Around some experimentations in combinatorial group theory

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1 Origins

In this lecture, considering the emphasis of this conference, I'll review some of my combinatorial and computational experimentations in group theory, influenced by Coxeter groups. My earliest group computer calculations, or better said, fiddling with groups, go back to the early 1970's when I used a stand-alone Todd-Coxeter program furnished by John Cannon.

Contents of the lecture.

- *Commutativity and Finiteness.*

  A nonsimplicity criterion; Weak Permutability and a finiteness criterion; Computations using GAP*;* Weakly commuting n copies of a group.

- *Linear groups in characteristic* 2*.*

The family $Y(m,n)$: finiteness and representations; Linear Groups over Laurent Polynomial Rings; Spinor Groups.

2 Commutativity and Finiteness

Commutativity in a group can be depicted by a graph having for vertices the elements of the group where two vertices are joined by an edge if and only if the corresponding elements commute. This graph has been useful in a number of instances. We refer to this graph for the purpose of visualization only.

2.1 A nonsimplicity criterion

It is an elementary, yet fundamental fact that finite p-groups are nilpotent. It follows easily that an automorphism of order a power of p of a non-trivial finite p-group has a nontrivial centralizer.

Consider the following combinatorial variant:

Question. *Suppose a finite group G contains a non-trivial elementary abelian p-subgroup $A = A_{p,k}$ of rank k such that every element of order p in G centralizes some nontrivial element in A. Does it follow that G contains a non-trivial normal p-subgroup?*

Precursors. *Suppose $p = 2$. Then, $A \cap O_2(G) \neq \{e\}$* when *(i) $k = 2$* **(***Shult 1970, Alperin 1972);* (ii) $k = 3$ **(***Sidki, 1976)*.

Modern. In 2007, by making significant use of the classification theory of finite simple groups, Aschbacher-Guralnick-Segev proved $A \cap O_2(G) \neq \{e\}$ for $p = 2$ and all k,

2.2 Weak Permutability and a Finiteness Criterion

In the above context, what are the possibilities for $W = < A, A^g >$? Suppose the weak commutation is determined by a bijection. Is it then possible to say something about the order of W? Is W a p-group?

The interaction between the two copies of A may be viewed more generally as a *weak form of permutablity* between two groups H, K, in the sense that $HK \cap KH$ contains a set of the same size as H.

Theorem (1980) *Let H, K be finite groups having equal orders n and let $f : H \to K$ be a bijection which fixes the identity. Then for any two maps $a : H \to K, b : H \to H$, the group*

$$G(H, K; f, a, b) = \left\langle H, K \mid hh^f = h^a h^b \text{ for all } h \in H \right\rangle$$

is finite of order at most $n \exp(n-1)$.

So, here $HK \cap KH$ contains the set $\left\{ hh^f \mid h \in H \right\}$. The essential idea of the proof already occurs in Sanov's finiteness of Burnside groups of exponent 4:

let $h \in H$, $k, k' \in K$ be non-trivial elements. Then

$$\begin{aligned} khk' &= k(hh^f)(\left(h^f\right)^{-1} k') \\ &= k(h^a h^b)(\left(h^f\right)^{-1} k') \\ &= (kh^a)\, h^b \,(\left(h^f\right)^{-1} k') \in KHK; \end{aligned}$$

note that the end elements k', $\left(h^f\right)^{-1} k'$ are different. As the word gets longer, we produce more equivalent forms with different end elements.

Let $t : H \to K$ be an isomorphism and write $K = H^t$. The exponential upper bound is justified by the group

$$G =< H, H^t \mid \left(hh^t\right)^2 = e \text{ for all } h \in H > .$$

This group G is the semidirect product of the augmentation ideal of the group algebra $\mathbb{Z}_2[H]$ by H, under the natural action; therefore its order is $2^{n-1}n$.

