Understanding Search Trees via Statistical Physics

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Sorting and Search

The Goal: Store data efficiently so that the search time is minimum

Ex: A random sequence of $N = 10$ integers: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

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Linear Sorting: Store the data sequentially onto a linear table

$$\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$$

Search for 7: Search proceeds sequentially by comparison

$$t_{\rm search} = 10 \sim O(N) \to \text{BAD}$$

Tree Sorting: of {6, 4, 5, 8, 9, 1, 2, 10, 3, 7}

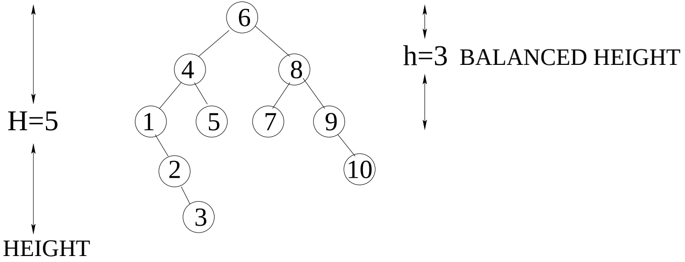

Figure: Binary Search Tree with $N = 10$ Elements.

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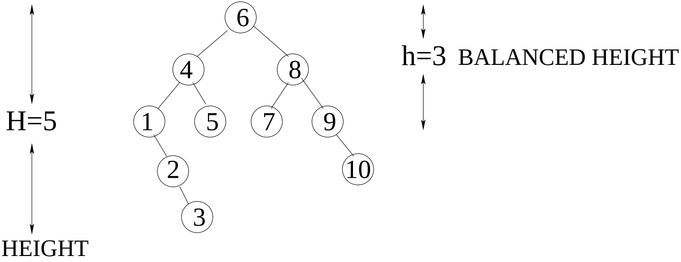

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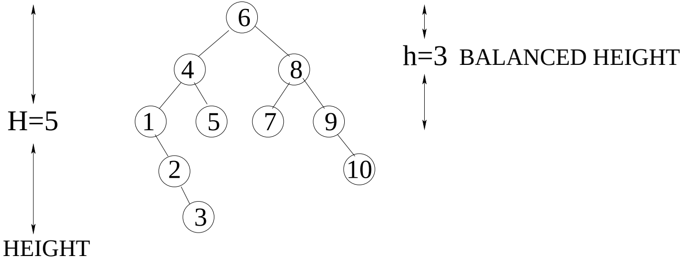

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Generalization to *m*-ary Search Trees: Muntz and Uzgalis '70

$m = 2 \rightarrow$ Binary Tree
Random Sequence: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$
Each node can contain atmost $(m-1)$ elements.

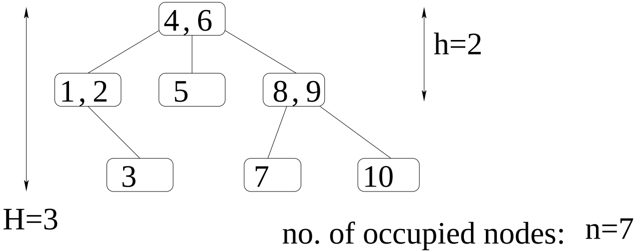

Figure: $m = 3$-ary Search Tree with $N = 10$ Elements

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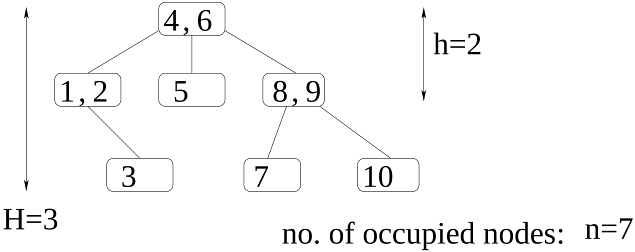

