Patterns Formed by Fast-growing Growing Sandpiles

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Work done with

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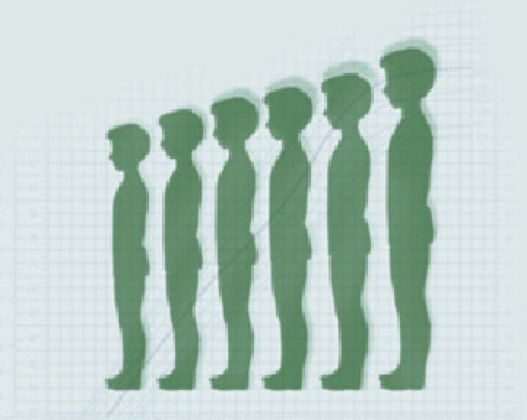
Outline

- Motivation
  - Proportionate growth
  - Pattern formation
  - Discrete analytic functions and tropical polynomials
- Definition and examples
- Compact growth
- Non-compact growth
- Tropical polynomials
- Summary

Proportionate Growth

- Animals grow in size, with different parts of body growing at roughly the same rate.
- Proportionate growth requires regulation, and/or communication between different parts.
- Most existing models of growth in physics literature
  DLA, Eden growth, KPZ growth, Invasion percolation ..
  can not model this.
- Same food becomes different tissues in different parts of the body.
- Mechanism in our model not the same as in biology

Proportionate Growth



Different body parts in animals grow roughly at the same rate.

Proportionate growth

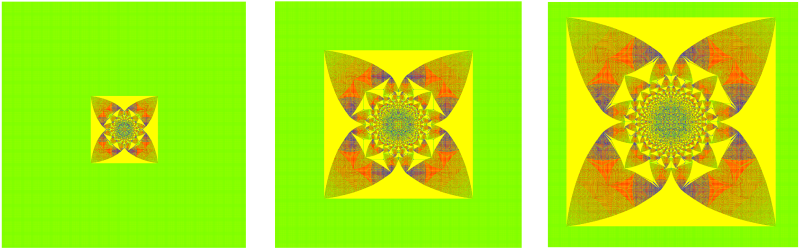


Figure: $N = (a) 4 \times 10^4$ (b) 2×10^5 (c) 4×10^5. Color code $0, 1, 2, 3 =$ R,B,G,Y

Diameter $\sim \sqrt{N}$.

Pattern formation

- Growing sandpiles give rise to beautiful complex patterns from simple local evolution rules
- Complete characterization of the asymptotic pattern in some cases
- Extra symmetry and robustness
- Effect of perturbations like boundaries and noise

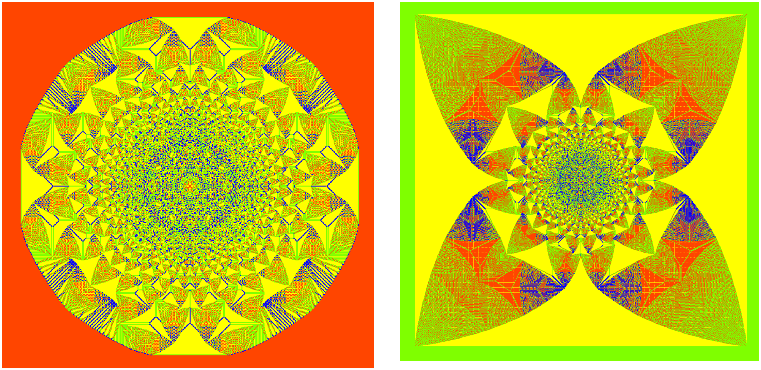


Figure: Patterns produced by adding 400000 particles at the origin, on a square lattice ASM, with initial state (a) all 0 (b) all 2.
Color code 0, 1, 2, 3 = R,B,G,Y

Mathematical motivations

Exact characterization of the pattern involves some interesting mathematics

Not fully understood

- Eigenfunctions of the Laplacian on discretized Riemann surface of many sheets
- Connection to tropical polynomials and tropical algebra
- A variational formulation using discrete piece-wise quadratic approximants?

