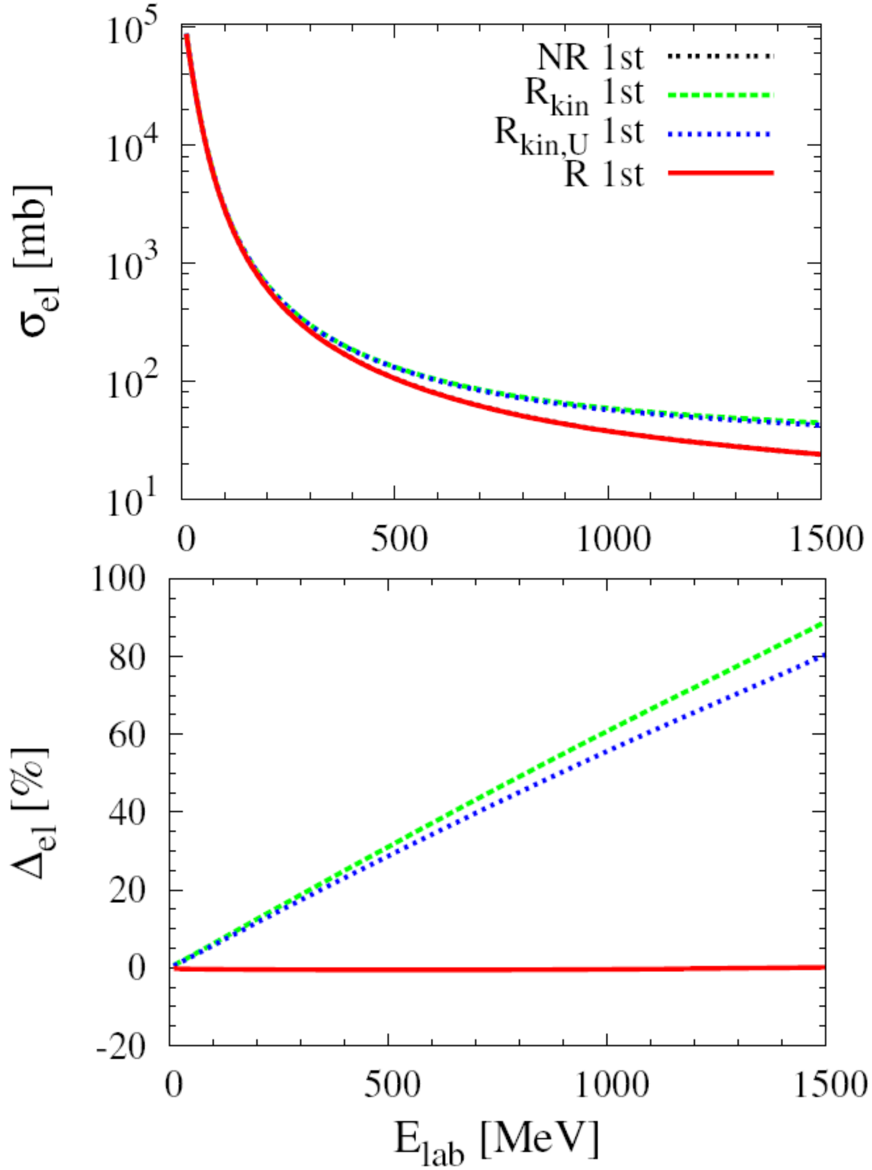
$$E = E_1 + E_2 + E_3$$
$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$$
$$M = \sqrt{E^2 - \mathbf{K}^2}$$
$$m_{23} = \sqrt{(E_2 + E_3)^2 - (\mathbf{k}_2 + \mathbf{k}_3)^2}$$

Relativistic kinematics

IA (1^{st} order)

$$T = tP$$

- Lorentz transformation Lab $\rightarrow$ c.m. frame) (3-body)
- Phase space factors in cross sections
- Poincarė-Jacobi momenta
- Permutations



Quantum Mechanics

Galilei Invariant: $H = \frac{\mathrm{K}^2}{2M} + h \quad ; \quad h = h_0 + v_{12}^{NR} + v_{13}^{NR} + v_{23}^{NR}$

Poincaré Invariant: $H = \sqrt{\mathrm{K}^2 + M^2} \quad ; \quad M = M_0 + V_{12} + V_{23} + V_{31}$

$$V_{ij} = M_{ij} - M_0 = \sqrt{\left(m_{0,ij} + v_{ij}\right)^2 + q_k^2} - \sqrt{m_{0,ij}^2 + q_k^2}$$