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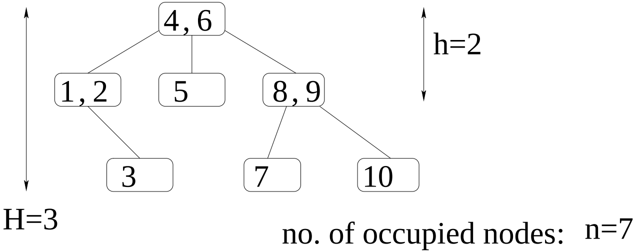

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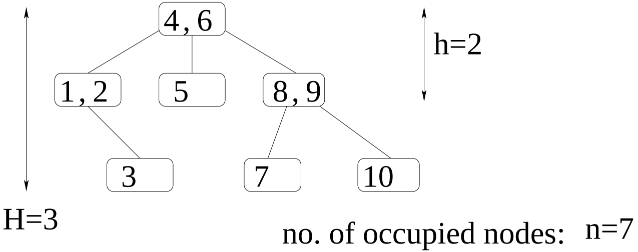

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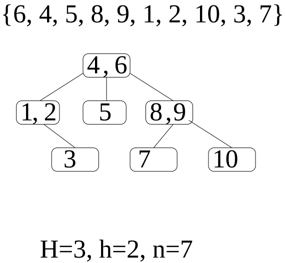
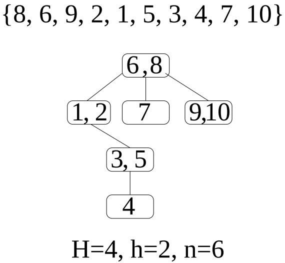
Random m-ary Search Tree Model: $RmST$

$N = 10$ data elements: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Each permutation $\rightarrow$ an m-ary tree.





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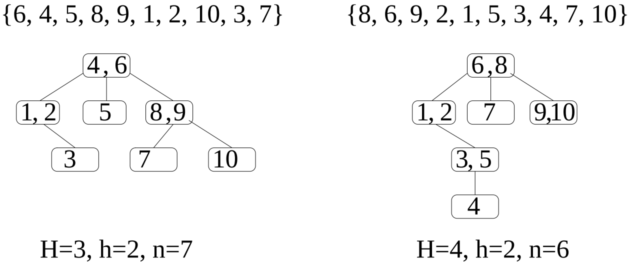

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Q: Statistics of HEIGHT H_N, BALANCED HEIGHT h_N and the no. of NON-EMPTY NODES n_N for RANDOM data of size N?

Asymptotic Results for RmST: for large data size N

(1) Height H_N:

- $\langle H_N \rangle \approx a_m \log(N) + b_m \log(\log(N))$ (??) $+ \ldots$

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Q: Significance of a_m and c_m? Correction terms?

Asymptotic Results for RmST: for large data size N...continued

(3) No. of NON-EMPTY Nodes n_N: No. of nodes required to store the data of size N.

$$\langle n_N \rangle \approx \alpha_m N + \ldots.$$

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A striking PHASE TRANSITION occurs for the Variance: $\nu_N = \langle \left(n_N - \langle n_N \rangle \right)^2 \rangle$.

$$\begin{aligned} \nu_N &\sim N && \text{for } m \leq 26 \\ &\sim N^{2\theta(m)} && \text{for } m > 26 \text{ (Chern \& Hwang, 2001).} \end{aligned}$$

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Q: Why 26? What is the mechanism of this Phase Transition and how generic is it? Can one calculate $\theta(m)$ exactly ?

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- Mapping between:

  Random m-ary Search Tree $\equiv$ Random FRAGMENTATION Process

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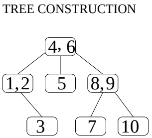
- $\longrightarrow$ A new type of Phase Transition

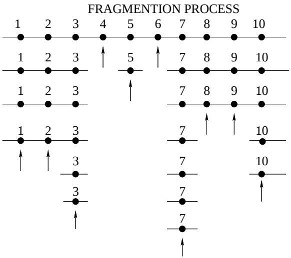
- $\longrightarrow$ generalization and new results for: Vector Data

The Mapping to a Fragmentation Process

Construction of the Tree → Dynamical Fragmention Process: Split an interval into m pieces with the break points chosen randomly. An interval can split iff it contains atleast one point.

