Condensation in temporally correlated zero-range dynamics

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joint work with Ori Hirschberg, David Mukamel, Weizmann Institute of Science

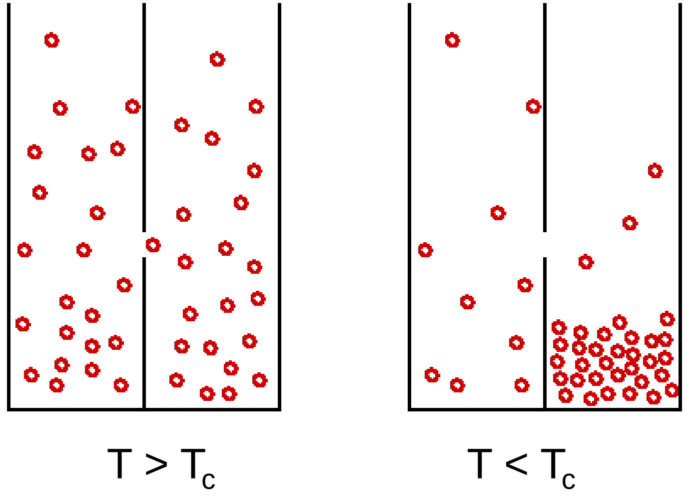
- Classical condensation phenomena
- Condensation transition in the zero-range process
- Non-Markovian ZRP dynamics
- Conclusions

1. Some classical condensation phenomena

Granular shaking:

N=100 plastic particles in box with two compartments separated by wall with slit
[Schlichting and Nordmeier '96, Eggers '99, Lohse '02]

Gaseous state



Condensed state

i) Strong shaking (fixed amplitude, 50 Hz frequency): ➔ Equal gaseous distribution

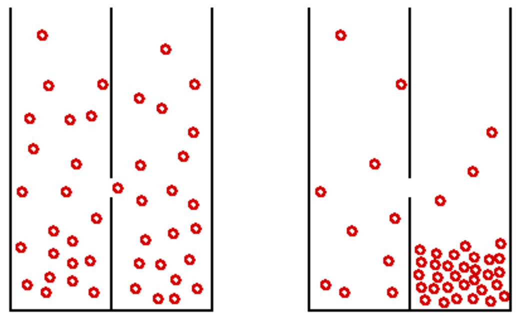
ii) Moderate shaking (same amplitude, 30 Hz): ➔ Condensation (with SSB)

Effective, frequency-dependent temperature leads to phase transition

Granular Clustering: L=5

http://stilton.tnw.utwente.nl/people/rene/clustering.html

Detlef Lohse, Devaraj van der Meer, Michel Versluis,
Ko van der Weele, René Mikkelsen



QuickTime™ and a
decompressor
are needed to see this picture.

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decompressor
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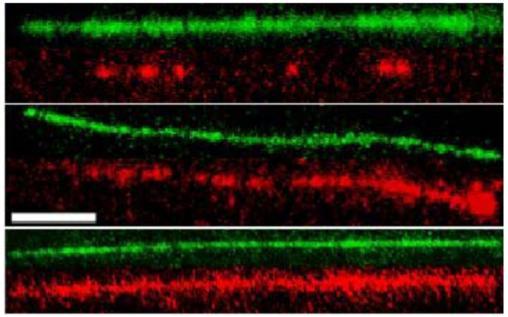
Time t = 0…12 sec

t approx. 1 min

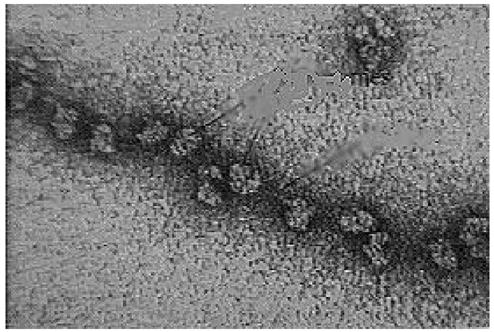
Single File Diffusion:

SFD: Quasi one-dimensional diffusive particle transport without passing

- diffusion in zeolites
- colloidal particles in narrow channels
- ion channels
- molecular motors and ribosomes
- gel electrophoresis
- one-dimensional interface growth
- automobile traffic flow
- …

