Integer partitions and exclusion statistics

Limit shapes and the largest part of Young diagrams

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Collaborators

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References

1. J. Stat. Mech. (2007) P10001
2. J. Math. Phys. Anal. Geom. **4**, 1 (2007)

Convex Polygons and their shapes

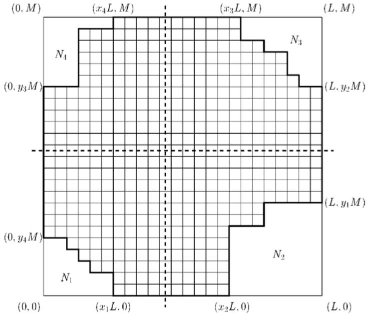

FIG. 1. A typical convex polygon on a square lattice and its bounding box is shown. All vertical and horizontal straight lines (dotted in the figure) intersect the polygon either 0 or 2 times. The convex polygon can be thought of as a rectangle from whose corners some squares have been removed by staircaselike paths.

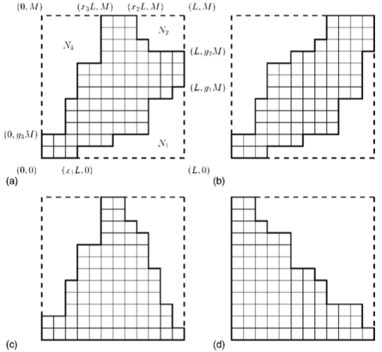

FIG. 4. Examples of (a) a directed convex polygon, (b) a staircase polygon, (c) a pyramidal polygon, and (d) Ferrers diagram on a square lattice.

Taken from [Rajesh & Dhar (2005)]

Interface between ordered phases

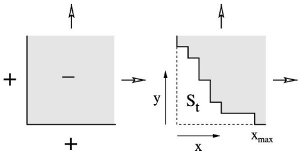

FIG. 1. A single corner interface (left) in the initial state and some time later (right). Gray denotes spin down (extending to ∞ in the $+x$ and $+y$ directions), and white denotes spin up. The evolving interface encloses an area S_t at time t.

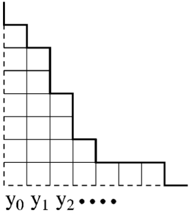

FIG. 3. The Young diagram that is based on the interface profile of Fig. 1. This diagram corresponds to a partition of the integer 22 into the set {7,6,4,2,1,1,1}.

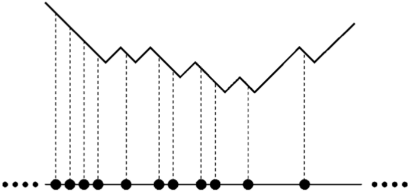

FIG. 10. The interface configuration of Fig. 3 rotated by 45° and the corresponding particle configuration.

Change in height

Total particle current

Taken from [Krapivsky, Redner, & Tailleur (2004)]

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   1. Integer partitions
   2. Integer partitions ⟷ ideal bosons/fermions
   3. Exclusion statistics
   4. Exclusion statistics ⟷ minimal difference partitions
2. **On minimal difference partitions**
   1. Grand partition function
   2. Limit shapes of Young diagrams
      1. Earlier results
      2. Our results
      3. Physical interpretation
   3. Largest part of Young diagrams
3. **Summary**

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Integer partitions

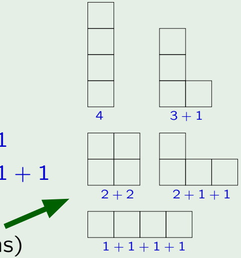
A partition of a positive integer E is a decomposition of E as a sum of a nonincreasing sequence of positive integers $\{h_j\}$, i.e.,

$$E = \sum_j h_j \quad \text{such that} \quad h_j \geq h_{j+1}, \quad \text{for} \quad j = 1, 2 \ldots .$$

Example

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

Young diagrams
(Ferrers diagrams)



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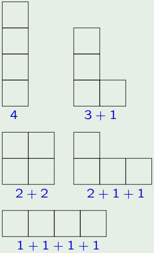

$\rho(E) := \#\{\text{partitions of } E\}$.

$$\begin{aligned} \rho(4) &= 5 \\ \rho(5) &= 7 \\ \rho(10) &= 42 \\ \rho(100) &= 190569292 \\ &\vdots \\ \rho(E) &\approx \frac{1}{4E\sqrt{3}} e^{\pi\sqrt{2E/3}} \end{aligned}$$

[Hardy & Ramanujan (1918)]

