Synchronization of Networks

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Plan of presentation

- Introduction
  - Synchronization
  - Networks
- Impulsive synchronization
- Synchronization of several coupled systems
  - Time varying networks
  - Delay / Anticipatory synchronization
- Applications
  - Parameter estimation
  - Extinction of Species

Introduction - Control

- An important property of a chaotic system is sensitivity to initial conditions. Two neighbouring trajectories tend to move away from each other exponentially.
- Hence one may think that it will be difficult to harness chaos.
- However, by a clever use of the properties of a chaotic system it is possible to harness chaos.

- OGY control - periodic orbits

  E. Ott, C. Grebogi and J. A. Yorke, Phys. Rev. Lett **64**, 1196 (1990).
- Synchronization - both periodic and chaotic orbits

  L. M. Pecora and T. A. Carroll, Phys. Rev. Lett **64**, 821 (1990).

Synchronization: Huygens (1665)

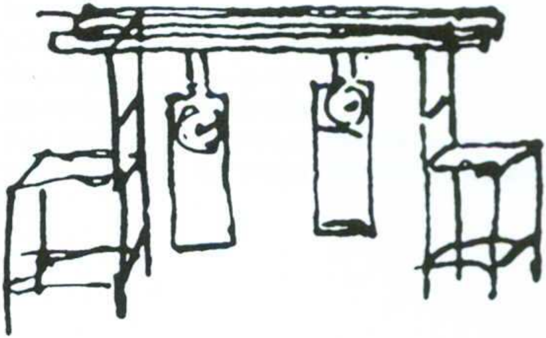
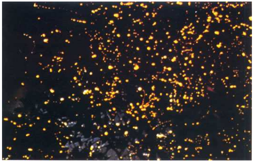


Figure 1.2. Original drawing of Christiaan Huygens illustrating his experiments with two pendulum clocks placed on a common support.

Two Pendulum clocks hung from the same support.

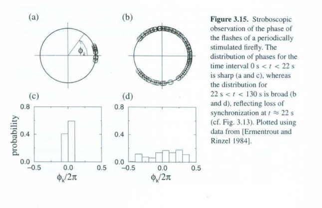
Synchronization: Firefly

Fireflies Sci. American May 76



Synchronus lightening

Synchronus lightening



Phase synchronization

Synchronization: Prey-Predator

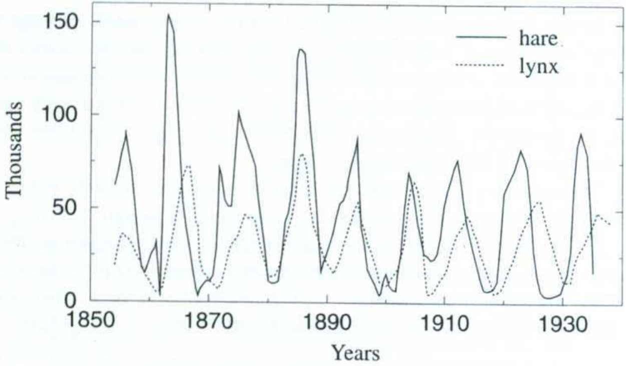


Figure 1.13. A classical set of data (taken from [Odum 1953]) for a predator–prey system: the Canadian lynx and snowshoe hare pelt-trading records of the Hudson Bay Company over almost a century. The notion of synchronization is not appropriate here because the lynxes and hares constitute a nondecomposable system.

Synchronization

- Synchronize the trajectory of one chaotic system with that of another chaotic system.
- Exact/perfect synchronization: $\mathbf{x} = \mathbf{x}^{'}$.
- The basic idea:
  - Control part of the system.
  - Introduce some coupling.

  Ideally the control must be such that its magnitude becomes zero in the synchronized state.

Synchronization

Synchronization with linear coupling Consider the dynamical system

$$\dot{x} = f(x)$$

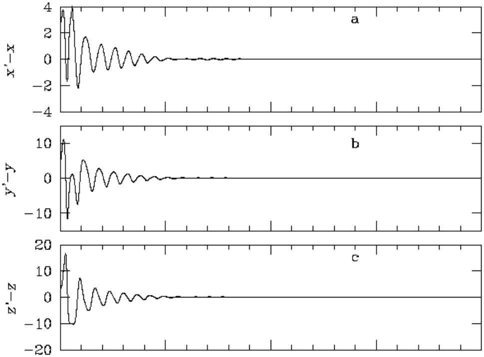
Couple two identical dynamical systems

$$\begin{aligned} \dot{x} &= f(x) + \epsilon_1 \Gamma(x' - x) \\ \dot{x'} &= f(x') + \epsilon_2 \Gamma(x - x') \end{aligned}$$

Under suitable conditions $|x - x'| \to 0$ as $t \to \infty$.

Synchronization - Lorenz System

$$\begin{aligned} \dot{x} &= \sigma(y-x)+\epsilon(x'-x) \\ \dot{y} &= x(z-r)-y \\ \dot{z} &= xy-bz \end{aligned} \qquad \begin{aligned} \dot{x}' &= \sigma(y'-x')+\epsilon(x-x') \\ \dot{y}' &= x'(z'-r)-y' \\ \dot{z}' &= x'y'-bz' \end{aligned}$$



The condition for synchronization: The transverse Lyapunov exponents are all negative.

