Majorana modes and non-abelian anyons in spin systems

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Workshop on Non-equilibrium Statistical Physics
1st February, 2010. IIT Kanpur.

Outline

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- Topological Quantum Computation
- Kitaev's Honeycomb Model
- Majorana Fermions
- Unpaired Majorana Modes and Non-Abelian Anyons
- The Tetrahedral Chain

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Topological Quantum Computation

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Quantum Computation

- Physically, a computer is a dynamical system. It starts from an initial state (encoding the data), evolves in time and reaches a final state (encoding the answer).
- Computers with quantum dynamics can solve some problems much faster than computers with the classical dynamics. eg: Prime factorisation (Shor, 1994), Search algorithm (Grover),.....
- The Hilbert space for a quantum computer is formulated as a tensor product of 2-level systems (qubits).
- The time evolution is,

$$|\psi\rangle_{fin} = T\left(e^{-i\int_{t_1}^{t_2} dt\ H(t)}\right)|\psi\rangle_{in}$$

The time dependence of the hamiltonian encodes the program.

Fault Tolerance

- The environment will always interact with the system and introduce errors.
- There are both "software" and "hardware" approaches to minimize errors.
- A hardware solution can be achieved if the environment cannot cause transitions in the subspace of the Hilbert space that represents the qubits.
- One possibility is to realise qubits as the states of topological defects so that the operators in the qubit subspace correspond to non-local observables.

Non-Abelian Anyons

- Vortices in the $\nu = 5/2$ FQHE states are believed to be non-abelian anyons (Moore and Read, 1991).
- There are two possible quantum states of the vortex excitation which are degenerate in energy.
- When the positions of two vortices are interchanged, the state is unitarily rotated in this degenerate subspace,

$$|\vec{R}_1, \vec{R}_2\rangle = U|\vec{R}_2, \vec{R}_1\rangle$$

- The U matrices form a unitary representation of the Braid group.

Topological Quantum Computation

- Qubits: Quantum states of non-abelian anyons.
- Computational operations (gates): Braiding of the anyons.

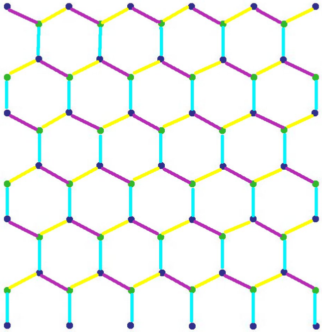
Hence,

- Physical systems where non-abelian anyons can be realised and manipulated are of interest.
- Since spin systems can be engineered and manipulated using cold atoms, realising non-abelian anyons in in spin systems is of interest.

Outline

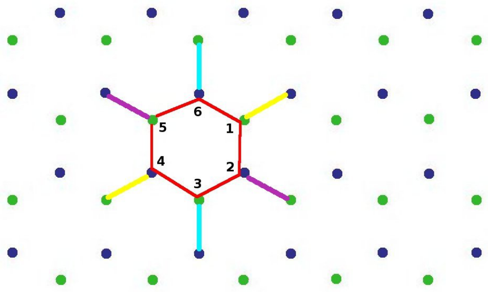
The hamiltonian

Alexei Kitaev,“Anyons in an exactly solved model and beyond", cond-mat/0506438, Annals of Physics.



$$H = J_x \sum_{<ij>} \sigma_i^x \sigma_j^x + J_y \sum_{<ij>} \sigma_i^y \sigma_j^y + J_z \sum_{<ij>} \sigma_i^z \sigma_j^z$$

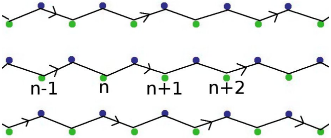
Conserved quantities



$$
\begin{aligned}
W_p &\equiv \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \\
[W_p, H] &= 0 = [W_p, W_{p'}] \\
W_p^2 &= 1
\end{aligned}
$$