Questions. *Is $2^{n-1}n$ in the theorem the real upper bound? What groups have orders exponential in n?*

The finiteness criterion holds more generally under the following conditions: let H, K be two finite groups of orders $n \geq m$, respectively, and let $f : H \to K$ be a surjection such that

(i) $f(e) = e, f^{-1}(e) = \{e\}$,

(ii) for all subsets S of K and all $h \in H$

$$\left| f(f^{-1}(S)h) \right| \geq |S|$$

This allows pairs (n, m) such that $m - 1$ divides $n - 1$.

Problem. *Study the dynamics of such functions f.*

- An example of a group with polynomial upper bound is $G = PSL(2, p)$ where p is an odd prime. The following

  $$< a, b \mid a^p = b^p = e, \left(a^i b^j\right)^2 = e \text{ for all } ij \equiv 1 \bmod p >$$

is a presentation of $PSL(2,p)$; where $a \rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $b \rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. A substantial reduction in the relations is possible: $i = 1, 2, 4$ are sufficient.

- Another example: let H, K be (not necessarily finite) groups *isomorphic* via t and define

  $$\chi(H) =< H, K \mid \left[h, h^t\right] = e \text{ for all } h \in H > .$$

  We studied this group in great generality. It has a section isomorphic to the Schur Multiplier of H and notably its subgroups $[H, H^t], [H, t]$ commute. Further works were done by Rocco and by Gupta-Rocco-Sidki. Subsequently, it was observed that this group was related to the so called noncommutative tensor square of groups, originating in homotopy theory.

2.3 Computations using GAP

Let A, B be isomorphic to $A_{p,m}$, elementary abelian p-group of rank m, $f : A \to B$ be a bijection fixing e, not necessarily an isomorphism. Denote the corresponding group G by $G\left(A_{p,m}; f\right)$. Then f can be seen as a permutation of A fixing e and this permutation may be chosen as a representative of a double coset from $SL(m,p) \backslash Sym\left(p^m - 1\right)/SL(m,p)$.

We give evidence to the conjecture that $G\left(A_{p,m}; f\right)$ is again a finite p-group in

Oliveira-Sidki, *On Commutativity and Finiteness in Groups*, Bull. Braz. Math. Soc. (2009).

The following computational data is obtained using a double coset program by Alexander Hulpke in GAP:

let c denote the nilpotency class and d the solvability degree of G;
(i) for $A_{2,3}$

$$SL(3,2)\backslash Sym\,(7)\,/SL(3,2) \text{ has 4 double cosets}$$

which produce 4 non-isomorphic groups and

f	$\left\|G\left(A_{2,3};f\right)\right\|$	c	d
$()$	2^{10}	3	2
$(6,7)$	2^{10}	3	2
$(6,7,8)$	2^{8}	2	2
$(5,6,7,8)$	2^{8}	2	2

(ii) for $A_{2,4}$,

$SL(4,2)\backslash Sym\,(15)\,/SL(4,2)$ has 3374 double cosets.

which produce groups having orders

$$2^9, 2^{10}, 2^{11}, 2^{12}, 2^{13}, 2^{15}, 2^{19}.$$

There are 5 representatives f for which the groups have maximum order. The corresponding groups are non-isomorphic and

f	$G\left(A_{2,4}; f\right)$	c	d
$()$	2^{19}	4	2
$(15,16)$	2^{19}	3	3
$(11,14)(15,16)$	2^{19}	5	3
$(9,11)(10,13)(12,14)$	2^{19}	5	3
$(9,12)(10,13)(11,14)$	2^{19}	4	2

.

(iii) for $A_{3,3}$, we consider instead permutations of the cyclic subgroups of A. Then

$PGL\left(3,3\right)\backslash Sym\left(13\right)/PGL\left(3,3\right)$ has 252 double cosets

The corresponding groups have orders $3^6, 3^7, 3^8, 3^9$, all have class at most 2.

Here, we see clearly a basic difference between $p = 2$ and p odd.