Different types of synchronization

- Exact synchronization: $\mathbf{x}^{'} = \mathbf{x}$.
- Phase synchronization: In general it is not easy to define a phase variable for chaotic systems. However, in many systems it is possible to introduce a suitable definition of a phase variable, i.e. for a Rössler system $\tan^{-1}(y/x)$ can be used as a phase variable.
- Generalized synchronization: There is a functional relation between the variables of the two systems.
- Delay synchronization: $\mathbf{x}^{'}(t) = \mathbf{x}(t - \tau)$.
- Anticipatory synchronization: $\mathbf{x}^{'}(t) = \mathbf{x}(t + \tau)$.

N coupled dynamical systems

- A network of N nodes and one oscillator on each node.
- The coupling between the oscillators is given by the edges of the network.

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}(\mathbf{x}^i(t)) + \sum_j G_{ij}\Gamma\mathbf{u}(\mathbf{x}^j(t)).$$

- $\mathbf{x}^i,\ i = 1, \ldots, N$, dynamical variables.
  m – dimension of each system. mN – total dimension.
- $\mathbf{f}$ – local dynamics, $\mathbf{u}$ – coupling function
  G – coupling matrix ($N \times N$)
  Γ – matrix defining the way the components are coupled ($m \times m$)

Single cluster synchronization (1CS)

- Definition of Single cluster synchronization (1CS)

$$\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N = \mathbf{x}$$

- Condition on the coupling matrix

$$\sum_j G_{ij} = g, \ \forall i.$$

- The synchronized state: It is a solution of

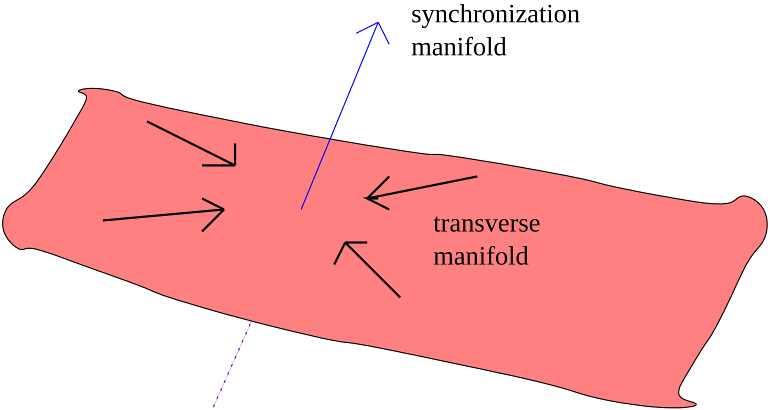
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + g\mathbf{u}(\mathbf{x}).$$

If $g = 0$ then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

Stability of 1CS

- We divide the phase space into two parts.
  - Synchronization manifold: Def. of Synchronization manifold is $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N = \mathbf{x}$.
    Dimension $= m$.
    Coupling matrix G has one eigenvector of the type $e_1^R = (1, \ldots, 1)^T$ with eigenvalue $\gamma_1 = g$ and defines the synchronization manifold.
  - Transverse manifold: All transverse directions.
    Dimension $= m(N-1)$.
- Condition for the stability of synchronized state:
  All Lyapunov exponents in the transverse directions must be negative.

Stability – Manifolds



Schematic diagram of synchronization and transverse manifolds.

Linear stability

- Linearization
  Let $\mathbf{z}^i = \mathbf{x}^i - \mathbf{x}$,
  $Z = (\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^N)$.

$$\dot{Z} \;=\; D\mathbf{f}\ Z + D\mathbf{u}\ Z\ G^T,$$

- Eigenvalues and eigenvectors of G^T
  $\gamma_k,\ e_k^L,\ \ k = 1, \ldots, N.$

$$\dot{Z} e_k^L = D\mathbf{f}\ Z e_k^L + D\mathbf{u}\ Z \gamma_k e_k^L.$$

Block diagonal form

Let $\phi_k = Ze_k^L$. (dimension m)

$$\dot{\phi}_k = [D\mathbf{f} + \gamma_k D\mathbf{u}]\phi_k,$$

where $k = 1, \ldots, N$.
For each k we can calculate the Lyapunov exponents.
For the stability of the synchronized state all transverse Lyapunov exponents ($k = 2, \ldots, N$) must be negative.

Time-varying networks

- Many natural networks have topologies changing with time.
- Periodical switches between couling matrices $G_1, G_2, \ldots, G_g$ with periods $\tau_1, \tau_2, \ldots \tau_g$.

$$G(t) = \sum_{i=1}^{g} G_i \chi_{[t_{i-1}, t_i]}$$

$\chi_{[t_{i-1}, t_i]}$ is an indicator function.