Abelian sandpile model

- Non-negative integer height z_i at sites i of a lattice
- Add rule: $z_i \rightarrow z_i + 1$
- Relaxation rule : if $z_i > z_c$, topple, and move one grain to each neighbor.

Complex patterns in sandpile models.
Rule for forming patterns:
Add N particles at one site on a periodic background, and relax.
Deterministic patterns. This is what we study here.

Compact growth

The key observation
S. Ostojic (2003).

- Diameter $\sim \sqrt{N}$
- Proportionate growth.
- Periodic height pattern in each patch. [ignoring Transients]
- Reduced coordinates $\xi = x/\sqrt{N}, \eta = y/\sqrt{N}$
  coarse-grained density $\rho(\xi, \eta)$ is constant within a patch.
- Define
  $\phi(\xi, \eta) = Lim_{N \to \infty}(1/N)[\ \#$ of topplings at $(\xi, \eta)]$

Basic lemma

ϕ is a quadratic function of ξ, η in each patch.

$$\text{Proof :} \qquad \phi(\xi,\eta) = Lim_{N\to\infty} \frac{1}{N} T(\sqrt{N}\xi, \sqrt{N}\eta)$$

Taylor expand $\phi(\xi,\eta)$ inside a patch about (ξ_0, η_0).

$$\phi(\xi_0+\Delta\xi, \eta_0+\Delta\eta) = \phi(\xi_0,\eta_0) + ..\Delta\xi + ..\Delta\eta + a_3\Delta\xi^2 + .. + K(\Delta\xi)^3 + ...$$

In terms of toppling number function $T(X, Y)$ this becomes

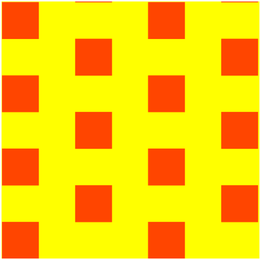
$$T(X_0 + \Delta X, Y_0 + \delta Y) = T(X_0, Y_0) + ..\Delta X + ..\Delta Y + a_3\Delta X^2 +$$

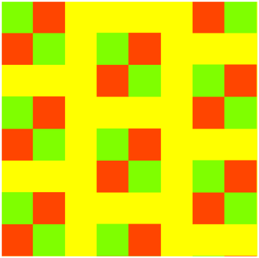
$$.. + K(\Delta X)^3/\sqrt{N}$$

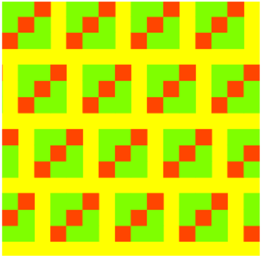
Since T is always an integer, it would jump by 1 at separations $N^{1/6}$, causing many defect lines. Hence

$$K = 0$$

Examples of periodic patterns in patches



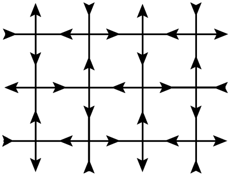




Compact growth: Simpler patterns

The square lattice pattern is not easy to characterize.

The F-Lattice.
Two arrows in and two out at each vertex.
Allowed stable heights are 0 and 1.



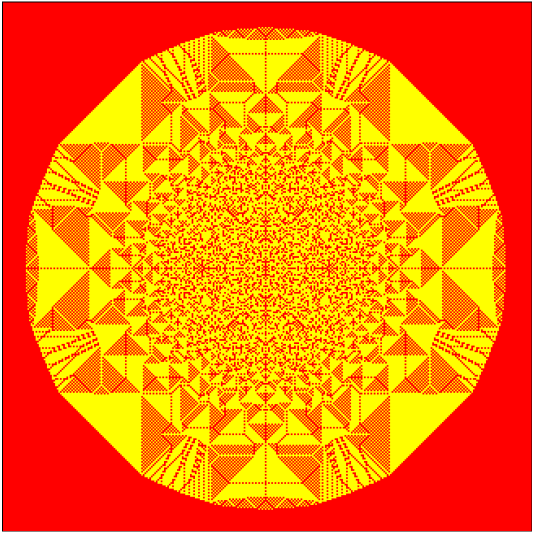


Figure: Pattern produced by adding 10^5 particles at the origin, on the F-lattice with initially empty lattice.

