On p-groups of Gorenstein-Kulkarni type

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Abstract

A finite p-group is said to be of Gorenstein-Kulkarni type if the set of all elements of non-maximal order is a subgroup of index p. This notion is motivated from the fact that 2-groups of Gorenstein-Kulkarni type arise naturally in the study of group actions on compact Riemann surfaces. This is joint work with Siddhartha Sarkar.

1. Genus spectra

Let X be a compact Riemann surface of genus $g \geq 2$. Then **Hurwitz's Theorem** [1893] says that the automorphism group $\mathrm{Aut}(X)$ of X is finite of order $|\mathrm{Aut}(X)| \leq 84(g-1)$. Hence only finitely many groups act as a group of automorphisms on any given X.

Conversely, let G be a finite group. Then **genus spectrum** of G is defined as

$$\mathrm{Spec}(G) := \{g \geq 2; G \text{ acts on a compact Riemann surface of genus } g\}.$$

The **Riemann-Hurwitz Formula** implies that $|\mathrm{Spec}(G)| = \infty$, while the structure of $\mathrm{Spec}(G) \subseteq \mathbb{N} \setminus \{1\}$ is elucidated by the following

Theorem: [Kulkarni, 1987] Let G be a finite group.
a) There is $N(G) \in \mathbb{N}$ such that

$$\mathrm{Spec}(G) \subseteq 1 + N(G) \cdot \mathbb{N};$$

the number $N(G)$ is called the **Kulkarni invariant** of G.
b) The **spectral gap** $(1 + N(G) \cdot \mathbb{N}) \setminus \mathrm{Spec}(G)$ is finite.
c) Let $G_p \in \mathrm{Syl}_p(G)$, for all primes p. Then

$$N(G) = \frac{1}{\gamma} \cdot \prod_p \frac{|G_p|}{\exp(G_p)},$$

where $\gamma = 1$ is G_2 is of **GK type**, and $\gamma = 2$ otherwise.

Kulkarni only considers 2-groups with respect to the GK property, where he calls these groups 'of type II'. We more generally look at arbitrary p-groups:

2. Groups of GK type

Definition: Let p be a prime, and let G be a finite p-group. Then G is said to be of **Gorenstein-Kulkarni (GK) type**, if the **GK kernel**

$$\mathcal{K}(G) := \{g \in G; |g| < \exp(G)\} \subseteq G$$

is a maximal proper subgroup of G. Note that $\mathcal{K}(G)$ is in general only a subset of G; we require it to be a, then necessarily characteristic, subgroup of index p.

Are groups of GK type an interesting class of groups? Compare to the following classes of p-groups: G is said to have the **maximal exponent property (MEP)** [Maclachlan, Talu, 1998] if $\mathcal{K}(G) \trianglelefteq G$ such that $G/\mathcal{K}(G)$ is abelian; and G is called **powerful**, if either p is odd and G/G^p is abelian, or $p = 2$ and G/G^4 is abelian. Then both groups of GK type and powerful groups have MEP, but no further implications between these properties hold in general.

Is there a classification of groups of GK type? Experiments with GAP using the SmallGroups database lead to:

3. GK trees

Iterating the process of taking GK kernels yields the characteristic series

$$R := \mathcal{K}^l(G) \lhd \cdots \lhd \mathcal{K}^{i+1}(G) := \mathcal{K}(\mathcal{K}^i(G)) \lhd \mathcal{K}^i(G) \lhd \cdots \lhd \mathcal{K}(G) \lhd \mathcal{K}^0(G) := G,$$

where $R := \mathcal{K}^l(G)$ is not of GK type and called the **GK root** of G; the number $l \in \mathbb{N}$ is called the **GK level** of G.

Conversely, given a p-group R not of GK type, let $\mathcal{T}(R)$ be the directed **GK tree** of R, having R and all groups of GK type with GK root R as its vertices, and having a directed edge $G \to \mathcal{K}(G)$ for all its non-root vertices G.

This immediately leads to a bunch of questions: Does $\mathcal{T}(R)$ always have an edge? If so, is $\mathcal{T}(R)$ infinite? When does $\mathcal{T}(R)$ have an infinite stem? If so, how many stems are there, and how is branching described? Are the stems given by 'the most abelian' of the groups in $\mathcal{T}(R)$? Can this be made precise? How do the off-stem branches of $\mathcal{T}(R)$ look like? Can their depth and breadth be described in terms of R? Is $\mathcal{T}(R)$ ultimately periodic? If so, is the period even equal to 1, and when does periodicity set in?

This is now work in progress...

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