Two-body interaction embedded in the 3-particle Hilbert space

$$m_{0,ij} = \sqrt{m_i^2 + p_{ij}^2} + \sqrt{m_j^2 + p_{ij}^2}$$

$$M_0 = \sqrt{m_{0,ij}^2 + q_k^2} + \sqrt{m_k^2 + q_k^2}$$

V_{ij} embedded in the 3-particle Hilbert space

$$V_{ij} = M_{ij} - M_0 = \sqrt{\left(m_{0,ij} + v_{ij}\right)^2 + q_k^2} - \sqrt{m_{0,ij}^2 + q_k^2}$$

need matrix elements : $\langle \vec{k} | V(\vec{p}) | \vec{k}' \rangle$

$$= v(\vec{k},\vec{k}') + \psi_b(\vec{k})(\sqrt{M_b^2+p^2} - M_b)\psi_b(\vec{k}') + \frac{1}{\omega - \omega'}\Big[(\sqrt{\omega^2+p^2} - \omega)\Re[t(\vec{k}',\vec{k};\omega)]$$

$$-(\sqrt{\omega'^2+p^2} - \omega')\Re[t(\vec{k},\vec{k}';\omega')]\Big] + \frac{1}{\omega - \omega'}\Big[\mathcal{P}\int d^3k'' \frac{(\sqrt{\omega''^2+p^2} - \omega'')}{\omega'' - \omega} t(\vec{k},\vec{k}'';\omega'')t^*(\vec{k}',\vec{k}'';\omega'')$$

$$-\mathcal{P}\int d^3k'' \frac{(\sqrt{\omega''^2+p^2} - \omega'')}{\omega'' - \omega'} t(\vec{k},\vec{k}'';\omega'')t^*(\vec{k}',\vec{k}'';\omega'')\Big].$$

H. Kamada,[1,*] W. Glöckle,[2,†] J. Golak,[2,3,‡] and Ch. Elster[4,§]

PHYSICAL REVIEW C **66**, 044010 (2002)

Two-Body Input: T1-operator embedded in 3-body system

$$T_1(\mathrm{p}',\mathrm{p};\mathrm{q}) = V(\mathrm{p}',\mathrm{p};\mathrm{q}) + \int d^3k'' \frac{V(\mathrm{p}',\mathrm{k}'';\mathrm{q})\, T_1(\mathrm{k}'',\mathrm{p};\mathrm{q})}{\sqrt{(2E(p'))^2 + q^2} - \sqrt{(2E(k''))^2 + q^2} + i\varepsilon}$$

Do not solve for V!

- Obtain fully off-shell matrix elements T_1(k,k',q) from half shell transition matrix elements by

  Solving a 1st resolvent type equation:

$$T_1(q) = T_1(q') + T_1(q)\,[g_0(q) - g_0(q')]\,T_1(q')$$

- For every single off-shell momentum point
- Proposed in
  - Keister & Polyzou, PRC 73, 014005 (2006)
- Carried out for the first time in **PRC 76, 1014010 (2007)** [PhD T. Lin]

⇒ Exact Boost

Obtain embedded 2N t-matrix $T_1(k,k',z')$ half-shell in 2-body c.m. frame first :

$$\langle \mathbf{k}|T_1(\mathbf{q};z')|\mathbf{k}'\rangle = \langle \mathbf{k}|V(\mathbf{q})|\,\mathbf{k}'^{(-)}\rangle$$
$$= \frac{2(E_{k'}+E_k)}{\sqrt{4E_{k'}^2+\mathbf{q}^2}+\sqrt{4E_k^2+\mathbf{q}^2}}\, t(\mathbf{k},\mathbf{k}';2E_{k'})$$

$$t(\mathbf{k},\mathbf{k}';2E_{k'}) = v(\mathbf{k},\mathbf{k}') + \int d\mathbf{k}'' \frac{v(\mathbf{k},\mathbf{k}'')t(\mathbf{k}'',\mathbf{k}';2E_{k'})}{E_{k'}-2\sqrt{m^2+k''^2}+i\epsilon}$$

Solution of the relativistic 2N LS equation with 2-body potential

Two-body Potential ?

Phase Equivalent 2-body t-matrices based on Transformation by Coester-Pieper-Serduke (CPS)

(PRC11, 1 (1975)) and given in Polyzou PRC 58, 91 (1998))

- Add interaction to square of non-interacting mass operator

$$M^2 = M_0^2 + u = 4mh \quad \text{with} \;\; h \equiv \frac{k^2}{m} + \frac{u}{4m} + m$$