- The time averaged $G(t)$ is

$$\bar{G} = \frac{1}{T} \sum_{i=1}^{g} G_i \tau_i$$

Time-varying networks

- Condition for synchronization:
  If the network synchronizes for the static time-average of the topology, i.e. with $\bar{G}$, then the network will synchronize with the time-varying topology if the time-variation is done sufficiently fast.
- It is interesting to note that the synchronized state can become stable even when the individual networks do not support the synchronized state.

Time-varying networks

Divide the time-varying networks into two classes.
I. Commuting Matrices:

$$[G_i, G_j] = 0, \text{ for i, j} = 1, \cdots, \text{g}.$$

The different coupling matricess G_i and also the average $\bar{G}$ have the same set of eigenvectors though the eigenvalues are different. It can be shown that if the switching is sufficiently fast

$$\bar{\lambda}_{kj} \approx \lambda_{kj} = \frac{1}{T} \sum_{i=1}^{g} \lambda_{kj}^{i} \tau_i$$

Thus, the time-varying case has the same stability range as that of the time-average case.

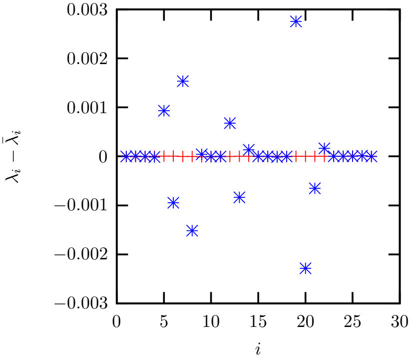
Time-varying networks

II. Noncommuting matrices: The different coupling matrices have different sets of eigenvectors. Hence, as we switch from one matrix to another the set of eigenvectors undergoes a rotation. This rotation has the effect of narrowing the spread of the Lyapunov exponents in the transverse manifold. Hence, largest transverse Lyapunov exponent decreses.

The time-varying case has a better stability than the time-average case.

(REA and C. K. Hu, Chaos **16**, 015117 (2006).)

Time-varying networks - Lyapunov exponents

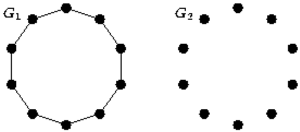


Difference between Lyapunov exponents of time-varying and time-average networks.

Time-varying networks - examples

Example Coupled Rössler systems.

- Ex. 1: The commuting class:
  Stability range of synchronized state, $\sigma \in (0.75, 2.30)$ for both t-varying and t-average case.

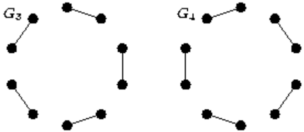


Time-varying networks - examples

Example Coupled Rössler systems.

- Ex. 2: The non-commuting class:
  Stability range of synchronized state
  - The t-varying case: $\sigma \in (0.70, 2.30)$
  - The t-average case: $\sigma \in (0.75, 2.30)$.

  Thus, the lower limit which corresponds to the long-wavelength instability gets extended for the t-varying case.

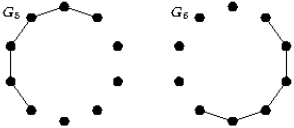


Time-varying networks - examples

Example Coupled Rössler systems.

- Ex. 3: The non-commuting class:
  Stability range of synchronized state
  - The t-varying case: $\sigma \in (0.75, 2.45)$
  - The t-average case: $\sigma \in (0.75, 2.30)$.

  Thus, the upper limit which corresponds to the short-wavelength instability gets extended for the t-varying case.



Critical coupling constant

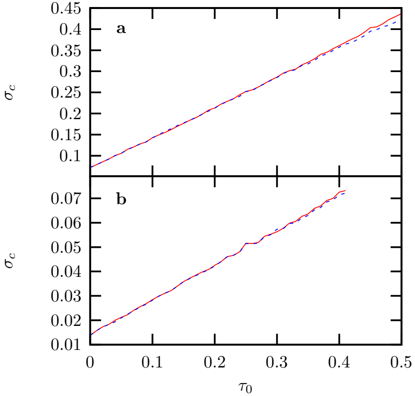
Expt 1:

- 1. Consider any network, A_N.
  2. Null network, A_{N0}, zero edges.
- Switch between $A_N \Longleftrightarrow A_{N0}$.
  Switching times τ_1 and τ_0.
- The critical coupling constant

$$\sigma_c \approx \bar{\sigma}_c = \sigma_{c0} + \frac{\tau_0}{\tau_1}\sigma_{c0}$$

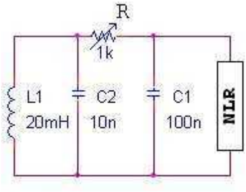
where A_N synchronizes for $\sigma > \sigma_{c0}$.