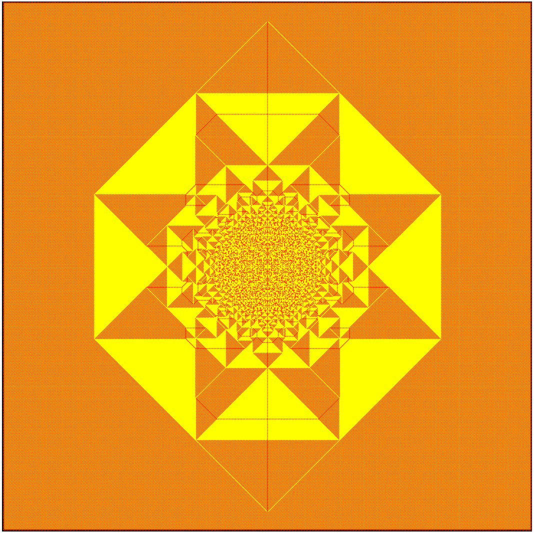


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard.

Characterizing the pattern on the F-lattice

Back-ground density 1/2

- Only two types of patches: densities 1/2 and 1.
- All boundaries are straight lines: slopes $0, \pm 1$, or ∞
- Each patch is 3- or 4- sided polygon

Quantitative characterization of the F-lattice pattern

The exact characterization involves four steps:

- Labelling patches using two integers (m, n). The adjacency graph is a discretized two-sheeted Riemann surface.
- Parameterize the potental in the (m, n) patch by

$$\phi_P(\xi,\eta) = \frac{1}{8}(m_P+1)\xi^2 + \frac{1}{4}n_P\xi\eta + \frac{1}{8}(1-m_P)\eta_P^2 + d_P\xi + e_P\eta + f_P$$

- Continuity of ϕ and derivatives implies that $d_{m,n}$ and $e_{m,n}$ both satisfy the equation

$$\psi_{m+1,n+1} + \psi_{m+1,n-1} + \psi_{m-1,n+1} + \psi_{m-1,n-1} - 4\psi_{m,n} = 0,$$

- Solve equations numerically on a large grid, to get the exact boundaries of patches

A variational formulation of the pattern selection problem

The general principle is called the (lazy man's) 'Least Action Principle':

"Don't do anything unless you have to".
The actual pattern minimizes the number of topplings to reach a stable configuration.

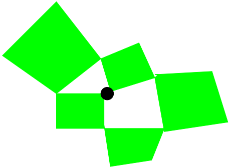
Proof is trivial for abelian models:

If a site is unstable, it will not stablize, until toppled.
Order of toppling does not matter.

Formulation as an electrostatics problem

We have $\nabla^2 \phi = +\overline{\delta z} - \delta(\xi, \eta)$
Positive point charge $+1$ at origin, and unit negative charge of areal density 1

Can we distribute the negative charge in such a way that the net potential is piecewise-quadratic, and exactly zero far away?



The answer, presumably unique, is the observed pattern on the F-lattice.
Other backgrounds have more choices of charge densities .

Robustness of the pattern

The arguments only depend on the existence of only two types of patches, and straight line boundaries.