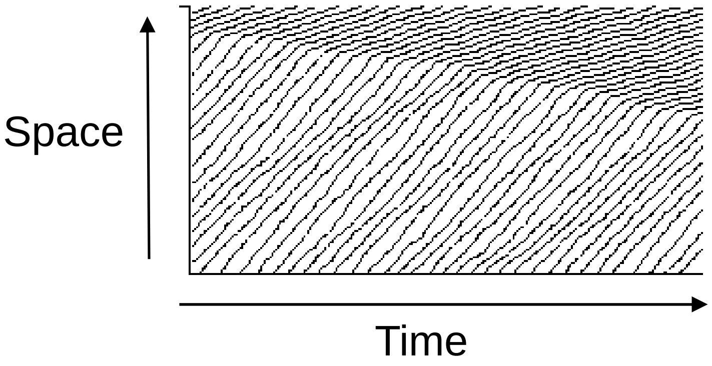


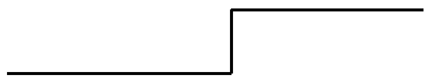
Three phases of kinesin transport (Chodhury et al.)



Polyribosome:
[http://omega.dawsoncollege.qc.ca/ray/protein/protein.htm
]

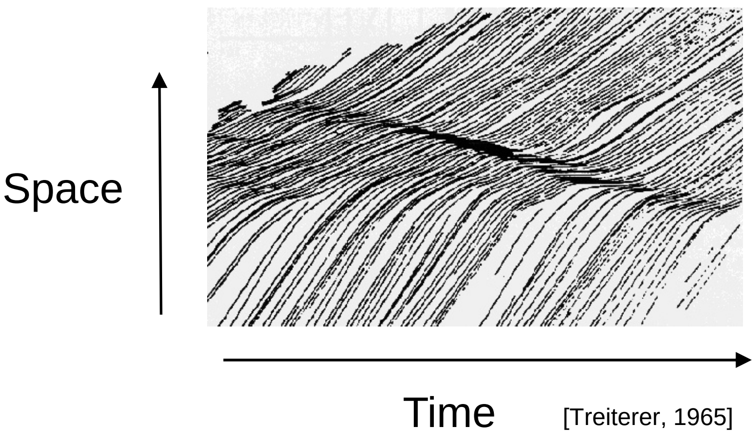
Traffic flow:

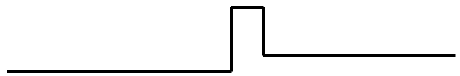




density profile

- Condensation = traffic jam





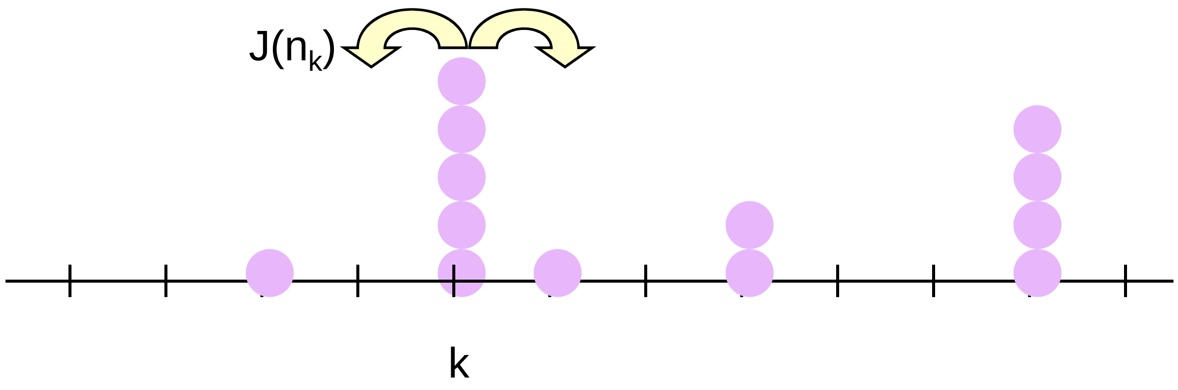
density profile

- Condensation = phase separation

2. Condensation transition in the zero-range process

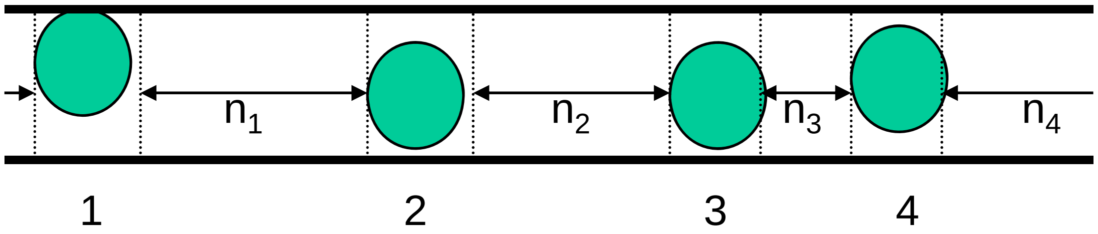
Zero-range process (ZRP) with symmetric nearest-neighbour hopping [Spitzer (1970)]