2.4 Weakly commuting n copies of a group

We go back to

$$\chi(H) = \left\langle H, H^t \mid hh^t = h^t h \text{ for all } h \in H \right\rangle$$

were t can be thought of having order 2.

- Let $H = A_{2,2}$. The following group

  $$\left\langle H, t \mid t^3 = 1, \left[h, h^t\right] = 1 \text{ for all } h \in H \right\rangle$$

  is infinite, an extension of $\mathbb{Z}^4$ by a finite group of order $2^{13}.3$.

In $\chi(H)$, the two subgroups $[H, H^t], [H, t]$ commute. We define the groups

$$\begin{aligned}
\kappa(H, n) &= \left\langle \begin{array}{c} H, t \mid t^n = 1, \left[h, h^{t^i}\right] = 1, \\ \left[[H,^{t^i} H^{t^j}], [H, t^k]\right] = 1 \\ \text{for } 1 \le i, j, k \le n-1 \end{array} \right\rangle, \\
\chi(H, n) &= \left\langle H^{t^i} \mid i = 1, ..., n-1 \right\rangle.
\end{aligned}$$

The group $\chi(H,n)$ preserves finiteness, and other group theoretic properties such as finitely generated.

3 Linear groups in characteristic 2

We will construct another group generated by a set of mutually weakly permutable groups.

Recall that when $f : A \rightarrow B$ is an onto isomorphism, the group

$$G =< A, B \mid \left(aa^f\right)^2 = e \text{ for all } a \in A > ,$$

is the semidirect product of the augmentation ideal $\omega_{2,n}$ of $GF(2)[A]$ by A.

- By a theorem of Coxeter,

$$
\begin{aligned}
G(m) &= < a_1, a_2, ..., a_m |\ a_k^3 = e \text{ for all } 1 \leq k \leq m, \\
\left(a_k a_l^{-1}\right)^2 &= e \text{ for all } 1 \leq k < l \leq m > .
\end{aligned}
$$

is finite only for $m \leq 3$.

- On the other hand, by Carmichael, the following

$$
\begin{aligned}
&< \ a_1, a_2, ..., a_m |\ a_k^3 = e \text{ for all } 1 \leq k \leq m, \\
(a_k a_l)^2 &= e \text{ for all } 1 \leq k \leq l \leq m >
\end{aligned}
$$

is a presentation of the alternating group $Alt(m+2)$.

We generalized this presentation in 1982 in the form:

$$
\begin{aligned}
Y(m,n) &= < a_1, a_2, ..., a_m |\ a_k^n = e \text{ for all } 1 \leq k \leq m, \\
\left(a_k^i a_l^i\right)^2 &= e \text{ for all } 1 \leq k < l \leq m,\ 1 \leq i \leq \frac{n}{2} > .
\end{aligned}
$$

Open Conjecture. *The group $Y(m,n)$ is finite, for all m,n.*

If n is odd greater than 1 then the group $Y(m,n)$ contains a subgroup isomorphic to the symmetric group $\mathbf{\Sigma}_m$, from which we obtain a new presentation

$$\begin{aligned} y(m,n) &= \; < a,\ \mathbf{\Sigma}_m \mid a^n = e, \\ \left[\tau_{12}^{a^i}, \tau_{12}\right] &= e \ \left(1 \le i \le \frac{n}{2}\right), \tau_{12}^{1+a+\ldots+a^{n-1}} = e, \\ \tau_{i,i+1} a \tau_{i,i+1} &= a^{-1} \ (2 \le i \le m-1) > \end{aligned}$$

which can be depicted as an extended Coxeter diagram.