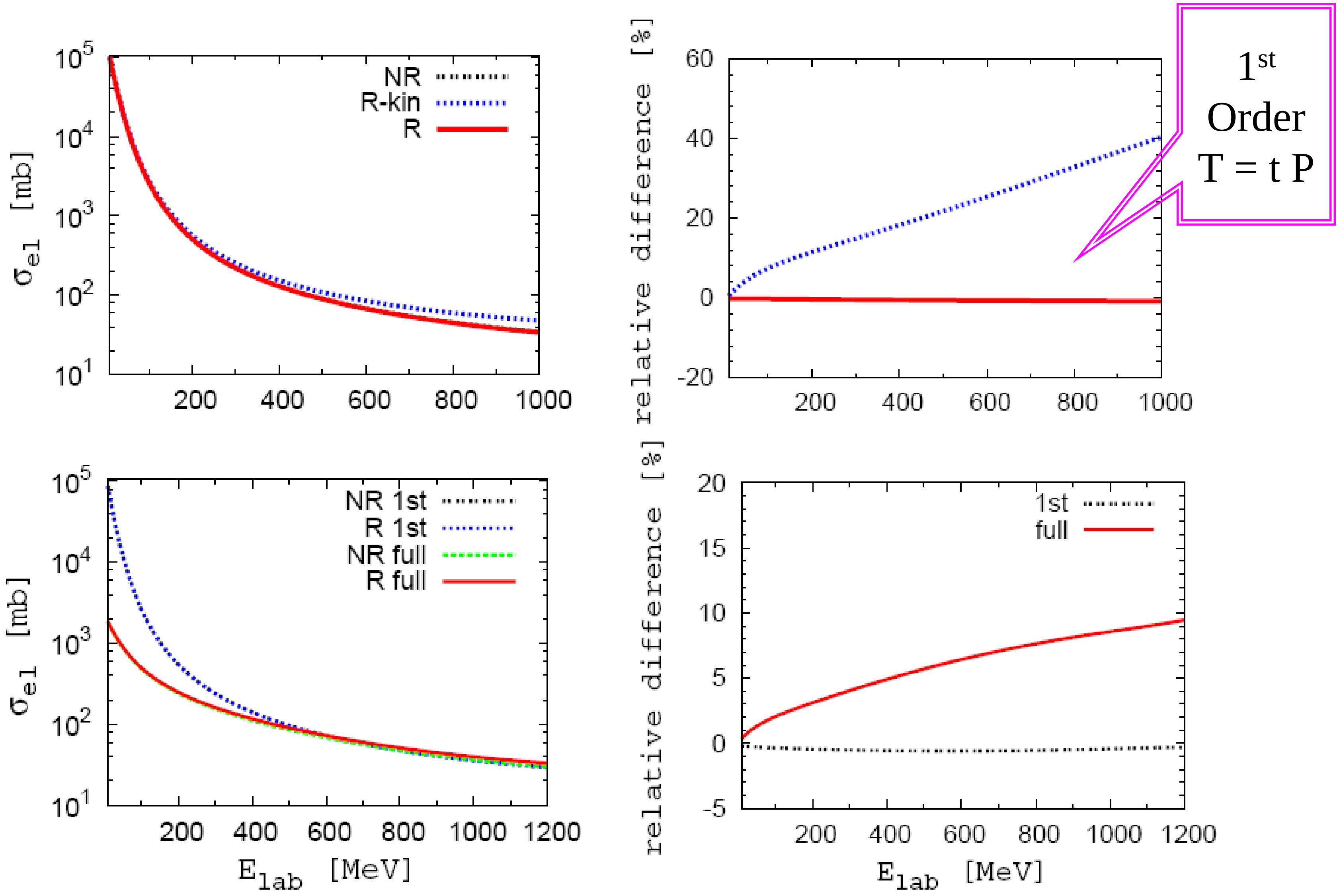
$$u = v^2 + \left\{ M_0^2 , v \right\}$$

- NO need to evaluate v directly, since M, M^2, h have the same eigenstates
- Relation between half-shell t-matrices

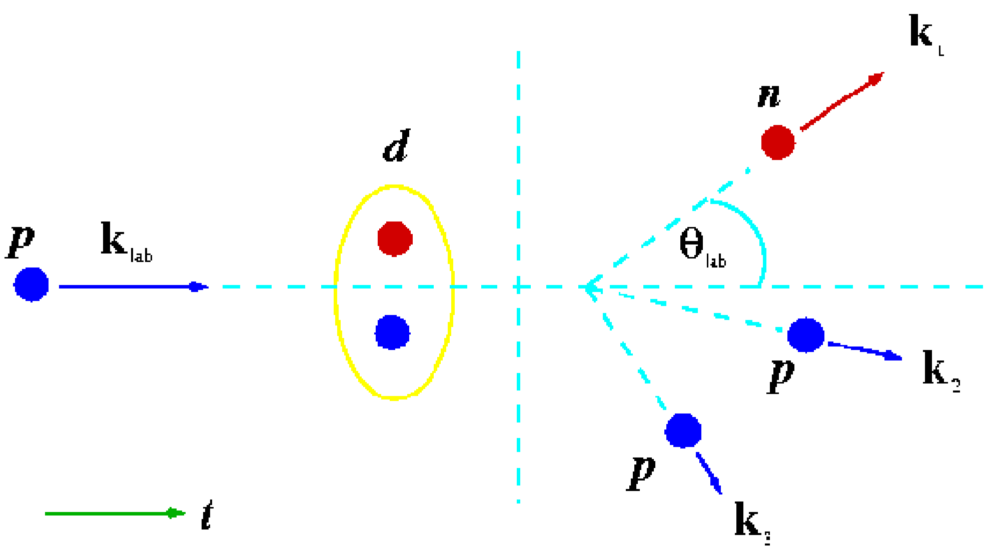
$$\langle k' | t_R(e(k)) | k \rangle = \frac{4m}{e(k)+e(k')} \langle k' | t_{NR}(k^2/m) | k \rangle$$