These can be found (by trial and error) in other cases also.
Then the asymptotic pattern is identical.
Some examples:

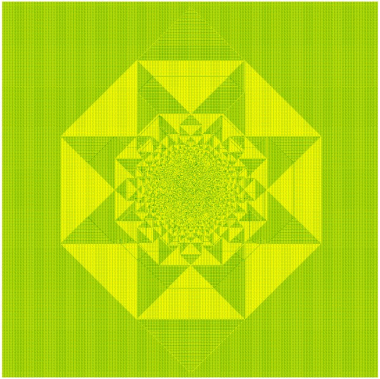


Figure: F-lattice with background density 5/8

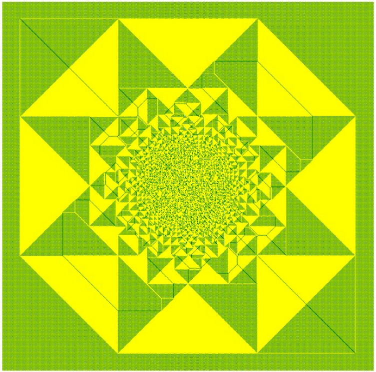


Figure: Manhattan lattice, with initial density 1/2, and 120, 000 particles

Robustness.

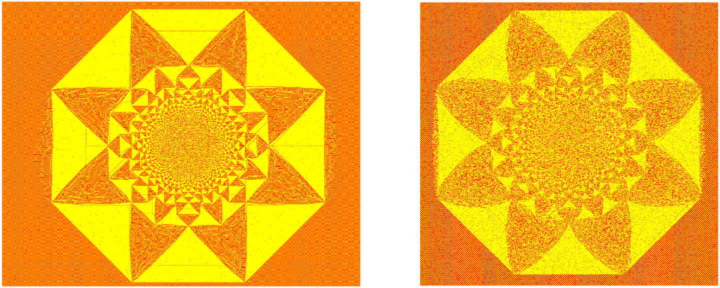


Figure: (a) 1% noise (b) 10%

Noise in the initial particle distribution.

Robustness.

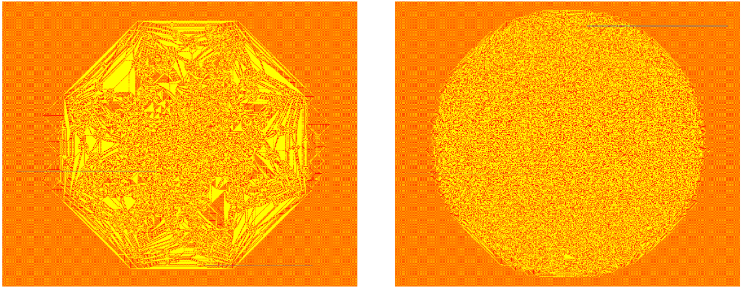


Figure: (a) 0.1% noise (b) 1%

Noise in the relaxation rule.

Fast growing sandpiles

- If the initial background density is low enough everywhere,
  $$\Lambda \sim N^{1/d}$$
- If many sites have large
  $$\Lambda = \infty \quad \text{for finite N}$$
- For an in-between set of periodic backgrounds
  $$\Lambda \sim N^{\alpha} \quad \text{for } 1/\text{d} < \alpha \leq 1$$

If $\Lambda \sim N^{\alpha}$, with $\alpha > 1/2$

The potential function for fast-growing sandpiles ϕ is piece-wise linear.

Proof:

$$\phi(\xi, \eta) = Lim_{N \to \infty} \frac{1}{N^\alpha} T(N^\alpha \xi, N^\alpha \eta)$$

Proof as before.

The slopes are rational numbers.