The group $y(3,n)$ affords a representation into $SL(2,F)$ where F is a field of characteristic 2 containing an element α of order n:

$$a \to \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \tau_{12} \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tau_{23} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Indeed, the representation can be extended to higher dimension by tensoring, as

$$a \to \begin{pmatrix} \alpha I_r & 0 \\ 0 & \alpha^{-1} I_r \end{pmatrix}, \tau_{12} \to \begin{pmatrix} I_r & 0 \\ I_r & I_r \end{pmatrix}, \tau_{23} \to \begin{pmatrix} 0_r & I_r \\ I_r & 0_r \end{pmatrix}$$

which creates space to allow representations of $y(m,n)$ for $m \geq 3$.

For $n \geq 5$, the linear groups thus generated turn out to involve orthogonal groups in higher dimensions. To exemplify,

$$\begin{aligned}
y(3,5) &\cong SL(2,16) \cong \Omega^-(4,4), \\
y(4,5) &\cong \Omega(5,4), \\
y(5,5) &\cong \Omega^-(6,4), \\
y(6,5) &\cong 4^6\Omega^-(6,4);
\end{aligned}$$

$$\begin{aligned}
y(3,7) &\cong \Omega^+(4,8), \\
y(4,7) &\cong \Omega(5,8).
\end{aligned}$$

The isomorphisms in this list were obtained by a combination of representation theory and computer coset enumeration. The periodicity of groups in the list is explained in my paper of 1982 and it is probably related to the Bott periodicity. The list was expanded in 1987 for larger (m,n) in collaboration with J. Neubuser and W. Felsch at RWTH in Aachen; for $n = 5$, the list reached $y(10, 5)$. During this conference E. O'Brien has reproduced and expanded the list even further to

$$y(12,5) \cong \Omega(11,4), y(8,7) \cong \Omega^+(7,8), y(5,11) \cong \Omega^+(6,32).$$

3.1 Linear Groups over a Laurent Polynomial Ring

We drop the relations $a_i^n = 1$ from $Y(3, n)$; that is, define

$$\begin{aligned} Y(3) &= < a_1, a_2, a_3 | \quad \left(a_k^i a_l^i\right)^2 = e \\ \text{for all } 1 &\leq \quad k < l \leq 3, \quad i \geq 1 > . \end{aligned}$$

.

Then $Y(3) \to SL(2, \mathbb{Z}_2[t, t^{-1}])$:

$$a \to \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \tau_{12} \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tau_{23} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let R be the ring of Laurent polynomials $k[t,t^{-1}]$ with coefficients from a general field k (commutative or not), $s = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in $SL(2,R)$ and the elementary subgroup $e(2,R) =< SL(2,k), s >$. We gave in J. Algebra 1990 a presentation of the group $e(2,R)$. On specializing R to $\mathbb{Z}_2[t,t^{-1}]$, the group $Y(3)$ is shown to be the normal closure of $< s >$ in $e(2,R)$. Now, since the congruence subgroup theorem holds for $\mathbb{Z}_2[t,t^{-1}]$, adding the relation $a^n = 1$ to the above presentation corresponds to making $t^n = 1$ and with this we reach

$$Y(3,n) \cong SL(2,\omega_{2,n})$$

where $\omega_{2,n}$ is the augmentation ideal of $\mathbb{Z}_2[C_n]$.

3.2 Spinor Groups

Another line of development was pursued by Claus Halkjaer in 1996, in his doctoral thesis at the University of Brasilia, with the title

"On Clifford Algebras $C(m)$ in prime characteristic and a class of geometric subgroups of $GL(2, C(m))$".

In it, the groups $y(m,n)$ were generalized to $y_p(m,n)$ where p is any prime number. This was done by making $\tau_{12}^p = e$ and by replacing the symmetric group $< \tau_{23}, ..., \tau_{m-1,m} >$ by one of its well-known central coverings. It was shown that $y_p(m,n)$ maps onto spinor groups in characteristic p, by using the same representation theoretic approach we had devised for $y(m,n)$. Here, the representations are interpreted as ones into $GL(2, C(m))$ where $C(m)$ are Clifford algebras in characteristic p. Though, for p odd, the relations do not guarantee anymore finiteness of the groups.