Critical coupling constant

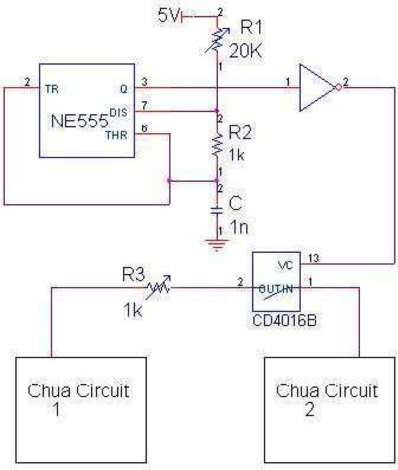


a. 2 nodes - coupled Rössler systems
b. 10 nodes completely connected

Chua circuits



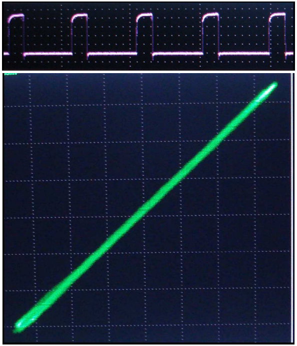
Single Chua circuit



Switching network of two Chua circuits

Chua circuits

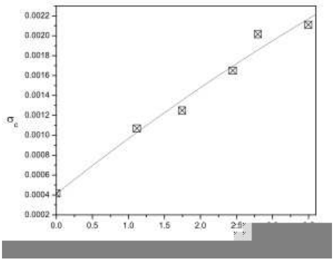
a



b

a. Control pulse

b. synchronization



σ_c vs τ_0

Critical coupling constant

Expt 2:

Two networks with critical coupling constants σ_{c1} and σ_{c2}.
switching times τ_1 and τ_2.

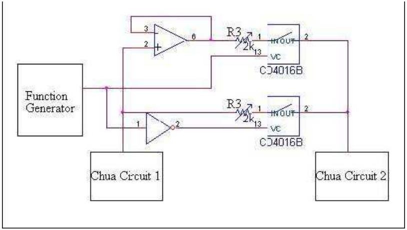
$$\sigma_c = \frac{\tau_1 \sigma_{c1} + r \tau_2 \sigma_{c2}}{\tau_1 + r \tau_2}$$

where $r = s_2/s_1$ and $s_i = (\frac{\partial \lambda_i}{\partial \sigma})_{\sigma = \sigma_{ci}}$

$$A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

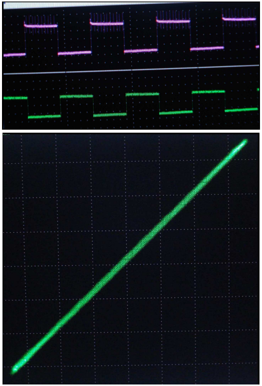
$$A_{21} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}; \;\; A_{22} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

Chua circuits



Unidirection $\leftrightarrow$ Bidirection

a

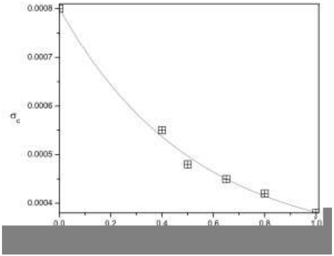


b

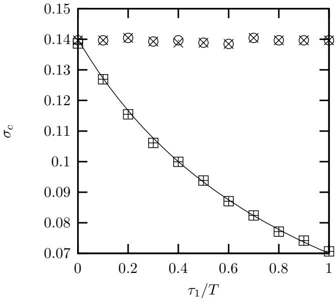
a. pulse using two switches

b. synchronization

Chua circuits



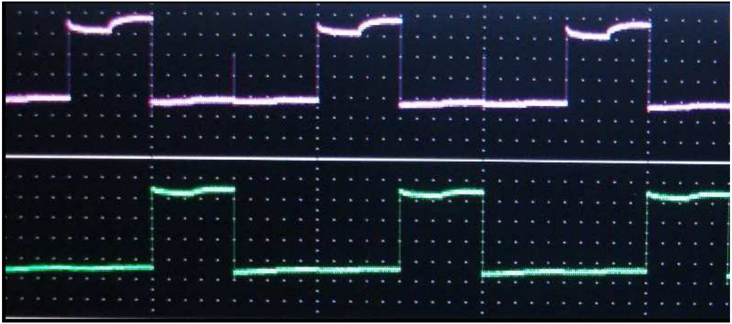
σ_c vs τ_1/T



Rössler system - numerical

Expt 3

For commuting matrices the order does not matter.
$A_2 \rightarrow A_{21} \rightarrow A_0 \rightarrow \ldots$
$A_2 \rightarrow A_0 \rightarrow A_{21} \rightarrow \ldots$



Two pulses for switching between three networks

Delay/Anticipatory synchronizarion

Two coupled systems

$$\begin{aligned}\dot{x} &= f(x)\\ \dot{y} &= f(y) + \epsilon\,(x_{t_1} - y_{t_2})\end{aligned}$$

where $z_{t_i} = z(t - t_i)$.
Under suitable conditions we have the synchronization as

$$y(t) = x(t + t_2 - t_1)$$

- $t_1 = t_2$ – Normal synchronization
- $t_1 > t_2$ – Delay Synchronization
- $t_1 < t_2$ – Anticipatory Synchronization

Variable delay

We assume that t_1 and t_2 vary with time.