Hexagonal Background $\ell = 1$

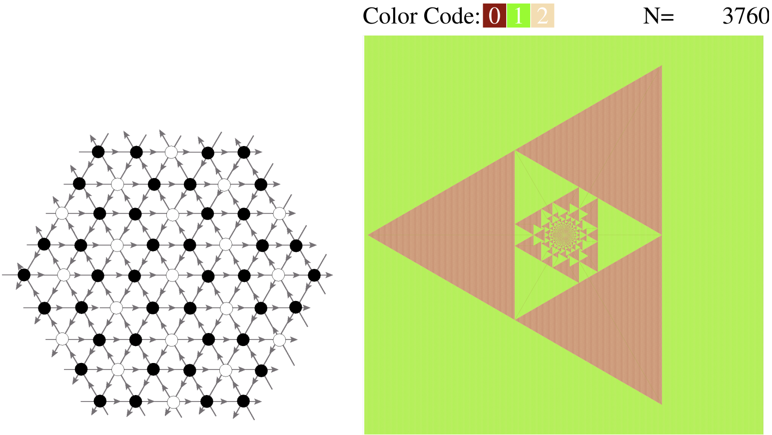
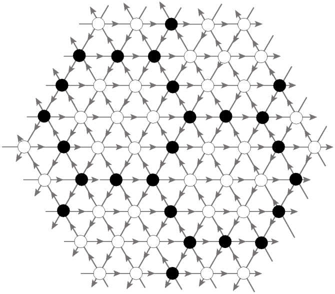
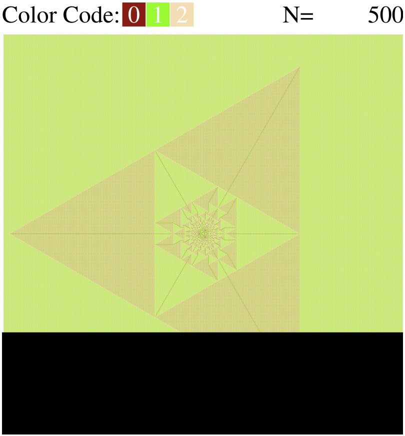


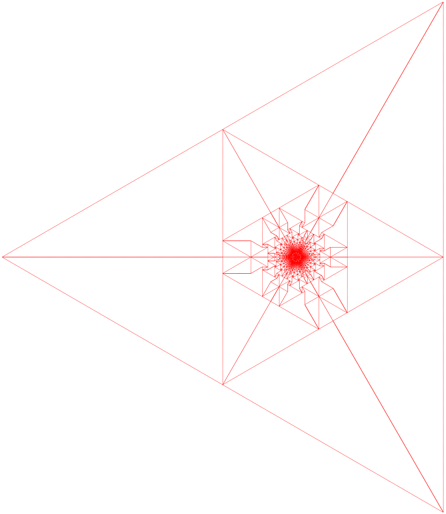
Figure: Unfilled circles=2 and filled circles=1