Conserved quantities have the properties of magnetic fluxes of a Z_2 gauge theory

Jordan-Wigner transformation for $S = \frac{1}{2}$



The disorder variables:

$$\mu_{i_l} = \prod_{m<l} \left(\sigma^z_{i_m}\right)$$

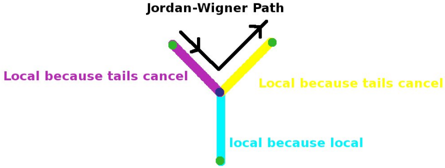
Jordan-Wigner fermions:

$$\xi_{i_n} = \sigma^x_{i_n}\mu_{i_n} \qquad \eta_{i_n} = \sigma^y_{i_n}\mu_{i_n}$$

$$\{\xi_{i_n}, \xi_{i_m}\} = 2\delta_{nm} = \{\eta_{i_n}, \eta_{i_m}\}$$

$$\{\xi_{i_n}, \eta_{i_m}\} = 0$$

Majoranisation



$$H = \sum_{\langle ij \rangle} i\xi_i\xi_j + \sum_{\langle ij \rangle} i\xi_i\xi_j + \sum_{\langle ij \rangle} i\xi_i u_{ij}\xi_j$$

$$u_{ij} = i\eta_i\eta_j$$

Hamiltionian of Majorana fermions, ξ_i, interacting with Z_2 gauge fields, u_{ij}, in the gauge,

$$u_{\langle ij \rangle} = u_{\langle ij \rangle} = 1$$

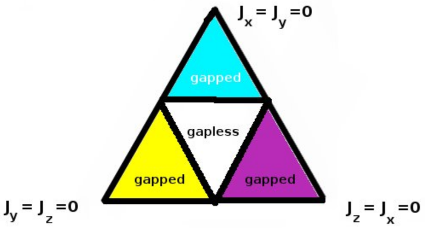
The exact solution

$$H = \sum_{\langle ij \rangle} i\xi_i\xi_j + \sum_{\langle ij \rangle} i\xi_i\xi_j + \sum_{\langle ij \rangle} i\xi_i u_{ij}\xi_j$$

$$\begin{aligned} [u_{ij}, u_{kl}] &= 0 \\ [u_{ij}, H] &= 0 \end{aligned}$$

- The Z_2 gauge fields, u_{ij}, are conserved and hence do not fluctuate.
- Hence the RVB mean field theory is exact !
  The problem reduces to a quadratic theory of Majorana fermions in the background of static Z_2 gauge field configurations.

The phase diagram



- The Chern number, ν, distinguishes the two phases. $\nu = 0$ in gapped phase, $\nu = \pm 1$ in gapless phase. Gapless excitations are Majorana fermions (spinons).
- Vortices (flux in one plaquette flipped) are anyonic excitations. Abelian in gapped phase and non-abelian in gapless phase.

A toy model for frustrated quantum magnets

- Infinitely degenerate classical ground state manifold like the Heisenberg model on Kagome and Pyrochlore lattices

  G. Baskaran, Diptiman Sen and R. Shankar, Phys. Rev. **B 78** 115116 (2008). D. Dhar, Kabir and Samarth Chandra (unpublished)

- A spin liquid ground state: Only non-zero 2-spin correlators:

$$\langle w|\sigma_i^x \sigma_{i+\hat{x}}^x|w\rangle \quad \langle w|\sigma_i^y \sigma_{i+\hat{y}}^y|w\rangle \quad \langle w|\sigma_i^z \sigma_{i+\hat{z}}^z|w\rangle$$

  G. Baskaran, Saptarshi Mandal and R. Shankar, PRL **98**, 247201 (2007)

- Low lying excitations are fermions (spinons).

Physics of Kitaev's Honeycomb Model similar to that expected in physically realisable frustrated quantum anti-ferromagnets.