Ex: Consider the data: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$ on a ($m = 3$)-ary tree





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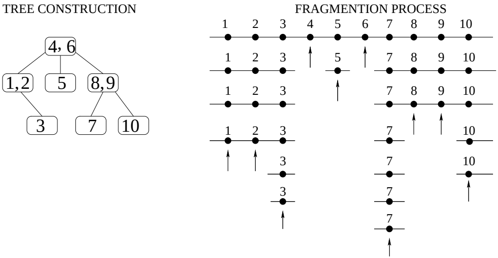
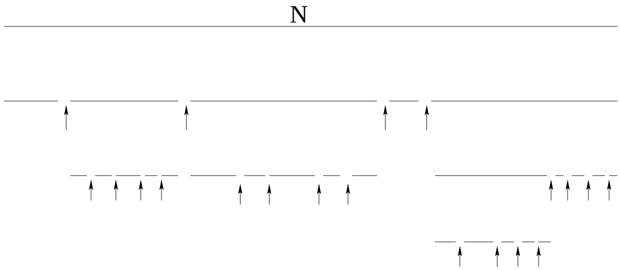


NOTE:

No. of NONEMPTY nodes n=7= No. of SPLITTING EVENTS

Fragmentation Process With a Stopping Time: Continuum Limit



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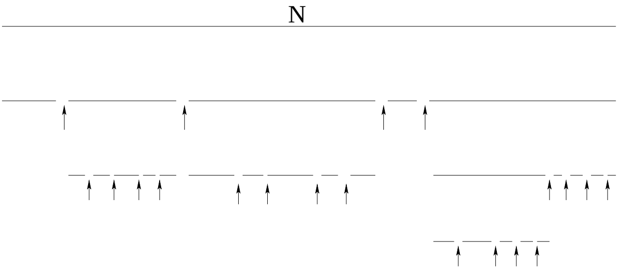

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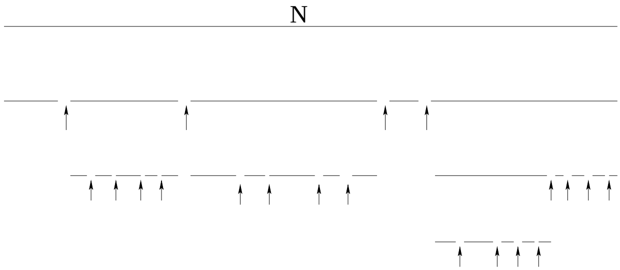

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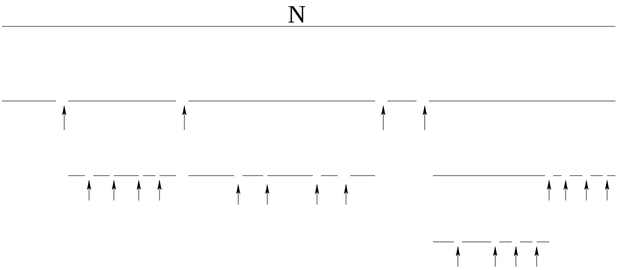

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4. Repeat the process till all pieces have length < 1 and then STOP.

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Height H_N:

- Prob[$H_N < n$]= Prob[$l_1 < 1$, $l_2 < 1$, ... after n steps] (No Stopping Time)

Balanced Height h_N:

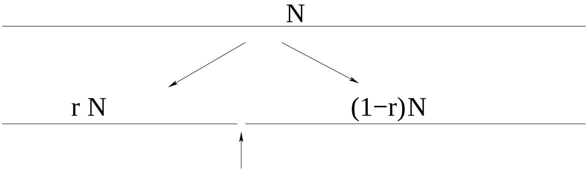
- Prob[$h_N > n$]=Prob[$l_1 > 1$, $l_2 > 1$, ... after n steps] (No Stopping Time)