- Relativistic and nonrelativistic cross sections are identical functions of the invariant momentum k

Total Cross Section for Elastic Scattering: Scalar Interaction



Breakup Scattering



Exclusive: Measure energy & angles of two ejected particles

V.Punjabi et al. PRC 38, 2728 (1998) – TRIUMF p+d @ 508 MeV

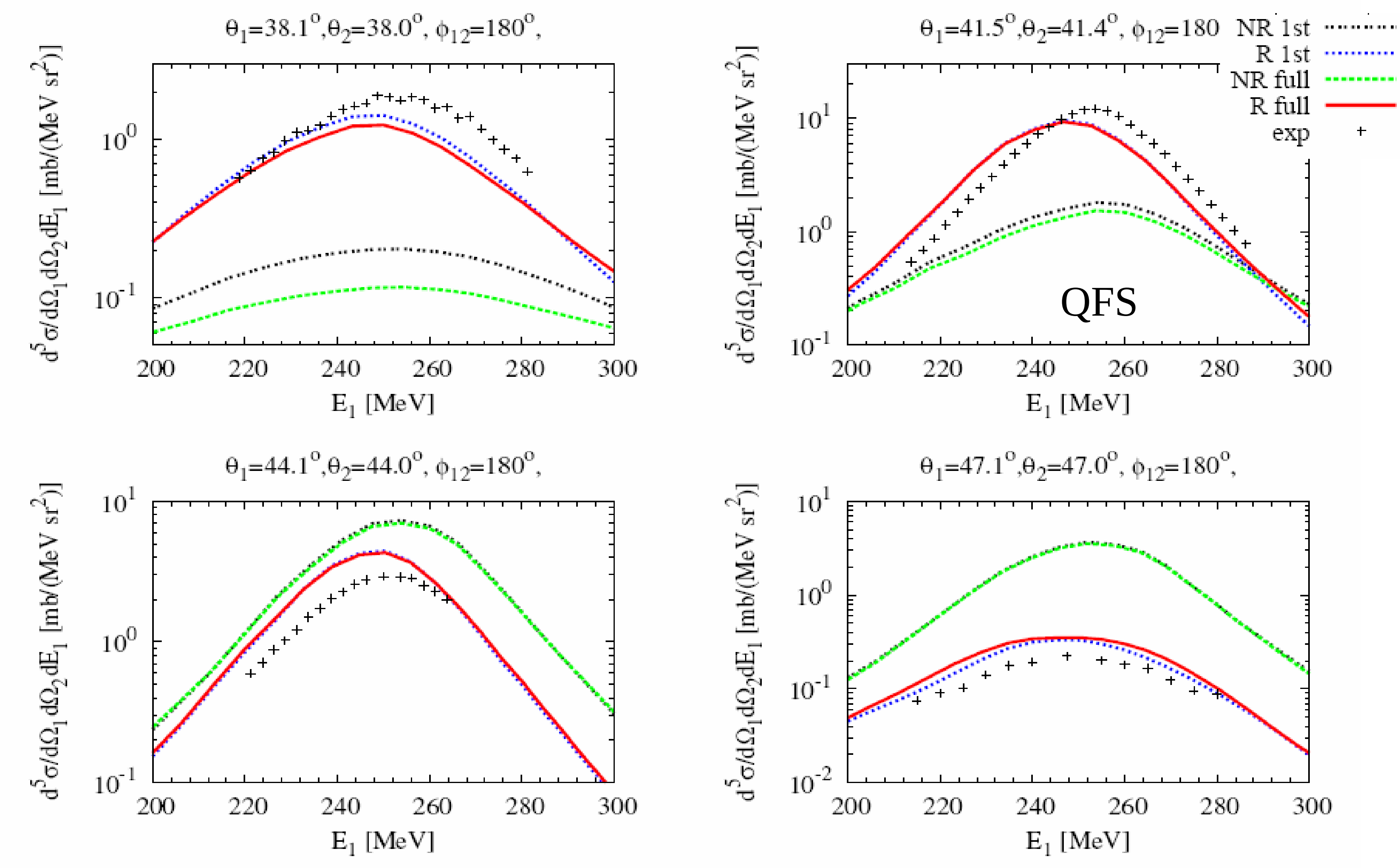
Outgoing protons are measured in the scattering plane

Exclusive Breakup Scattering

(symmetric configuration)