The Institute of Mathematical Sciences

Outline

Majorana operators

Consider a system of N fermions,

$$\left\{c_i, c_j^\dagger\right\} = \delta_{ij}$$
$$\left\{c_i, c_j\right\} = 0$$
$$\left\{c_i^\dagger, c_j^\dagger\right\} = 0$$

Define $2N$ hermitian Majorana operators,

$$\xi_{i1} \equiv \frac{1}{2}\left(c_i + c_i^\dagger\right)$$
$$\xi_{i2} \equiv \frac{1}{2i}\left(c_i - c_i^\dagger\right)$$
$$\left\{\xi_{ia}, \xi_{jb}\right\} = 2\delta_{ij}\delta_{ab}$$

General Hamiltonian

$$H = \sum_{ij} \frac{i}{2} \xi_{ia} A_{ia,jb} \xi_{jb}$$

$$A^*_{ia,jb} = A_{ia,jb} = -A_{jb,ia}$$

The eigenvalues of A are purely imaginary and come in pairs,

$$A\phi^n = i\epsilon_n \phi^n, \qquad A(\phi^n)^* = -i\epsilon_n (\phi^n)^*, \quad n = 1, \dots N$$

$$\phi^n = \phi^{nR} + i\phi^{nI}$$

$$\left(\phi^{nR}\right)^T \phi^{mR} = \delta_{nm} \quad \left(\phi^{nR}\right)^T \phi^{mI} = 0 \quad \left(\phi^{nI}\right)^T \phi^{mI} = \delta_{nm}$$

Diagonalisation

$$\xi_{ia} = \sum_n \left(\alpha_n \phi_{ia}^{nR} + \beta_n \phi_{ia}^{nI}\right)$$

$$\alpha_n = \sum_{ia} \phi_{ia}^{nR} \xi_{ia} \qquad \beta_n = \sum_{ia} \phi_{ia}^{nI} \xi_{ia}$$

$$H = \sum_n \epsilon_n \left(i\beta_n \alpha_n\right)$$

$$\left(i\beta_n \alpha_n\right)^2 = 1$$

Ground state:

$$i\beta_n \alpha_n |GS\rangle = -|GS\rangle, \ \ \forall n \neq 0$$

- If N_0 of the $\epsilon_n s$ are zero, the ground state is 2^N fold degenerate, corresponding to $i\beta_{0n}\alpha_{0n} = \pm 1$

Unpaired modes

- The eigenfunctions of $\epsilon = 0$ modes can always be chosen to be real

$$A\phi^{0I} = 0, \ \ \left(\phi^{0I}\right)^{*} = \phi^{0I}, \ \ I = 1, \ldots, 2N_0$$

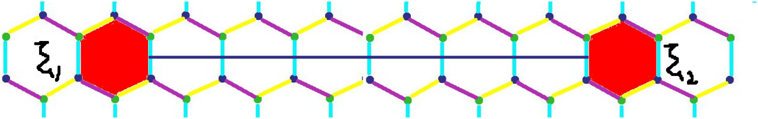
- The degenerate ground state manifold corresponds to different ways of "pairing" the zero modes

$$\gamma_I \equiv \sum_{ia} \phi^{0I}_{ia} \xi_{ia}$$

- Since there is no fixed way of pairing the zero modes, we will refer to them as unpaired modes.

Outline

Unpaired Majorana modes in the Kitaev Honeycomb Model



- Ground state of the Kitaev model corresponds to $W_p = +1, \ \forall p$
- $W_p = -1 \ \equiv$ vortex at plaquette p.
- Each vortex has an trapped unpaired Majorana mode.
- If there is only one vortex, the second unpaired mode is at the edge of the sample.
- If the system is gapped (by modifying the model), the unpaired mode is localised.