Number of Nonempty Nodes n_N ($m > 2$):

- Prob[$n_N = n$]= Prob[there are n SPILLITING EVENTS till the end of the Fragmentation process] (With Stopping Time)

Analysis of HEIGHT H_N

P(n,N)= Prob[$H_N < n$]= Prob[$l_1 < 1,\ l_2 < 1, \ldots$ after n steps starting with initial length N] (No Stopping)



Recursion: $P(n,N) = \int_0^1 P\,(n-1, rN)\,P\,(n-1,(1-r)N)\,dr$
$\longrightarrow$Nonlinear and starts with $P(n,1) = \theta(n-1)$.

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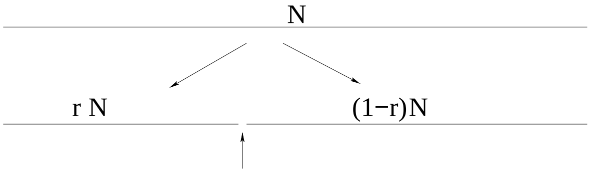

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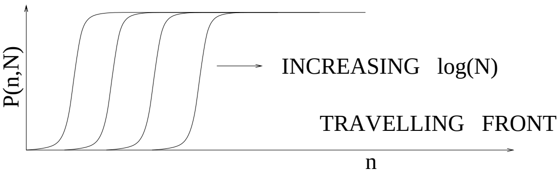

Travelling Front in Fisher/KPP Equation

Fisher/KPP equation: Population Dynamics, Branching Process, ...

$\partial_t \phi(x,t) = \partial_x^2 \phi(x,t) + \phi - \phi^2$ [Initial Cond: $\phi(x,0) = \theta(-x)$]

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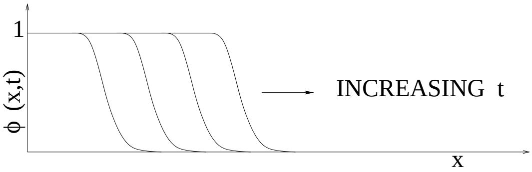

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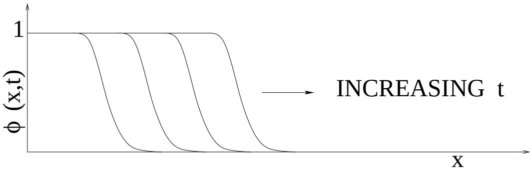

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$$x_f(t) \sim \mathrm{v}\ t + \ldots .$$

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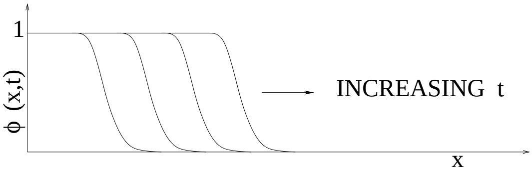

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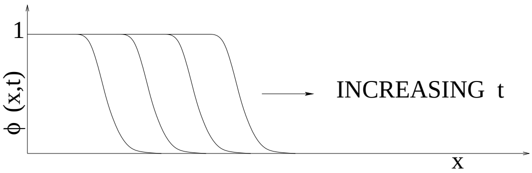
Q: How to determine the Front Velocity v?

Kolmogorov's Velocity Selection Principle:

$\partial_t \phi(x,t) = \partial_x^2 \phi(x,t) + \phi - \phi^2$



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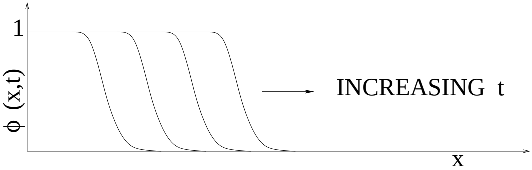

Linearize near the tail $\rightarrow$ $\phi(x,t) \sim \exp[-\lambda(x - vt)]$

DISPERSION RELATION: $v(\lambda) = \lambda + \frac{1}{\lambda}$ $\rightarrow$ minimum at $\lambda^* = 1$.