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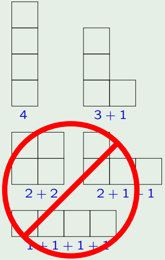

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Partitions into distinct parts

$$E = \sum_j h_j \quad \text{such that } h_j > h_{j+1}.$$

$\rho(4) = 2$

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Example

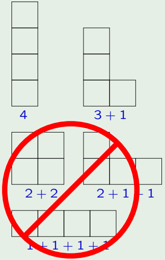
$4 = 4$

$= 3 + 1$

$=$ ~~$2 + 2$~~

$=$ ~~$2 + 1 + 1$~~

$=$ ~~$1 + 1 + 1 + 1$~~



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Partitions into distinct parts

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$$\rho(100) = 444793$$
$$\vdots$$
$$\rho(E) \approx \frac{1}{4} \cdot \frac{1}{3^{1/4} E^{3/4}} e^{\pi\sqrt{E/3}}$$

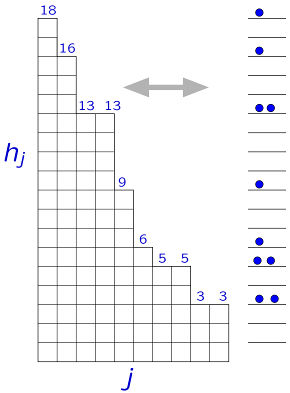
see [Abramowitz & Stegun (1972)]

Integer partitions

↕

Bose/Fermi statistics

Integer partitions ⟷ ideal bosons/fermions



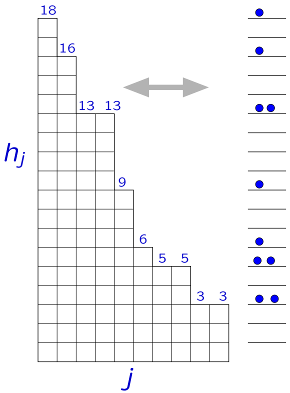
$$91 = 18 + 16 + 13 + 13 + 9 + 6 + 5 + 5 + 3 + 3$$

$n_i := \#\{\text{columns whose heights } = i\}$.

$$E = \sum_j h_j = \sum_{i=1}^{\infty} n_i \epsilon_i \quad \text{with } \epsilon_i = i.$$

Number of parts $N = \sum_{i=1}^{\infty} n_i$.

Integer partitions ⟷ ideal bosons/fermions



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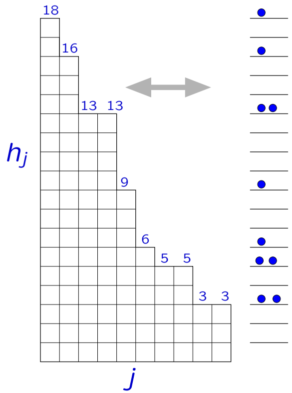

$91 = 18 + 16 + 13 + 13 + 9$
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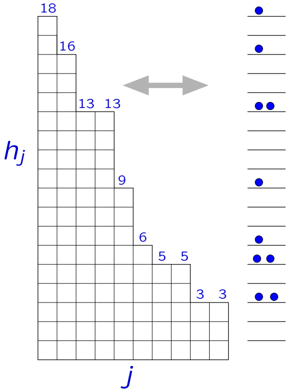

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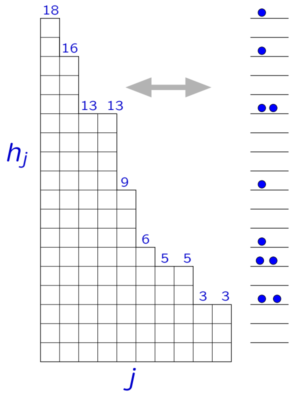
Number of ways of partitioning E

$$\rho(E) = \sum_{\{n_i\}} \delta\left(E - \sum_{i=1}^{\infty} n_i \epsilon_i\right)$$

Integer partitions ⟷ ideal bosons/fermions



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$\rho(E, N)$ → Number of ways of partitioning E into N parts

$$\rho(E, N) = \sum_{\{n_i\}} \delta\left(E - \sum_{i=1}^{\infty} n_i \epsilon_i\right) \delta\left(N - \sum_{i=1}^{\infty} n_i\right)$$

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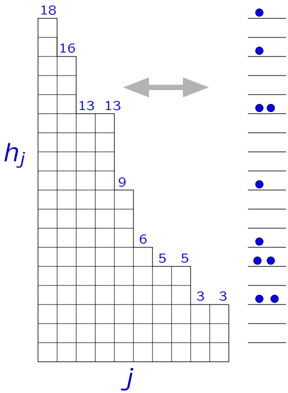