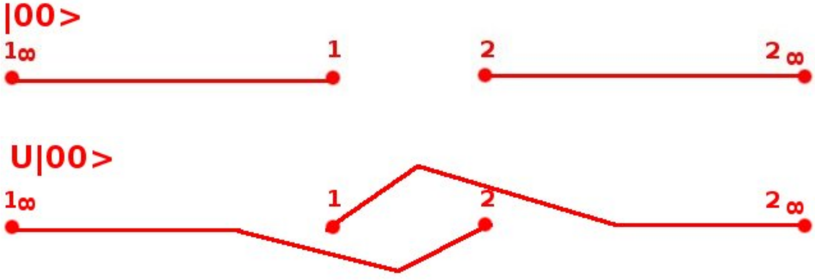
Non-Abelian Anyons

Consider a situation with 4 unpaired modes, two in the bulk, γ_1, γ_2 and two at the edge, $\gamma_{1\infty}$, $\gamma_{2\infty}$.
If the ground state is,

$$i\gamma_1\gamma_{1_\infty}|00\rangle = |00\rangle, \qquad i\gamma_2\gamma_{2_\infty}|00\rangle = |00\rangle$$

The exchange operation

$$
\begin{aligned}
U &\equiv e^{\frac{\pi}{4}\gamma_1\gamma_2} \\
U^\dagger \gamma_1 U &= \gamma_2 \\
U^\dagger \gamma_2 U &= -\gamma_1 \\
U|00\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
UU|00\rangle &= |11\rangle
\end{aligned}
$$



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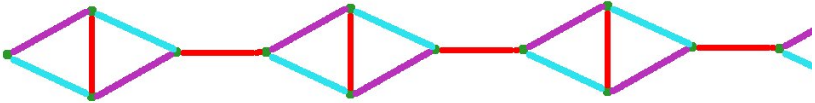
The Tetrahedral Chain

Motivation

(Abhinav Saket, S.R. Hassan and R. Shankar)

- It is of interest to find ways to create and manipulate unpaired modes.
- Kitaev model can be generalised in a variety of ways to get similar (Z_2 gauge theories with conserved fluxes) exactly solvable models any dimension.
- 1-d models are easier to play around with.
- So construct and explore 1-d models with unpaired Majorana fermions.

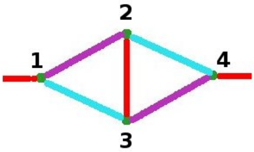
The Model



$$H = -J_x \sum_{<ij>} \sigma_i^x \sigma_j^x - J_y \sum_{<ij>} \sigma_i^y \sigma_j^y - J_z \sum_{<ij>} \sigma_i^z \sigma_j^z$$

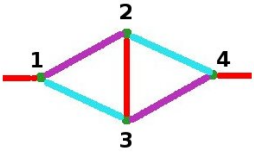
Conserved fluxes:

$$W^L = \sigma_1^x \sigma_2^z \sigma_3^y \quad W^R = -\sigma_4^x \sigma_2^y \sigma_3^z$$



$$H = \sum_{\langle ij \rangle} i\xi_i \xi_j + \sum_{\langle ij \rangle} i\xi_i \xi_j + \sum_{\langle ij \rangle} i\xi_i u_{ij} \xi_j$$

Local Symmetry and Degeneracy



Local Symmetry at $J_y = J_z$:

$$\sigma_2^a \leftrightarrow \sigma_3^a$$

$$\sigma_i^y \rightarrow \sigma_i^z, \qquad \sigma_i^z \rightarrow -\sigma_i^y$$

Implies every eigenstate is 2^{N_p} fold degenerate

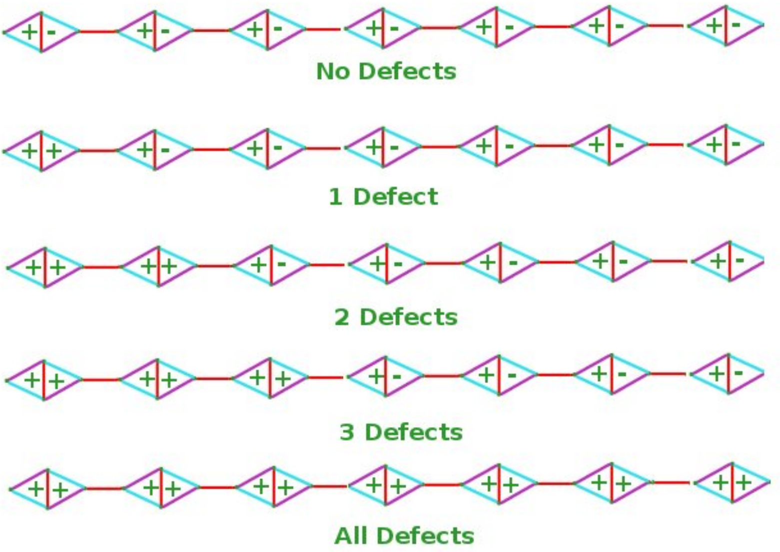
Degeneracy can be lifted, in fact any flux sector can be made the ground state by adding "chemical potential" terms,

$$H = H + H_w \quad H_w = \sum_n \left(\mu_n^L W_n^L + \mu_n^R W_n^R \right)$$

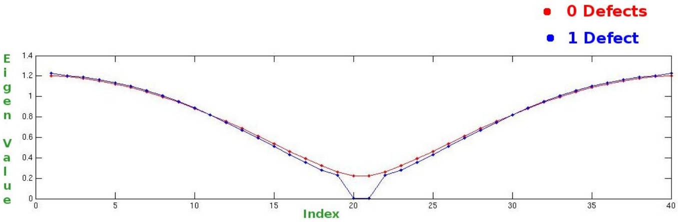
Ground state

- At $J_x = J_y = J_z = 1$ and $\mu_p^{L,R} = 0$ the ground state is in the flux sector, $W_p^L = 1, \; W_p^R = -1$ (and symmetry related sectors).
- Numerically checked by explicit evaluation in all sectors for the 20 site system.
- Numerically checked for 800 site system for translationally invariant states.
- Fermionic sector gapped with a gap of 0.25

Defects

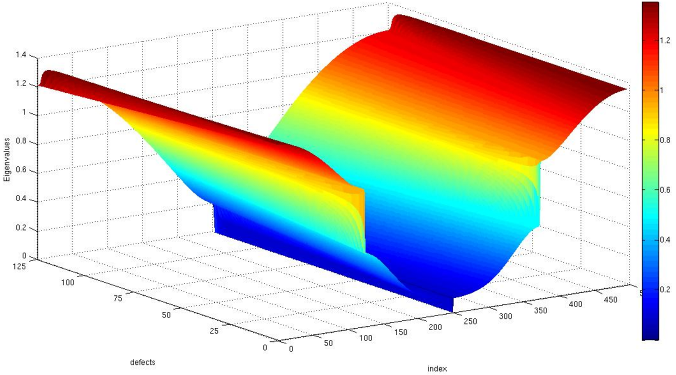


Unpaired Modes



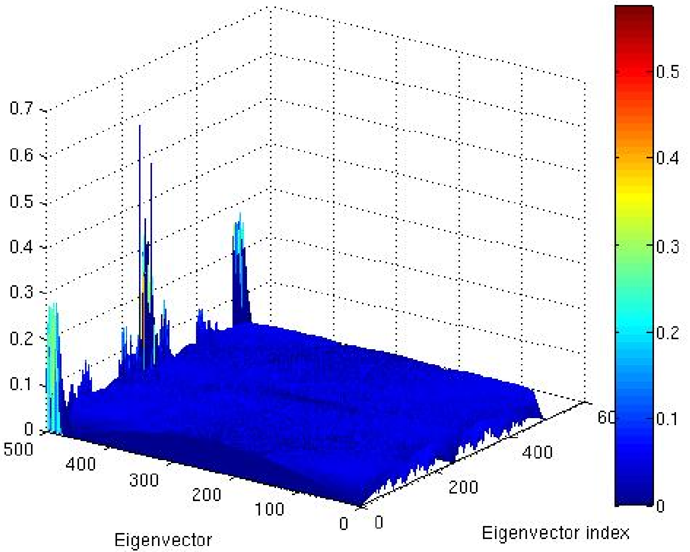
No zero energy modes for 0 Defects.
1 zero energy mode for 1 Defects.