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= f(y) + \epsilon \sum_{m=0}^{\infty} \Gamma\left(x_{t_1} - y_{t_2}\right) \chi_{(m\tau,(m+1)\tau)} \end{aligned}$$

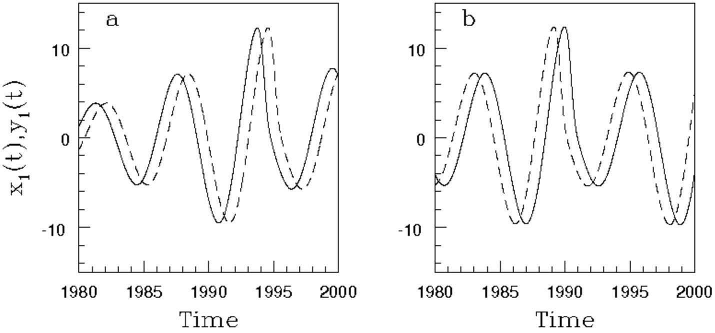
We choose the following time dependence

$$t_i = \tau_i + t - m\tau, \quad i = 1, 2.$$

$$t - t_i = m\tau - \tau_i$$

Here, τ is the reset time.

Delay/Anticipatory synchronization



delay anticipatory

Coupled Rössler systems

Linear stability analysis

- Three time scales: $\tau_1,\ \tau_2,\ \tau$.
- The transverse system: $\Delta = y - x_{\tau_1 - \tau_2}$.

$$\dot{\Delta} = f^{'}(x_{\tau_1-\tau_2})\Delta - \epsilon \sum_{m=0}^{\infty} \chi_{(m\tau,(m+1)\tau)} \Delta_m$$

  where $\Delta_m = \Delta(m\tau - \tau_2)$.
- Approximate solution can be obtained.
  - $f^{'}$ is some effective constant, say λ.
  - Δ is a scalar.
- In the interval $m\tau \leq t < (m+1)\tau$, the solution is

$$\Delta = \alpha\Delta_m + C_m e^{\lambda t}$$

  where $\alpha = \epsilon/\lambda$, and C_m is an integration constant.

Linear stability analysis

- For $0 \le \tau_2 \le \tau$: We get the recursion relation

$$\Delta_{m+1} = a\Delta_m - b\Delta_{m-1}$$

where $a = \alpha(1 - e^{\lambda(\tau-\tau_2)}) + e^{\lambda\tau}$ and $b = \alpha e^{\lambda\tau}(1 - e^{-\lambda\tau_2})$.

- This gives a 2-d map

$$\begin{pmatrix} \Delta_{m+1} \\ \Delta_m \end{pmatrix} = \begin{pmatrix} a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_m \\ \Delta_{m-1} \end{pmatrix}$$

- Delay/anticipatory synchronized state: $\Delta = 0$.

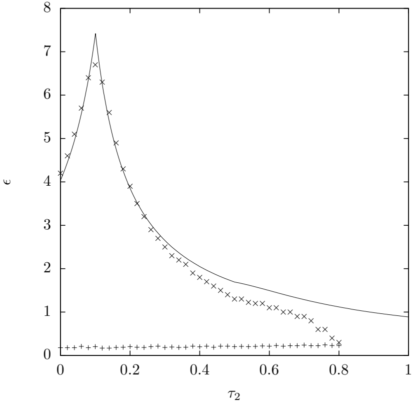
Linear stability analysis

- For $k\tau < \tau_2 < (k+1)\tau$: $k+2$ dimensional map.

$$\begin{pmatrix} \Delta_{m+k+1} \\ \Delta_{m+k} \\ \Delta_{m+k_1} \\ \vdots \\ \Delta_m \end{pmatrix} = \begin{pmatrix} c & 0 & \dots & b_1 & b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_{m+k} \\ \Delta_{m+k-1} \\ \Delta_{m+k-2} \\ \vdots \\ \Delta_{m-1} \end{pmatrix}$$

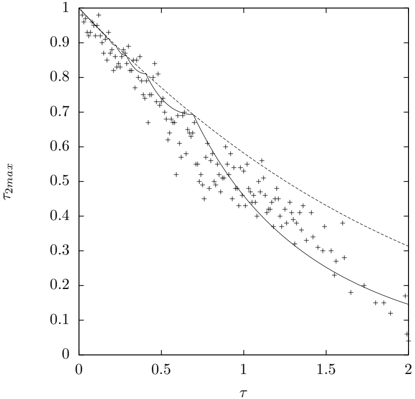
where $c = e^{\lambda\tau}$, $b_1 = \alpha(e^{\lambda(\tau-\tau_2^{'})} - 1)$ and $b_0 = \alpha e^{\lambda\tau}(1 - e^{-\lambda\tau_2^{'}})$.

Stability region



Stability region of the synchronized state of two chaotic Rössler systems in the parameter plane $\tau_2 - \epsilon$.

Stability region



The maximum τ_{2max} for stable anticipatory synchronization, in two coupled Rössler systems as a function of the reset time τ.

Oscillator Death - Quiescent state of neurons

Hindmarsh-Rose model

Parameter estimation from time series

Definition of the problem

- Given the form of the dynamical equations

$$\dot{x} = f(x, \mu)$$

  μ is a set of parameters.
  x is d-dimensional.
- Two possibilities:
  - The time series of a scalar variable is given.
  - The time series of all the variables are given.
- To determine μ.

Parameter estimation

Error minimization: Static method, master–slave approach.