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The grand partition function:

$$\mathcal{Z}(\beta, z) = \sum_{N=0}^{\infty} z^N \sum_E \mathrm{e}^{-\beta E} \rho(E,N) = \begin{cases} \prod_{i=1}^{\infty} \left(1 - z\mathrm{e}^{-\beta i}\right)^{-1} & \text{(bosons)} \\ \prod_{i=1}^{\infty} \left(1 + z\mathrm{e}^{-\beta i}\right) & \text{(fermions)} \end{cases}$$

Exclusion statistics (thermodynamics)

If the grand partition function of a quantum gas

$$\mathcal{Z}(\beta, z) = \sum_{N=0}^{\infty} z^N \sum_E \mathrm{e}^{-\beta E} \underbrace{\rho(E, N)}$$

→ micro-canonical partition function

can be expressed as an integral representation

A $$\ln \mathcal{Z}(\beta, z) = \int_0^\infty \underbrace{\tilde{\rho}(\epsilon)} \ln y_p \left(z\mathrm{e}^{-\beta\epsilon} \right) \, \mathrm{d}\epsilon,$$

→ single particle density of states

where

B $$y_p(x) - x\, y_p^{1-p}(x) = 1,$$

then the gas is said to obey exclusion statistics with parameter $0 \le p \le 1$.

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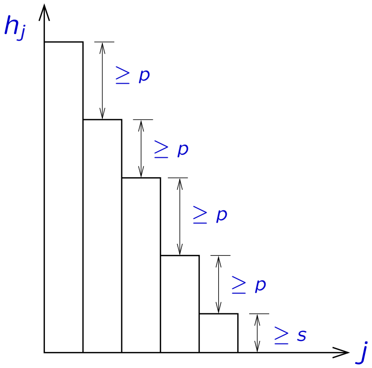
then the gas is said to obey exclusion statistics with parameter $0 \leq p \leq 1$.

1. $p = 0$ → $y_0(x) = \dfrac{1}{1-x}$ → Bose statistics.
2. $p = 1$ → $y_1(x) = 1 + x$ → Fermi statistics.
3. $0 < p < 1$ → fractional exclusion statistics.

Exclusion statistics

Minimal difference partitions



$$E = \sum_j h_j \quad \text{such that} \quad h_j - h_{j+1} \geq p$$

$$p = 0, 1, 2, \ldots$$

$$\rho(E, N, \ell, s) := \# \left[\begin{array}{l} \text{minimal difference partitions of } E \text{ into } N \text{ parts} \\ \text{such that the largest part } \leq \ell \text{ and the smallest part } \geq s \end{array} \right]$$

$$= \sum_{\{h_j\}} \delta\left(E - \sum_{j=1}^{N} h_j\right) \cdot \left[\prod_{i=1}^{N-1} \theta\big(h_j - h_{j+1} - p\big)\right] \cdot \theta\big(\ell - h_1\big) \cdot \theta\big(h_N - s\big)$$

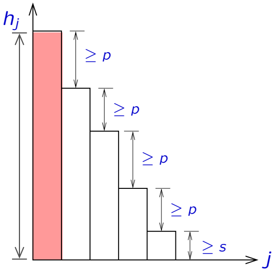
Outline

Grand partition function

$$\rho(E, N, \ell, s) := \# \left[\begin{array}{l} \text{minimal difference partitions} \\ \text{of } E \text{ into } N \text{ parts such that} \\ \text{the largest part } \leq \ell \text{, and} \\ \text{the smallest part } \geq s \end{array} \right]$$

$$= \rho(E, N, \ell - 1, s) + \rho(E - \ell, N - 1, \ell - p, s)$$

$$= \rho(E, N, \ell, s + 1) + \rho(E - s, N - 1, \ell, s + p)$$



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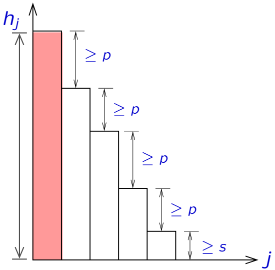

$$\mathcal{Z}(\beta, z, \ell, s) := \sum_{N=0}^{\infty} z^N \sum_E \mathrm{e}^{-\beta E} \rho(E, N, \ell, s)$$

$$= \mathcal{Z}(\beta, z, \ell - 1, s) + z\mathrm{e}^{-\beta \ell} \mathcal{Z}(\beta, z, \ell - p, s)$$