Unpaired Modes

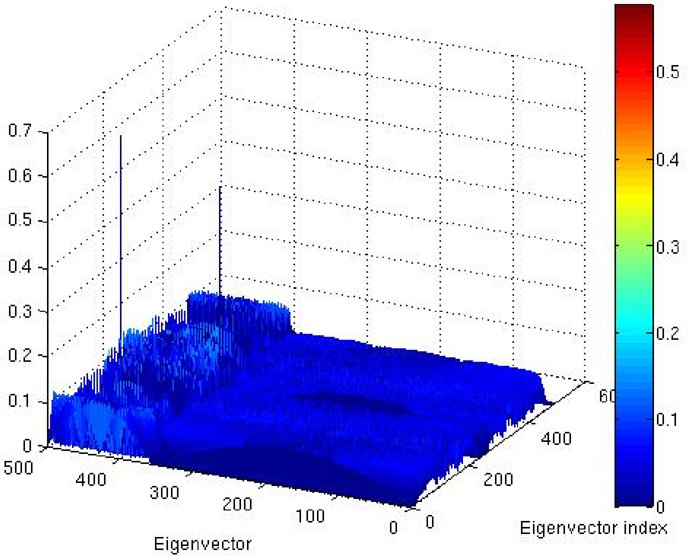


No zero energy modes for 0 Defects.
1 zero energy mode for $n \neq 0$ Defects.