Hexagonal $\ell = 2$

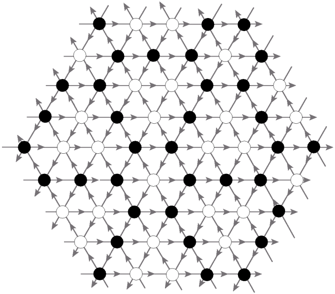


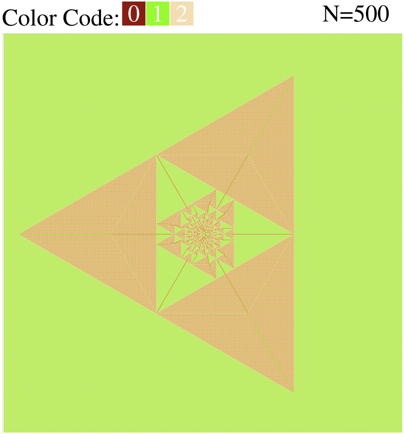


Patch boundaries

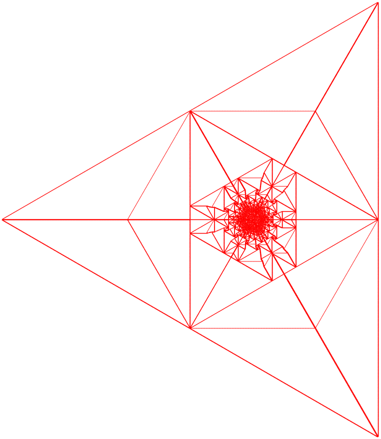


Triangular background $\ell = 4$

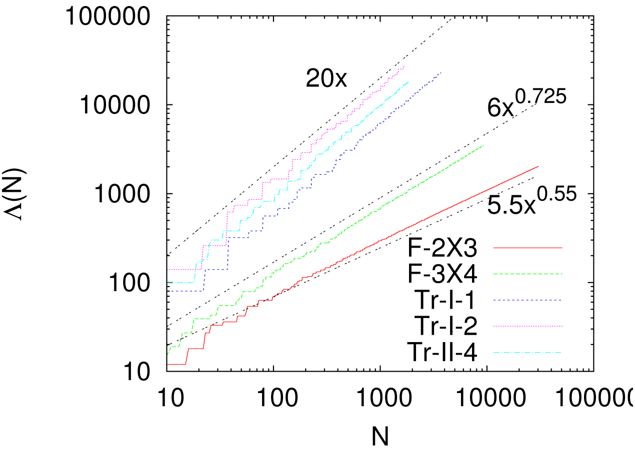




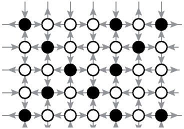
Patch boundaries

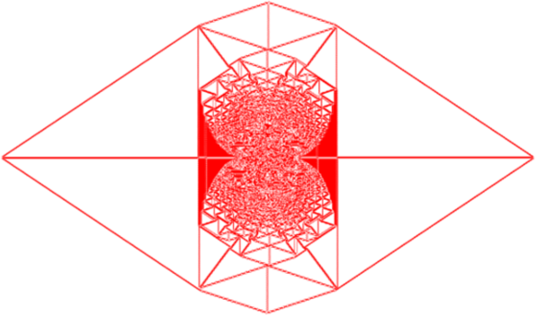


Graph of Λ vs N

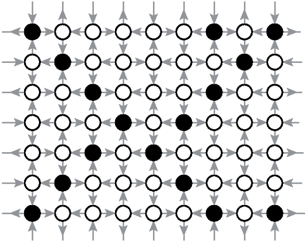


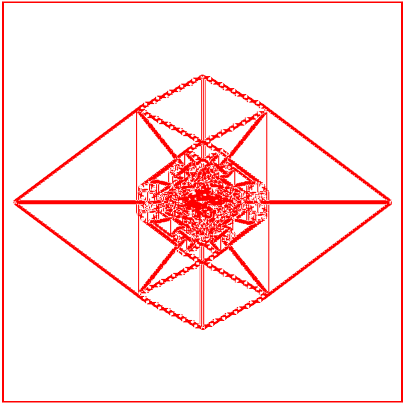
Pattern on F-lattice showing $\alpha = 0.55$





Pattern on F-lattice showing $\alpha = 0.725$





Characterizing the triangular non-compact pattern

Analysis is similar to previous case, actually simpler.

The potential function in different patches is given by

$$\phi_P = a_p\, \xi + b_P\, \eta + f_p$$

a_P and b_p are determined by matching slope discontinuity to line charge densities.

Then, f_p satisfies a Laplace's equation on the adjacency graph.

Connection to Tropical Mathematics

Define

$$a \oplus b = Max[a, b]$$

$$a \otimes b = a + b$$

Then standard properties of usual addition and multiplication (commutative, identity, distributive ..) contiue to hold.

Example: $3 \oplus 5 \oplus 2 = 5$

$3 \otimes 4 = 7$

Tropical polynomials: $a \otimes x \otimes x \oplus b \otimes x \oplus c$

Example: $x \otimes x \oplus 2 \otimes x \oplus 5 = Max[2x, x + 2, 5]$.

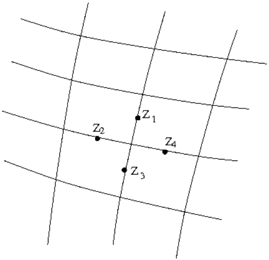
Fundamental theorem of tropical algebra.