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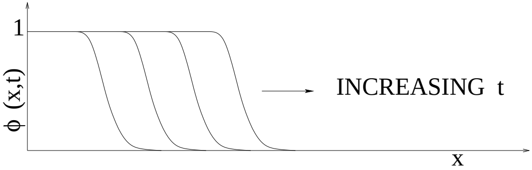

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For sharp initial condition, Front velocity $v = v(\lambda^*) = 2$.

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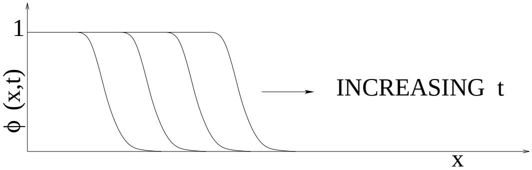

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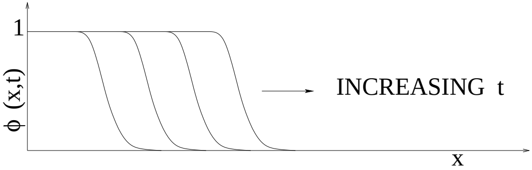

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(Bramson, van Saarloos, Brunet & Derrida, ...)

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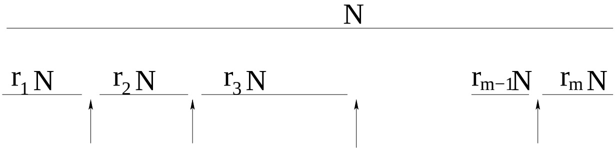
- Same strategy holds for the Balanced Height h_N

(P.L. Krapivsky & S.M., D.S. Dean and S.M., 2000-2006)

No of Non-Empty Nodes:



$$r_1 + r_2 + r_3 + \ldots\ldots + r_m = 1$$

No. of Non-empty nodes $n(N)$ in the tree $\equiv$ Total no. of Splitting Events in the fragmentation process till the Stopping Time, starting with the initial length N

Recursion:

$$n(N) \equiv n(r_1 N) + n(r_2 N) + n(r_3 N) + \cdots + n(r_m N) + 1; \qquad \sum_i^n r_i = 1$$

The marginal distribution of any fragment: $\eta_m(r) = (m-1)(1-r)^{m-2}$

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- Variance: $\nu(N) = \langle (n(N) - \mu(N))^2 \rangle$ satisfies another integral equation:

$$\nu(N) = m \int_{1/N}^{1} \nu(rN) \eta(r) dr + \langle (S - \langle S \rangle)^2 \rangle$$

where the Source Function $S = \sum_{i=1}^{m} \mu(r_i N)$.

These integral equations can be solved analytically (Dean & S.M.)

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For $m > m_c = 26.0461...$, $\theta(m) = \lambda_2(m)$ (Dean and S.M., 2002).

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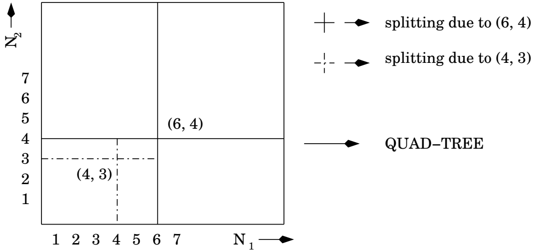

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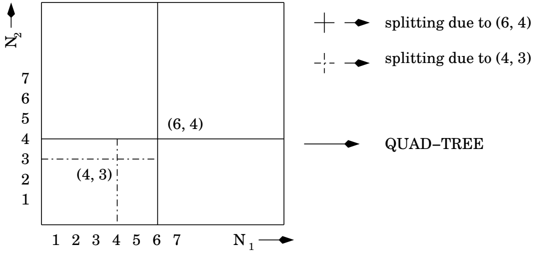

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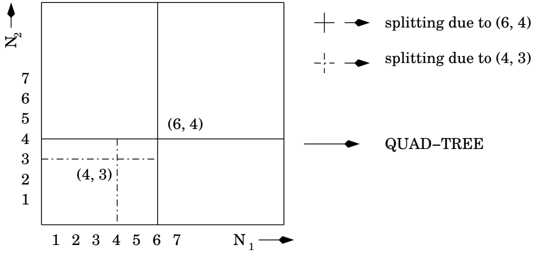

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Is there a PHASE TRANSITION in the variance of n_N?