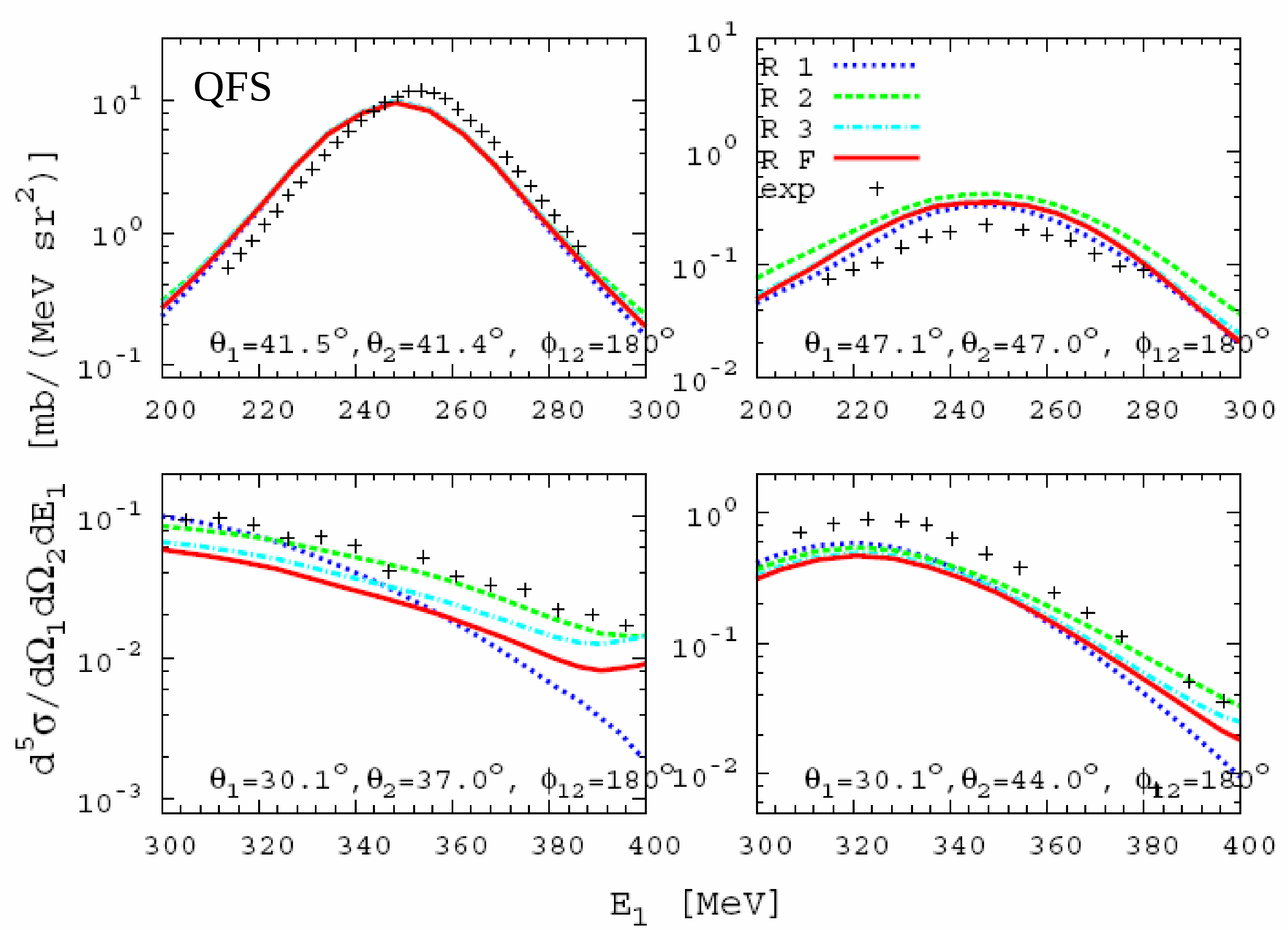
E_{lab} = 508 MeV

(V.Punjabi et al. PRC 38, 2728 (1998)