- The master system gives the time series say $x_1(t)$.
- The slave system knows the equations but does not know one of the parameter, say r. Assume an arbitrary value r'.
- Define synchronization error

$$E_s = \frac{1}{T}\int_0^T (y-x)^2 dt$$

- Evaluate E_s as a function of r'. For $r' = r$, we get $E_s = 0$.
- Method requires a large number of calculations.

U. Parlitz, Phys. Rev. Lett. **76**, 1232 (1996).

Parameter estimation

Adaptive control: Dynamic method

- The master system gives the time series say $x_1(t)$.
- The slave system knows the equations but does not know one of the parameter, say r. Assume an arbitrary value r'.
- Introduce a dynamic equation for r'.

$$\dot{r}' = h(y_1 - x_1)$$

- The function $h(0) = 0$ and it is defined in such a way that asymptotically $r' \to r$ and also $y \to x$.

A. Maybhate and REA, Phys. Rev. E **59**, 284 (1999); **61**, 6461 (2000).

Lorenz equations + parameter

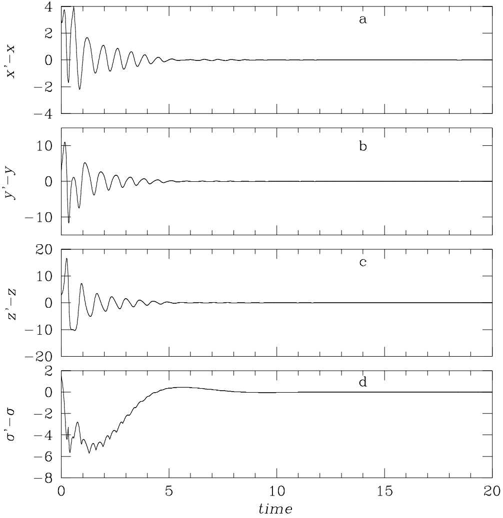
Lorenz equations for the master system

$$
\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= x_1(r - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - b x_3
\end{aligned}
$$

Lorenz equations for the slave system

$$
\begin{aligned}
\dot{y}_1 &= \sigma(y_2 - y_1) - \epsilon(y_1 - x_1) \\
\dot{y}_2 &= y_1(r - y_3) - y_2 \\
\dot{y}_3 &= y_1 y_2 - b y_3 \\
\dot{\sigma}' &= -\delta(y_1 - x_1)(y_2 - y_1)
\end{aligned}
$$

Parameter estimation - Lorenz



Problems of Synchronization methods

- Total time: Condition

$$T > \tau_s$$

T – the total time of the time series data.
τ_s – the time scale for synchronization.

- Synchronizing variable:
(a) The slave system must be able to synchronize.
(b) Only time series data of the variables which can lead to synchronization can be used.

Modified Newton-Raphson method

Given a dynamical system

$$\dot{x} = f(x, \mu)$$

Construct an auxiliary system

$$\dot{y} = f(y, \nu)$$

Here ν are the guess values of the parameters μ.
The difference vector

$$w(t) = y(t) - x(t)$$

We look for the solution of the equation

$$w(t) = 0$$

Modified Newton-Raphson method

We combine the standard Newton-Raphson method and an Euler expansion for time evolution.

$$-(\delta y^k)_i = (W^k)_i = \sum_j (A^{k-1} \cdots A^0)_{ij} (W^0)_j, \quad i = 1, \cdots, d.$$

$\rightarrow$ d independent linear equations for m unknown quantities $\delta\nu$. To get m equations, we write equations for $W^1, W^2, \ldots, W^k$ so that $kd \geq m$.

Modified Newton-Raphson method

Numerical procedure:

1. Take some guess values for the parameters ν.
2. Use equations above to yield $\delta\nu$.
3. The process is iterated by taking the new improved guess values as $\nu + \delta\nu$.

Note: The total duration of the time series: $k\Delta t$.
The procedure leads to numerical problems if m, the number of unknown parameters is large.

- Multiple solutions
- Divergence

Embedding

Embedding with suitable time delays:

- Choose some initial times $t_1, \ldots, t_n$.
  E.g. $0, \tau, \ldots, (n-1)\tau$.
- The final times $t_1 + k\Delta t, \ldots, t_n + k\Delta t$.
- For each pair $(t_i, t_i + k\Delta t)$, construct d-equations.
  (condition: $m \leq nd$.)
- Numerical procedure same as before.

Total time duration: $t_n - t_1 + k\Delta t$.