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- $\langle h_N \rangle \approx 0.37336\ldots \log(N) + \frac{0.89374...}{D} \log(D \log(N)) + \ldots$

No. of Non-empty Nodes n_N : $\langle n_N \rangle \approx \frac{2}{D} V$ where $V = N^D$.

Variance ν_N has a Phase Transition:

$$\begin{aligned} \nu_N &\sim V && \text{for } D \leq D_c \\ &\sim V^{2\theta(D)} && \text{for } D > D_c \end{aligned}$$

$D_c = \frac{\pi}{\arcsin\left(\frac{1}{\sqrt{8}}\right)} = 8.69362\ldots$

$\theta(D) = 2\cos(\frac{2\pi}{D}) - 1 \rightarrow$ increases continuously with D for $D > D_c$

(D.S. Dean & S.M, 2002)

Probability Distribution of the no. of Non-Empty Nodes n_N:

$P[n_N] \to$ GAUSSIAN for $D < D_c = 8.69362\ldots$

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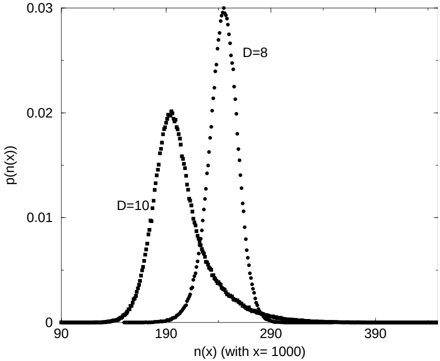

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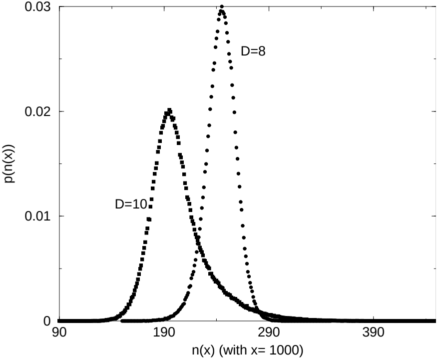

Further work in Computer Science: Janson '2005-'2008, Chern et. al. 2007,...

Exact vs. Rigorous

Chern et. al., ACM Trans. on Algorithms, 3, 1-51 (2007)

Phase Changes in Random Fragmentation Models. The same phase change phenomenon as leaves in random quadtrees was first observed in Dean and Majumdar [2002], where they proposed *random continuous fragmentation models* to explain *heuristically* the phase changes in random search trees. Their continuous model corresponding to quadtrees is as follows. Pick a point in $[0, x]^d$ uniformly at random ($x \gg 1$), which then splits the space into 2^d smaller hyperrectangles. Continue the same procedure in the subhyperrectangles whose volumes are larger than unity. The process stops when all subhyperrectangles have volumes less than unity. They argue heuristically that the total number of splittings undergoes a phase change: "While we can rigorously prove that the distribution is indeed Gaussian in the subcritical regime [$d \leq 8$], we have not been able to calculate the full distribution in the super-critical regime [$d \geq 9$]"; see Dean and Majumdar [2002].

Recently, Janson [2005] showed that the same type of phase change can be constructed by considering the number of nodes at distance ℓ satisfying $\ell \equiv j$ mod d, $0 \leq j < d$, in random binary search trees, or equivalently, the number of nodes using the $(\ell + 1)$-st coordinate as discriminators in random k-d trees, where $\ell \equiv j$ mod d. In all these problems, *periodicity* plays a key role in phase changes.

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Perspectives: Lots of beautiful open problems in Sorting and Search that may be possible to resolve using a variety of statistical physics techniques.

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