A piecewise -linear convex function can be represented as a tropical polynomial.

Hence useful for describing the function $\phi(\xi, \eta)$.

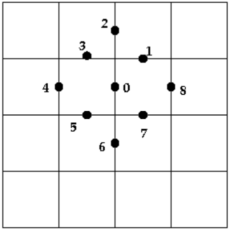
Discrete Analytic Functions

Functions defined only on discrete points in the complex $z-$ plane.



Discrete Cauchy-Riemann conditions:

$$\frac{F(z_1) - F(z_3)}{z_1 - z_3} = \frac{F(z_2) - F(z_4)}{z_2 - z_4}$$



On a square grid :

$$\Delta F_{13} + \Delta F_{35} + \Delta F_{57} + \delta F_{71} = 0$$

is equivalent to

$$\Delta F_{02} + \Delta F_{04} + \Delta F_{06} + \Delta F_{08} = 0$$

Discrete Laplace Equation.
Sum, but not product, of discrete analytic functions is also DA

simple discrete analytic functions are constant, z, z^2, z^3, $z^4 - z\bar{z}, \ldots$

Define DA function $F_{1/2}(z)$, which varies as $\sqrt{z}$ for large $|z|$, and $F(0) = 0$

The function $d(m,n) + ie(m,n)$ which characterizes the pattern for F-lattices is $cF_{1/2}(m + in)$.

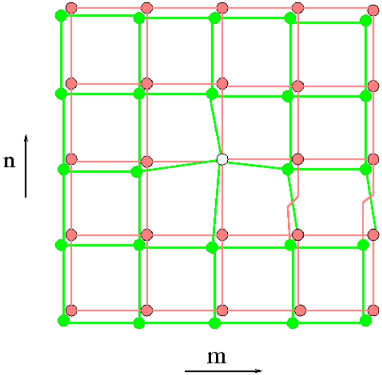


Figure: A discretized two sheeted Riemann surface for $F_{1/2}(z)$

Discrete Quadratic Approximants

Example of discrete approximants:

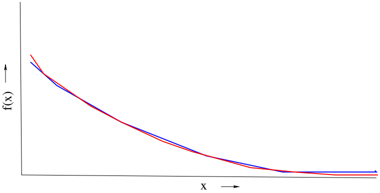


Figure: Approximate $f(x)$ by piece-wise linear functions with integer slopes

An iterative formulation

- Start with a trial pattern.
- Determine the corresponding $\phi(\xi, \eta)$
- Determine the "best" piece-wise quadratic approximants to $\phi(\xi, \eta)$ using the given set of quadratic functions ϕ_P.
- The correspond charge density is piece-wise constant. Remove singularities at boundaries.
- Determine corresponding potential $\phi^{(1)}(\xi, \eta)$.
- Iterate

If the process converges, we get the asymptotic pattern.

Summary

- We can fully characterize quantitatively patterns with only two types of patches.
- Additional (8-fold rotational) symmetry, and robustness to small noise in initial background.
- Pattern in the presence of a lines of sinks
- A large class of patterns with $\Lambda > \sqrt{N}$.
  Quantitative exact characterization in some cases.

Thank You.

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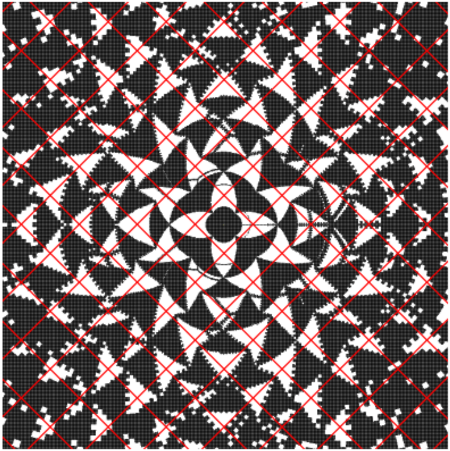

Figure: $z' = 1/z^2$ transform of original figure.