Example: Rössler

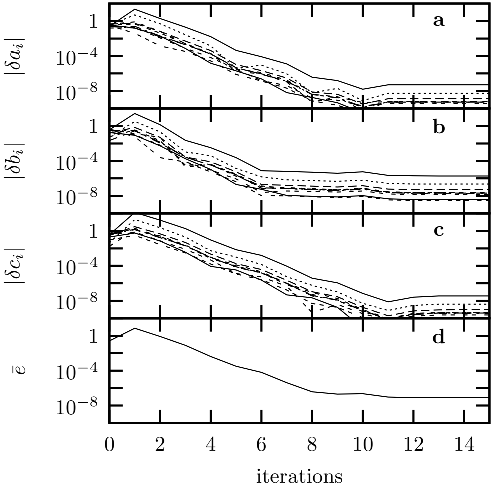
Rössler System

$$
\begin{aligned}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= b + x_3(x_1 - c)
\end{aligned}
$$

Rewrite with all quadratic terms

$$
\begin{aligned}
\dot{x}_1 &= a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_1^2 + a_5 x_2^2 + a_6 x_3^2 \\
&\quad + a_7 x_1 x_2 + a_8 x_2 x_3 + a_9 x_3 x_1 \\
\dot{y}_1 &= b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_1^2 + b_5 x_2^2 + b_6 x_3^2 \\
&\quad + b_7 x_1 x_2 + b_8 x_2 x_3 + b_9 x_3 x_1 \\
\dot{z}_1 &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1^2 + c_5 x_2^2 + c_6 x_3^2 \\
&\quad + c_7 x_1 x_2 + c_8 x_2 x_3 + c_9 x_3 x_1
\end{aligned}
$$

Example: Rössler



Extinction: Introduction

- Extinction: 99% species that ever existed on the surface of the earth are now extinct.
  1. What are the reasons of this extinction?
  2. Why the species die everywhere, and not survive in some location or patches? (Rescue effect)
- Second question: Spatial synchronization as a possible answer.

REA and G. Rangarajan, Phys. Rev. Lett. **96**, 258102 (2006).

REA and G. Rangarajan, unpublished.

Extinction of species

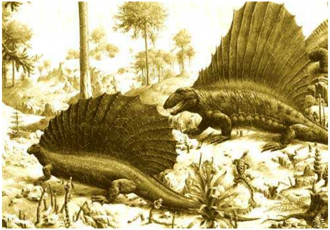
Some facts.

- More than twenty million species exist on the earth today.
- More than 99% species that ever existed on the earth are now extinct.
- Presently more than 1000 animal species are endangered.
- There are several mass extinction events. A large fraction of the species die in a small period. (Permian-Triassic extinction: Killed about 95% of the species.)
- Sixth extinction: It is feared that today we are either close to or at the beginning of a mass extinction event caused by man's activities.

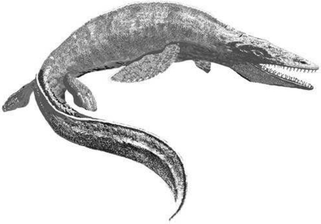
Major mass extinction events

Extinction Event	Million years ago	Likely Cause
Cretaceous-Tertiary (KT)	65	asteroid hits $\rightarrow$ large scale weather disturbance
End Triassic	199–214	massive floods of lava
Permian-Triassic	251	asteroid hit
Late Devonian	364	unknown
Ordovician-Silurian	439	a drop in sea levels as glaciers formed, then by rising sea levels as glaciers melted.

KT Extinction

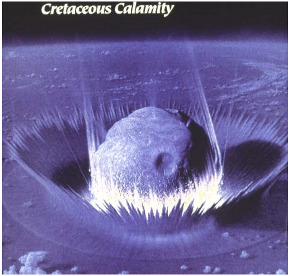


Edaphosaurus,Dimetrodon



Platecarpus

Extinction - causes



KT extinction

Asteroid Hit

Ordovician-Silurian Extinction

Falling and rising sea level

Rescue Effect

- Two possible ways to rescue a population
  1. Variation of population: Different patches have different populations $\rightarrow$ take different times for extinction. (case I)
  2. Variation of external threat/ forcing: Intensity of an external threat will be minimum or zero in some patches. Populations may survive in such patches. (case II)
- Rescue effect: Population survives in some isolated patches and leads to the revival of the species.

How does a species become extinct throughout the world?

Why does the rescue effect not operate?

Population dynamics

A species located at different locations or patches.

- $P_i(t)$ – the population at $i-$th patch, $i = 1, \dots N$.
- $Q(t)$ – an external variable (e.g. lava, asteroid hit, climate etc.).

Case 1: External variable affects all the populations

$$\begin{aligned}
\frac{dP_i}{dt} &= f_1(P_i(t)) + \epsilon_1 g_1(P_i(t), Q(t)) \\
&\quad + \delta \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} h(P_i, P_j) \\
\frac{dQ}{dt} &= f_2(Q(t)) + \frac{\epsilon_2}{N} \sum_i g_2(P_i(t), Q(t)),
\end{aligned}$$

Synchronization and extinction

We define the following states

- Spatial Synchronization: $P = P_1 = P_2 = \cdots = P_N$.
- Extinction: $P_i = 0$ for all i.

We study the stability of both the states.