- Stochastic microscopic particle hopping model for large scale hydrodynamic behaviour
- Cluster of size n ⇔ occupation number in ZRP
- particle flux $J(n_k)$ between compartments ⇔ hopping rate in ZRP



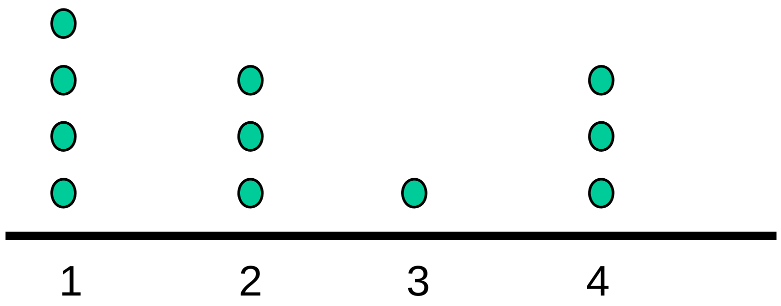
Mapping of single-file diffusion to zero range process:

- Label particles consecutively



- Map particle label to lattice site

- Map discretized interparticle distance to particle number



Condensation transition ➔ Proposed to explain condensation phenomena

- Granular shaking
- Network rewiring
- Accumulation of wealth

Mapping to single-file dynamics (one-dimensional):

- Phase separation in multi-component systems
- Traffic flow

➔ Generic model for condensation in complex systems

Exact grand canonical stationary distribution of zero-range process [Spitzer, (1970)]

➔ Product measure with marginals P(n) and local partition function Z

$$P(\vec{n}) = \prod_{i \in \Lambda} P(n_i)$$

$$P(n) = \frac{1}{Z} z^n \prod_{k=1}^{n} J^{-1}(k), \qquad Z = \sum_{n=0}^{\infty} \tilde{P}(n)$$

- Fugacity z determines (fluctuating) density
- Well-defined for fugacities within radius of convergence z* (that depends on flux function)
- Canonical ensembles for any N by projection on fixed N

Spatially homogeneous systems

1) Asymptotically vanishing flux J(n) → 0: ➔ $z^*=0$ and hence $\rho_c = 0$
   (strong condensation)

2) Consider generic case where for large n

$$J(n) = A\,(1 + b/n^{\sigma})$$

➔ radius of convergence of partition function: $z < z^* = A$

➔ at z^* one has finite density ρ_c for $\sigma < 1$

→ For $\sigma = 1$: ➔ $P(n) \sim 1/n^b$

$$\rho(z^*) = \begin{cases} \infty & \text{for } b \leq 2 \\ \rho_c = 1/(b-2) & \text{for } b > 2 \end{cases}$$

Interpretation of critical density for b>2 or $\sigma < 1$ for canonical ensemble:

• Above critical density all sites except one (background) are at critical density

• One randomly selected site carries remaining O(L) particles

➔ Classical analogue of Bose-Einstein condensation
[Evans ’96, Ferrari, Krug ’96, O’Loan, Evans, Cates, ’98, Jeon, March ‘00]

➔ Single random condensation site
[Grosskinsky, GMS, Spohn, ’05, Ferrari, Landim, Sisko ’07, Loulakis, Armendariz ‘08, Evans, Majumdar ‘08]]

➔ Continuous condensation transition ($\rho_{bg} = \rho_c$)

➔ Coarsening as precursor of condensation
[Grosskinsky, GMS, Spohn, ’05; Godreche ‘05]

Remarks:

- Product measure stationary for ZRP on arbitrary graph

- Single-file dynamics (n = interparticle distance) ➔ 1d phase transition?

  Thermally activated jumps: $J(n) \sim \exp(-\beta E(n)) \sim \exp(b/n^{\sigma})$

  $E(n) = a + b/n^{\sigma} + \ldots$ ➔ Long range interaction in 1d!

- Basic mechanism of condensation:

  Growth of large domains on the expense of small domains

  ➔ Asymptotically decaying J(n) with critical decay exponent $\sigma = 1$

  ➔ In this case, condensation depends on interaction strength b

3. Non-Markovian ZRP dynamics

Complex systems: Markovian property (lack of memory) may be unjustified

(e.g. colloidal particles in a fluid: power law tail in velocity autocorrelation)

➔ Introduce memory term (on microscopic level)

- Is condensation stable w.r.t. memory?
- Can memory induce condensation?