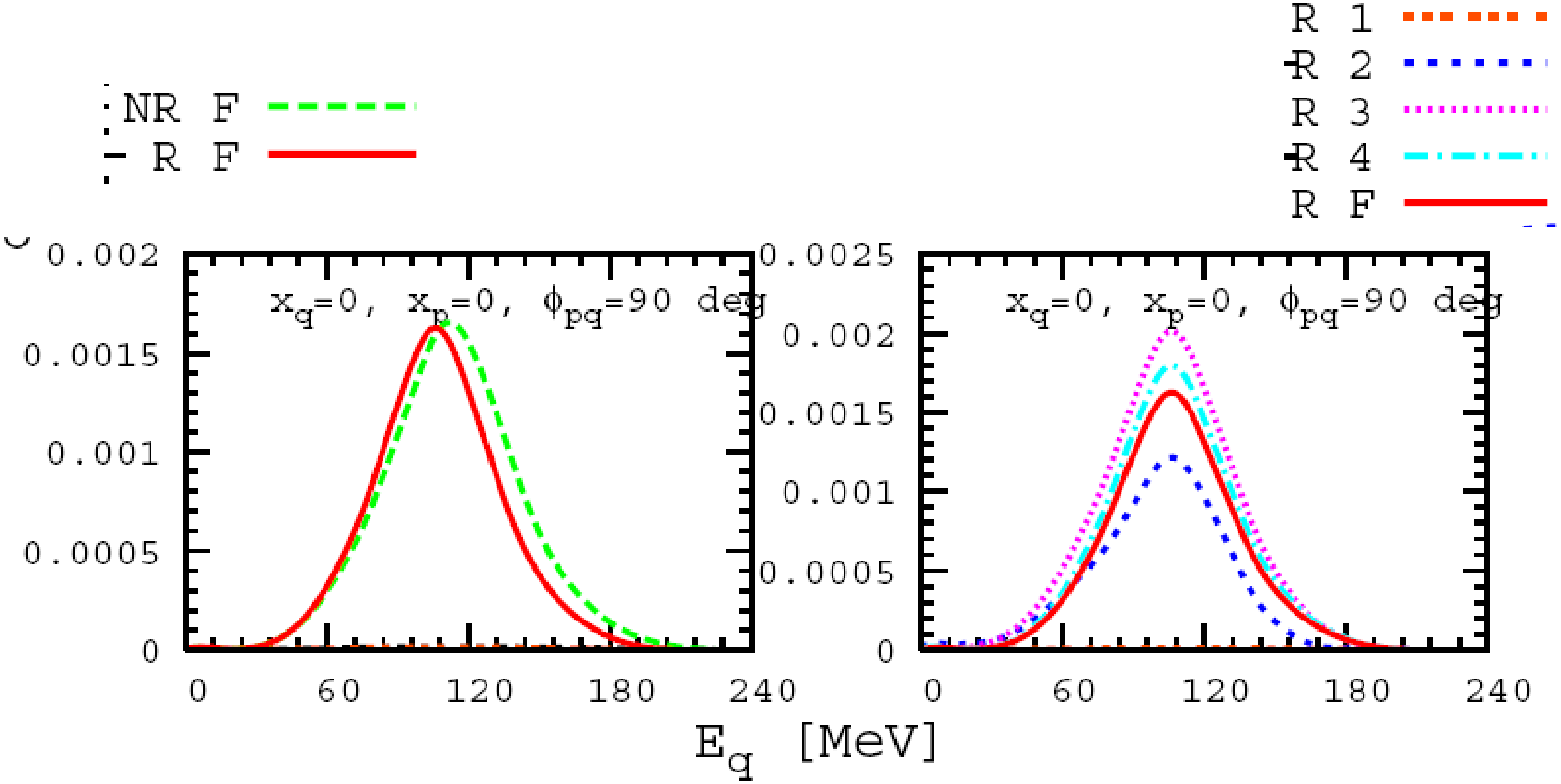


Exclusive Breakup Scattering

E_{lab} = 508 MeV



Exclusive Breakup Scattering
Space-Star



E_{lab} = 508 MeV

Poincaré Invariant Faddeev Calculations

- **Carried out up to 2 GeV for elastic and breakup scattering for scalar interactions**
  - Solved Faddeev equation in vector variables = NO partial waves
- **Relativistic effects are important at 500 MeV and higher**
  - Relativistic total elastic cross section increases up to 10% compared to the non-relativistic
  - Relativistic kinematics determines QFS peak positions in inclusive and exclusive breakup
  - Breakup: Relativistic effects very large dependent on configuration
- **Above 800 MeV projectile energy:**
  - multiple scattering series converges after ~2 iterations
  - In breakup QFS conditions 1st order calculations sufficient

Including spin and isospin degrees of Freedom: General Form of NN interaction

- **Space (e.g. momenta)** $\vec{p}\,'-\vec{p}, \quad \vec{p}\,'+\vec{p}, \quad \vec{p}\,'\times\vec{p}$
  - Basis: vector variables

- **Spin – Operators** $\vec{\sigma}_1 \quad \text{and} \quad \vec{\sigma}_2$

- **Isospin - Operators** $\vec{\tau}_1 \quad \text{and} \quad \vec{\tau}_2$

Idea: form scalar functions with the vector variables

handle operators analytically

General Form of NN interaction, cont'd

Allow explicit isospin dependence: $\langle t' m_t' \mid V \mid t m_t \rangle = \delta_{tt'} \delta_{m_t m_t'} V^{t m_t}$

Spin momentum operator structure invariant under rotation, parity, time-reversal

$$
\begin{aligned}
w_1(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= 1 \\
w_2(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \\
w_3(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= i\,(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot (\mathrm{p} \times \mathrm{p}') \\
w_4(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot (\mathrm{p} \times \mathrm{p}')\ \boldsymbol{\sigma}_2 \cdot (\mathrm{p} \times \mathrm{p}') \\
w_5(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot (\mathrm{p}' + \mathrm{p})\ \boldsymbol{\sigma}_2 \cdot (\mathrm{p}' + \mathrm{p}) \\
w_6(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathrm{p}', \mathrm{p}) &= \boldsymbol{\sigma}_1 \cdot (\mathrm{p}' - \mathrm{p})\ \boldsymbol{\sigma}_2 \cdot (\mathrm{p}' - \mathrm{p})
\end{aligned}
$$