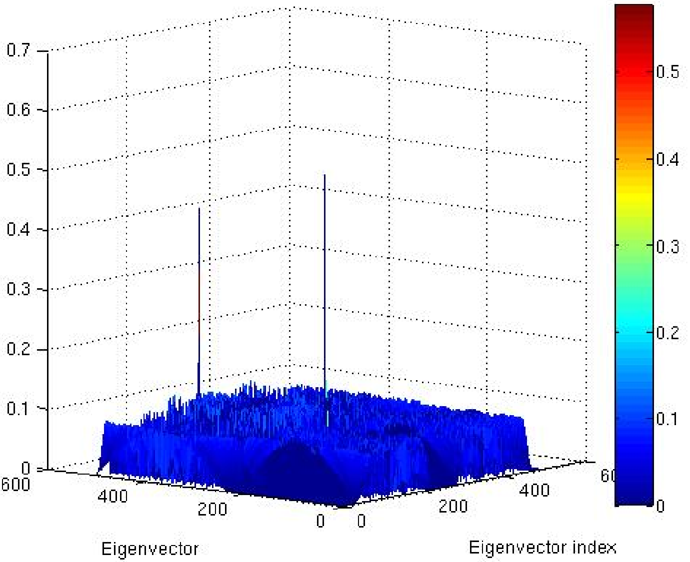
Eigenvectors with 5 defects



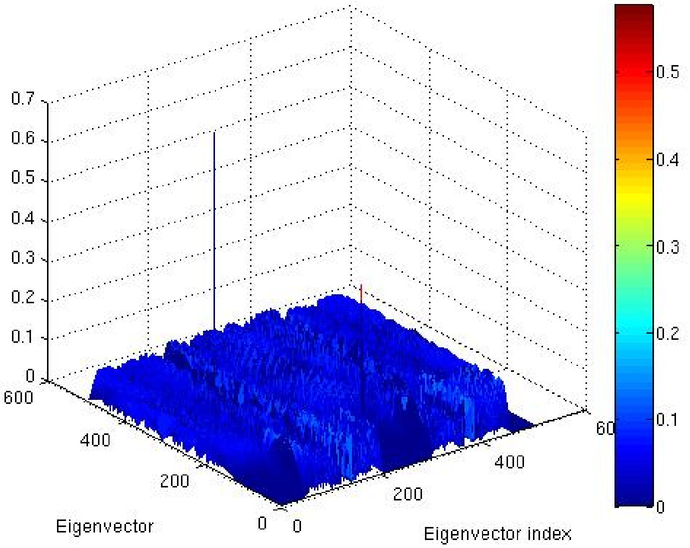
Eigenvectors with 35 defects



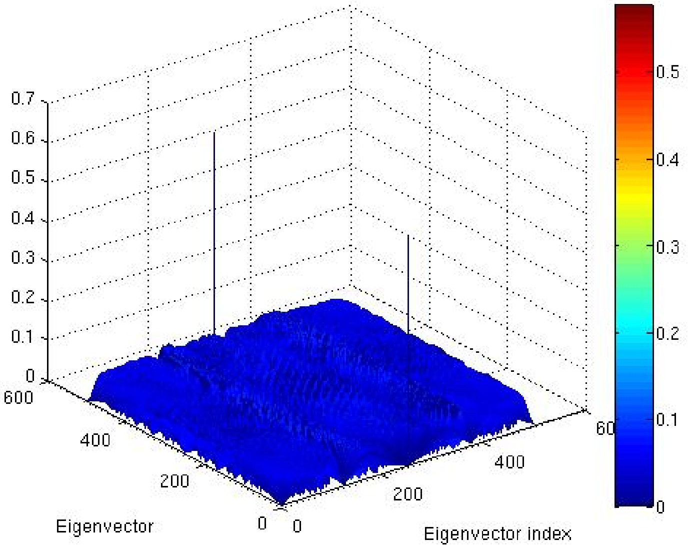
Eigenvectors with 65 defects



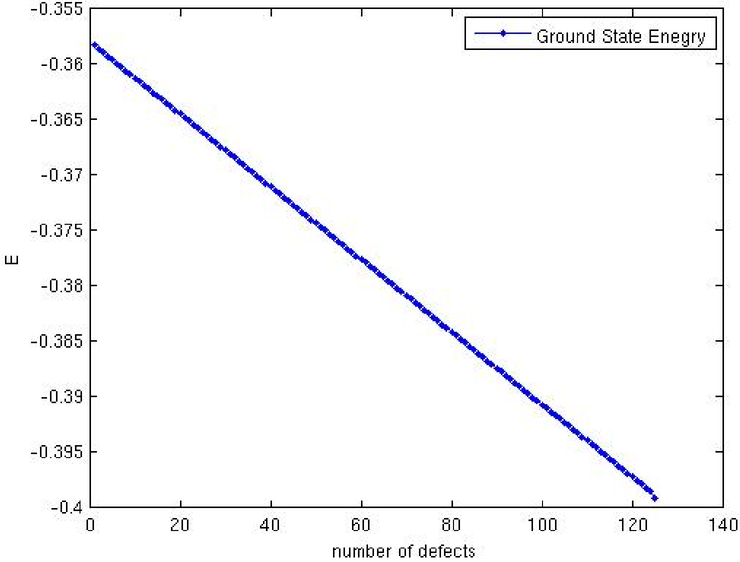
Eigenvectors with 95 defects



Eigenvectors with all defects



Defect energy



Conclusions

Results:

- The tetrahedral model has localised unpaired Majorana modes.
- These modes can be moved around by tuning the local chemical potential which correspond to 3-spin operators.

Ongoing work:

- Can these results be understood analytically ?
- Can the local chemical potential arise as effective interactions from a magnetic field term ?

THANK YOU !

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