Example: AHR model for probe particle in a driven fluid:

- strongly correlated non-Markovian jumps with effective jump rate
- domain size distribution (distance between probes) identical to ZRP
- no condensation, but "almost" (huge mean domain size)

Our approach to model non-Markovian dynamics:

- make jump rates dependent on “age” of site i (integer clock τ_i)
  ==> $u(n,\tau)$
- age measured since last arrival (reset $\tau(k) = 0$ at arrival of particle)
- discrete increments τ_i --> $\tau_i + 1$ at exponential random times
- clock increment independent of n_i, but in general depending on other clocks

➔ Joint dynamics $(n(k),\tau(k))$ is Markovian

➔ Particle hopping $n(k)$ by itself is non-Markovian and zero range

1) Special case: On-off model with interaction of clocks

- Consider on-off case $\tau = 0,1$

- Asymmetric nearest neighbour jumps

$$u(n,\tau) = \begin{cases} 0 & \tau = 0 \quad (\text{"off" state}) \\ u(n) & \tau \geq 1 \quad (\text{"on" state}) \end{cases}$$

- Clock increment depending on target site

- Exact results:

- Stationary distribution factorizes into-site marginals $P(n) = P_0(n) + P_1(n)$

- P(n) same form as Markovian ZRP with effective hopping rate

$$u_{eff}(n) = c\, u(n) / (c + u(n))$$

==> Shift in critical b for condensation

2) Generic model without clock interaction

- make jump rates dependent on “age” of site i (integer clock τ_i)
  ==> $u(n,\tau)$

- age measured since last arrival (reset $\tau(k)=0$ at arrival of particle)

- discrete increments τ_i --> $\tau_i + 1$ at exponential random times (independent of n_i and other clocks)

$$(n_i,\tau_i),(n_j,\tau_j) \xrightarrow{u(n_i,\tau_i)} (n_i-1,\tau_i),(n_j+1,\tau_j=0)$$
$$(n_i,\tau_i) \xrightarrow{c} (n_i,\tau_i+1),$$

Consider two cases:

A) Mean field dynamics: Uniform random target site j (fully connected graph)

B) Totally asymmetric nearest neighbour dynamics (1-d periodic lattice)

A) Mean field dynamics:

- Uniform random target site j: Mean Field (MF) dynamics
- approximate factorization for large L
- focus on single site with incoming “mean-field” current J

$$\frac{dP(n,\tau)}{dt} = -P(n,\tau)[J + c + u(n,\tau)] +$$
$$JP(n-1)\delta_{\tau,0} + cP(n,\tau-1) + u(n+1,\tau)P(n+1,\tau)$$

with average occupation number $P(n) \equiv \sum_{\tau} P(n,\tau)$

and current $J = \sum_{n,\tau} u(n,\tau)P(n,\tau)$

Stationary distribution:

- set time-derivative to zero
- define mean hopping rate

$$\bar{u}(n) \equiv \frac{\sum_\tau P(n,\tau)\, u(n,\tau)}{\sum_\tau P(n,\tau)}$$

==> single-site marginal

$$P(n) = P(0) J^n \bar{f}(n) \quad \text{with} \quad \bar{f}(n) = \prod_{i=1}^{n} \bar{u}(i)^{-1}$$

Same form as usual Markovian ZRP with hopping rate ū(n)

and current J=z

Shift of condensation transition:

- Critical current

$$J_c = \frac{c}{2}\left(\sqrt{1+\frac{4}{c}} - 1\right)$$

==> b has to be larger than 2, condensation transition for

$$b > \frac{4}{c}\left(\sqrt{1+4/c} - 1\right)^{-1}$$

Memory destroys condensation for b close to 2

On-off model:

- Consider on-off case $\tau = 0,1$

$$u(n,\tau) = \begin{cases} 0 & \tau = 0 \quad (\text{"off" state}) \\ u(n) & \tau \geq 1 \quad (\text{"on" state}) \end{cases}$$