Most general expression for any NN potential:

$$
V^{t m_t} \equiv \sum_{j=1}^{6} v_j^{t m_t}(\mathbf{p}', \mathbf{p})\ w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p})
$$

where $v_j^{t m_t}(\mathbf{p}', \mathbf{p})$ is a scalar function of |p'|, |p|, and p'·p

Examples:

χEFT LO potential:

$$V_{LO} = -\frac{1}{(2\pi)^3}\frac{g_A^2}{4F_\pi^2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{q}\,\boldsymbol{\sigma}_2\cdot\mathbf{q}}{\mathbf{q}^2+M_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \frac{C_S}{(2\pi)^3} + \frac{C_T}{(2\pi)^3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,,$$

$$\mathbf{q} = \mathbf{p}\,' - \mathbf{p}$$

χEFT NLO potential:

$$\begin{aligned} V_{NLO} &= -\frac{1}{(2\pi)^3}\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{384\pi^2F_\pi^4}L^{\tilde{\Lambda}}(q)\left[4m_\pi^2(5g_A^4-4g_A^2-1)+\mathbf{q}^{\,2}(23g_A^4-10g_A^2-1)+\frac{48g_A^4m_\pi^4}{4m_\pi^2+\mathbf{q}^{\,2}}\right] \\ &- \frac{1}{(2\pi)^3}\frac{3g_A^4}{64\pi^2F_\pi^4}L^{\tilde{\Lambda}}(q)\left(\boldsymbol{\sigma}_1\cdot\mathbf{q}\,\boldsymbol{\sigma}_2\cdot\mathbf{q}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,\mathbf{q}^{\,2}\right) \\ &+ \frac{C_1}{(2\pi)^3}\mathbf{q}^{\,2}+\frac{C_2}{(2\pi)^3}\mathbf{k}^2+\left(\frac{C_3}{(2\pi)^3}\mathbf{q}^{\,2}+\frac{C_4}{(2\pi)^3}\mathbf{k}^2\right)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ &+ \frac{C_5}{(2\pi)^3}\frac{i}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{q}\times\mathbf{k}+\frac{C_6}{(2\pi)^3}\mathbf{q}\cdot\boldsymbol{\sigma}_1\,\mathbf{q}\cdot\boldsymbol{\sigma}_2+\frac{C_7}{(2\pi)^3}\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\boldsymbol{\sigma}_2\,, \end{aligned} \tag{C2}$$

$$\mathbf{k} = \tfrac{1}{2}(\mathbf{p}\,'+\mathbf{p})$$

NN t-matrix:

$$t^{tm_t} = V^{tm_t} + V^{tm_t} G_0 t^{tm_t}$$

$$t^{tm_t} \equiv \sum_{j=1}^{6} t_j^{tm_t}(\mathbf{p}', \mathbf{p})\ w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p})$$

$$V^{tm_t} \equiv \sum_{j=1}^{6} v_j^{tm_t}(\mathbf{p}', \mathbf{p})\ w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p})$$

$$\begin{aligned}\sum_{j=1}^{6} t_j^{tm_t}(\mathbf{p}', \mathbf{p})\,\mathrm{w}_j(\mathbf{p}', \mathbf{p}) &= \sum_{j=1}^{6} v_j^{tm_t}(\mathbf{p}', \mathbf{p})\mathrm{w}_j(\mathbf{p}', \mathbf{p}) \\ &+ 2\mu \lim_{\epsilon\to 0} \sum_{k,j=1}^{6} \int d\mathbf{p}'' \frac{v_k^{tm_t}(\mathbf{p}', \mathbf{p}'')\mathrm{w}_k(\mathbf{p}', \mathbf{p}'')t_j^{tm_t}(\mathbf{p}'', \mathbf{p})\,\mathrm{w}_j(\mathbf{p}'', \mathbf{p})}{p^2 + i\epsilon - p''^2}\end{aligned}$$

Project with w_k from the left
and perform the trace in NN spin space

$$\sum_j A_{kj}(\mathbf{p}', \mathbf{p}) t_j^{tm_t}(\mathbf{p}', \mathbf{p}) = \sum_j A_{kj}(\mathbf{p}', \mathbf{p}) v_j^{tm_t}(\mathbf{p}', \mathbf{p})$$
$$+ \int d^3p'' \sum_{jj'} v_j^{tm_t}(\mathbf{p}', \mathbf{p}'') G_0(p'') \; t_{j'}^{tm_t}(\mathbf{p}'', \mathbf{p}) \; B_{kjj'}(\mathbf{p}', \mathbf{p}'', \mathbf{p})$$

All functions are scalar !

$$A_{kj}(\mathbf{p}', \mathbf{p}) \;\equiv\; \mathrm{Tr}\Big(w_k(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p}) \; w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p}) \Big)$$

$$B_{kjj'}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) \;\equiv\; \mathrm{Tr}\Big(w_k(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p}) \; w_j(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}', \mathbf{p}'') \; w_{j'}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{p}'', \mathbf{p}) \Big)$$