Linear Stability

The Jacobian ($\delta = 0$)

$$J = \begin{pmatrix} \frac{\partial f_1(P)+\epsilon_1 g_1(P,Q)}{\partial P} & \cdots & 0 & \frac{\partial \epsilon_1 g_1(P,Q)}{\partial Q} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{\partial f_1(P)+\epsilon_1 g_1(P,Q)}{\partial P} & \frac{\partial \epsilon_1 g_1(P,Q)}{\partial Q} \\ \frac{\partial \epsilon_2 g_2(P,Q)}{\partial P} & \cdots & \frac{\partial \epsilon_2 g_2(P,Q)}{\partial P} & \frac{\partial f_2(Q)+\epsilon_2 g_2(P,Q)}{\partial Q} \end{pmatrix}$$

Manifolds

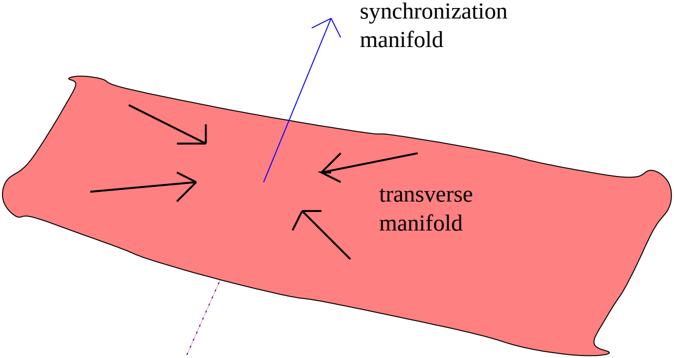
- Synchronization manifold: It is defined by (P, Q) and has dimension two. The eigenvalues are obtained from the matrix

$$J = \begin{pmatrix} \frac{\partial(f_1(P)+\epsilon_1 g_1(P,Q))}{\partial P} & \frac{\partial \epsilon_1 g_1(P,Q)}{\partial Q} \\ N\frac{\partial \epsilon_2 g_2(P,Q)}{\partial P} & \frac{\partial(f_2(Q)+\epsilon_2 g_2(P,Q))}{\partial Q} \end{pmatrix}$$

- Transverse manifold It has dimension $(N-1)$. The eigenvalues are degenerate and are given by

$$\frac{\partial(f_1(P)+\epsilon_1 g_1(P,Q))}{\partial P}$$

Stability – Manifolds



Schematic diagram of the manifolds.

Condition for the stability of synchronized state: All Lyapunov exponents in the transverse directions must be negative.

Stability conditions

The stability conditions and the time constants

Spatial Synchronization

$$\begin{aligned}\lambda_s &= \langle \frac{\partial}{\partial P}(f_1(P) + \epsilon_1 g_1(P, Q))\rangle < 0 \\ \tau_s &= 1/\lambda_s\end{aligned}$$

Extinction

$$\begin{aligned}\lambda_e &= \langle \frac{\partial}{\partial P}(f_1(P) + \epsilon_1 g_1(P, Q))_{P=0}\rangle < 0 \\ \tau_e &= 1/\lambda_e\end{aligned}$$

Growth and decay terms

We expand function $f_1(P)$ in terms of P.

$$f_1(P) = aP + bP^2 + \mathcal{O}(P^3).$$

- First term - must be a growth term, i.e. $a > 0$.
- Second term - observations indicate and also the population models assume that this term is a decay term, i.e. $b < 0$.

We find that

$$\begin{aligned} \tau_s &< \tau_e, \quad b < 0 \\ \tau_s &> \tau_e, \quad b > 0 \end{aligned}$$

Synchronization and extinction

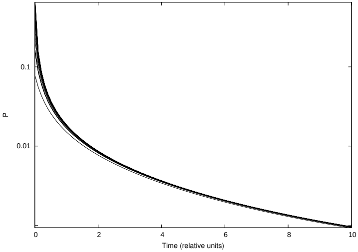
- $b < 0$, $\tau_s < \tau_e$
  - Spacial synchronization will occur before extinction.
  - Hence during extinction populations in different patches will die almost simultaneously.
  - Rescue effect cannot revive the population.
- $b > 0$, $\tau_s > \tau_e$
  - Populations will not synchronize before extinction.
  - The rescue effect is possible.

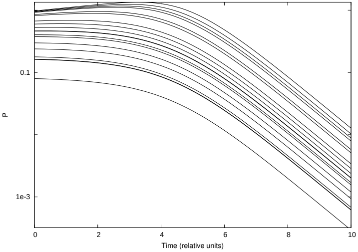
(Note: In this case the next higher order term in the expansion is required to get stable solution.)

Synchronization and extinction

- The term bP^2 normally comes from competition. Hence in general $b < 0$.
- A cooperation between the members of the species can help to make b less negative or even positive. In this case the rescue effect may operate.
- Thus the species can have a natural resistance to extinction if $b > 0$.

Numerical Demonstration





$b < 0$ $\qquad\qquad$ $b > 0$.

Populations of patches vs time

(Parameters $N = 100,\ \epsilon_1 = 4.8,\ \epsilon_2 = 1.0, Q^* = 0.5,$
$u = 0.1,\ f_2(Q) = -u(Q - Q^*)$)

Concluding Remarks

- Synchronization is an important phenomena of coupled dynamical systems.
- We can observe synchronization in time varying networks.
- Synchronization with variable delay is possible.
- We can use synchronization for estimating parameters.
- We establish a close connection between extinction and spatial synchronization. Spatial synchronization precedes extinction when $b < 0$, thus avoiding rescue effect.