- Mean hopping rate

$$\frac{1}{\bar{u}(n)} = \frac{P_{\text{off}}}{J} + \frac{1}{u(n)}$$

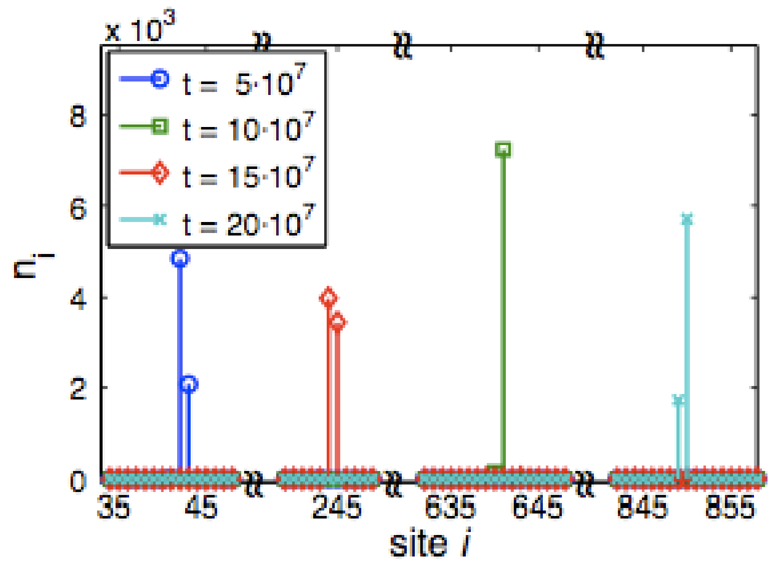
==> for generic Markovian rates u ~ 1 + b/n

$$\bar{u}(n) \sim \frac{c+J}{c+J+1}\left(1 + \frac{b_{\text{eff}}}{n}\right)$$

Same form as usual Markovian ZRP with interaction parameter b_{eff}

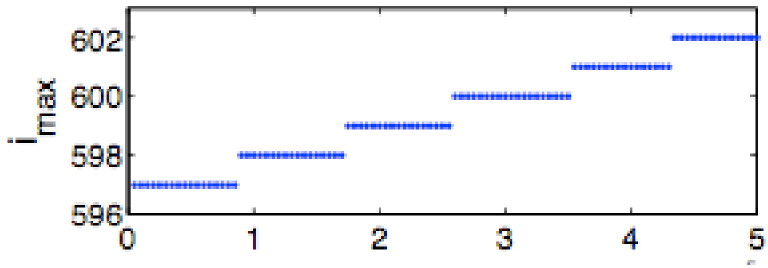
B) Totally asymmetric on-off model with periodic boundary conditions:

- mean field approximation not good
- condensate typically occupies two sites

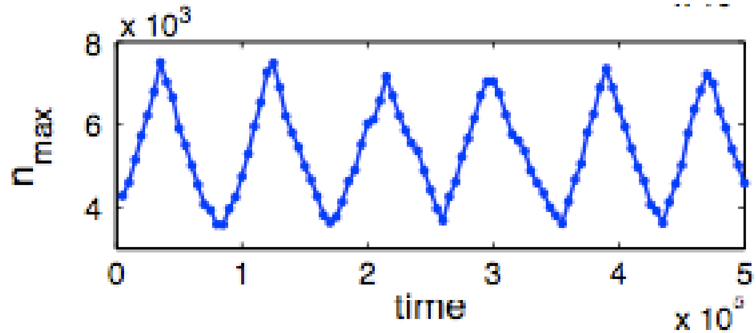


- condensate moves

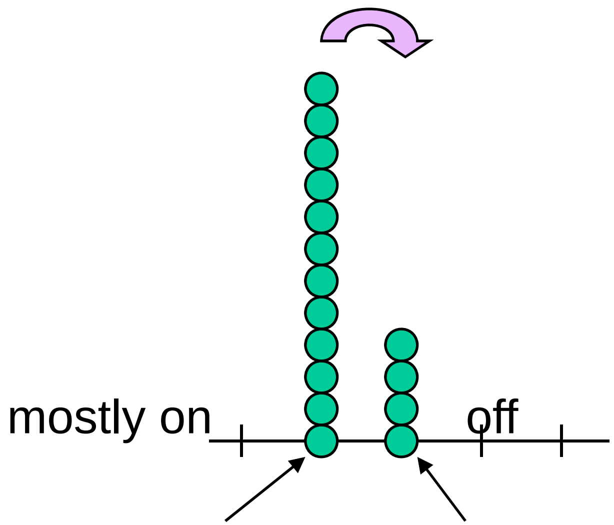
Motion of condensate:



- position of most occupied site



- occupation of most occupied site



==> slinky motion, speed $v = 1/(N-N_c) \sim 1/L$

Conclusions

1. Construction of family ZRP with memory

2. Exactly solvable case with coupled clocks: product measure, modified ZRP hopping rates that affect condensation

3. General mean field dynamics with uncoupled clocks: modified ZRP hopping rates that affect condensation (large L)

4. Totally asymmetric On-off model with nearest neighbour hopping in one dimension:
   - condensate occupies two sites
   - slinky motion with finite velocity ~ 1/L

➔ Similar conclusions for heterogeneous Single-File Diffusion with long range interaction

Acknowledgments

Thanks to:

- Weizmann Institute of Science