NN t-matrix consists of 6 coupled eqs of scalar functions

Structure of some of the A_{kj} and $B_{kjj'}$:

$$A_{26}(\mathbf{p}', \mathbf{p}) = 4(\mathbf{p}' - \mathbf{p})^2$$
$$A_{33}(\mathbf{p}', \mathbf{p}) = -8(\mathbf{p} \times \mathbf{p}')^2$$
$$A_{56}(\mathbf{p}', \mathbf{p}) = 4(p'^2 - p^2)^2$$

14 non-vanishing

$$B_{261}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = 4(\mathbf{p}' - \mathbf{p}'')^2$$
$$B_{612}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = 4(\mathbf{p}' - \mathbf{p})^2$$
$$B_{133}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = -8(\mathbf{p}'' \times \mathbf{p}') \cdot (\mathbf{p} \times \mathbf{p}'')$$
$$B_{331}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = -8(\mathbf{p} \times \mathbf{p}') \cdot (\mathbf{p}'' \times \mathbf{p}')$$
$$B_{313}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = -8(\mathbf{p} \times \mathbf{p}') \cdot (\mathbf{p} \times \mathbf{p}'')$$
$$B_{145}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = 4\{(\mathbf{p} \times \mathbf{p}') \cdot \mathbf{p}''\}^2$$
$$B_{155}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = 4\{(\mathbf{p}' + \mathbf{p}'') \cdot (\mathbf{p}'' + \mathbf{p})\}^2$$
$$B_{551}(\mathbf{p}', \mathbf{p}'', \mathbf{p}) = 4\{(\mathbf{p}' + \mathbf{p}) \cdot (\mathbf{p}' + \mathbf{p}'')\}^2$$

148 non-vanishing

Road to Realistic NN Forces

- Work with operator representation of states
  - Operators built from spin & position vectors
  - Allows to carry out spin algebra analytically
  - Relative momenta are contained in scalar coefficient functions
- Deuteron **(Fachruddin, Elster, Glöckle, PRC 63, 054003 (2001)**

$$\Psi_{m_d} \quad = \quad \left[\phi_1(p) + \left(\boldsymbol{\sigma}_{(2)} \cdot \boldsymbol{p}\, \boldsymbol{\sigma}_{(3)} \cdot \boldsymbol{p} - \frac{1}{3}p^2\right) \phi_2(p)\right] |1 m_d\rangle$$

with $\phi_1 = \dfrac{1}{\sqrt{4\pi}} u(q)$ and $\phi_2 = \dfrac{3}{4q^2} \dfrac{1}{\sqrt{2\pi}} d(q)$

and $\left|\Psi_{md}\right|^2 = \int_0^\infty dq\, q^2 \left(u^2(q) + d^2(q)\right)$

^{3}H and ^{3}He :

Fachruddin, Glöckle, Elster,
Nogga PRC 69, 064002 (2004)

$$\psi_{tT}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{8} \phi_{tT}^{(i)}(\boldsymbol{p},\boldsymbol{q})\; O_i |\chi^m\rangle$$

$$|\chi^m\rangle \;=\; |(0\tfrac{1}{2})\tfrac{1}{2}m\rangle$$

$$\begin{aligned}
O_1 &= 1 \\
O_2 &= \boldsymbol{\sigma}(23)\cdot\boldsymbol{\sigma}_{(1)} \\
O_3 &= \boldsymbol{\sigma}_{(1)}\cdot(\hat{p}\times\hat{q}) \\
O_4 &= \boldsymbol{\sigma}(23)\cdot\hat{p}\times\hat{q} \\
O_5 &= \boldsymbol{\sigma}(23)\cdot\hat{q}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{p} \\
O_6 &= \boldsymbol{\sigma}(23)\cdot\hat{p}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{q} \\
O_7 &= \boldsymbol{\sigma}(23)\cdot\hat{p}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{p} \\
O_8 &= \boldsymbol{\sigma}(23)\cdot\hat{q}\;\boldsymbol{\sigma}_{(1)}\cdot\hat{q}.
\end{aligned}$$

Apply this to Faddeev equations outlined in

Glöckle, Elster, Golak, Skibinski Witala, Kamada, FBS 47, 25 (2010)

Glöckle, Fachruddin, Elster, Golak Skibinski, Witala, EPJ A43, 339 (2010)

Isospin states : $\left|\left(0\tfrac{1}{2}\right)\tfrac{1}{2}\right\rangle, \left|\left(1\tfrac{1}{2}\right)\tfrac{1}{2}\right\rangle, \left|\left(1\tfrac{1}{2}\right)\tfrac{3}{2}\right\rangle$

$$\boldsymbol{\sigma}(23) \equiv \tfrac{1}{2}(\boldsymbol{\sigma}_{(2)} - \boldsymbol{\sigma}_{(3)})$$

Thank you