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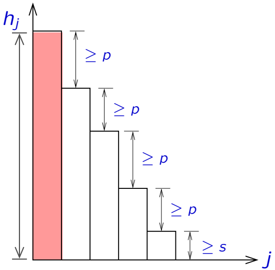

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$$= \mathcal{Z}(\beta,z,\ell,s+1) + z\mathrm{e}^{-\beta s}\mathcal{Z}(\beta,z,\ell,s+p)$$

Ansatz: $\mathcal{Z}(\beta,z,\ell,s) \approx \exp\left[\beta^{-1}\,\Phi(\beta\ell,\beta s,z)\right]$ as $\beta \to 0$.

$$\ln \mathcal{Z}(\beta,z,\ell,s) \xrightarrow{\beta\to 0} \frac{1}{\beta}\int_{\beta s}^{\beta\ell} \ln y_p\left(z\mathrm{e}^{-\epsilon}\right)\, \mathrm{d}\epsilon \quad \text{where} \quad y_p(x) - x\, y_p^{1-p}(x) = 1.$$

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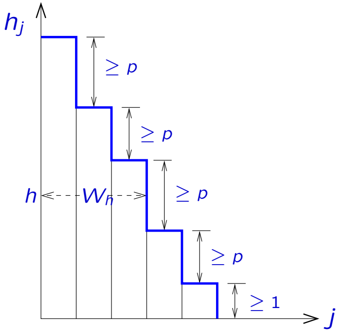
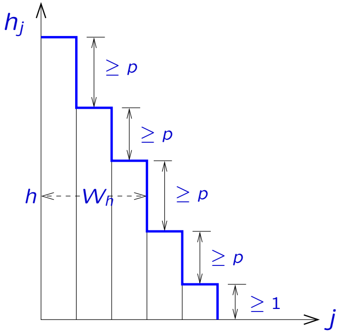


Let

$$X = \lim_{E\to\infty} \left[\frac{W_h}{\sqrt{E}}\right],$$

$$Y = \lim_{E\to\infty} \left[\frac{h}{\sqrt{E}}\right].$$

The limit shape is given by the XY curve.



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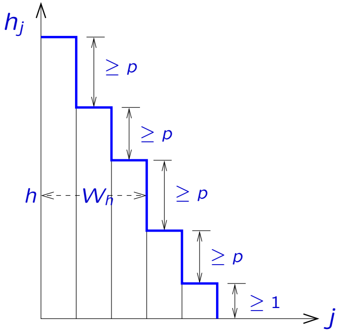
Earlier results

- $p = 0$ (unrestricted partitions):

$$\mathrm{e}^{-b(0)X} + \mathrm{e}^{-b(0)Y} = 1, \quad b(0) = \frac{\pi}{\sqrt{6}}$$

[Temperley (1952)]

Limit shapes



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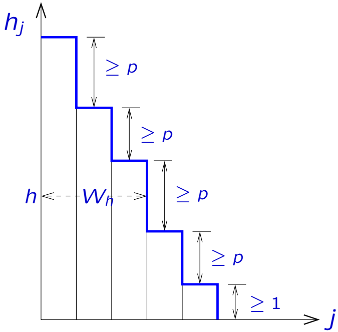
[Temperley (1952)] , [Vershik and collaborators]

- $p = 1$ (partitions into distinct parts):

$$e^{b(1)X} - e^{-b(1)Y} = 1, \quad b(1) = \frac{\pi}{\sqrt{12}}$$

[Vershik and collaborators]

Limit shapes



Let

$$X = \lim_{E\to\infty}\left[\frac{W_h}{\sqrt{E}}\right],$$

$$Y = \lim_{E\to\infty}\left[\frac{h}{\sqrt{E}}\right].$$

The limit shape is given by the XY curve.

Earlier results

- $p = 0$ (unrestricted partitions):

$$\mathrm{e}^{-b(0)X} + \mathrm{e}^{-b(0)Y} = 1, \quad b(0) = \frac{\pi}{\sqrt{6}}$$

[Temperley (1952)] , [Vershik and collaborators]

- $p = 1$ (partitions into distinct parts):

$$\mathrm{e}^{b(1)X} - \mathrm{e}^{-b(1)Y} = 1, \quad b(1) = \frac{\pi}{\sqrt{12}}$$

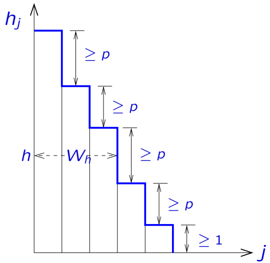
[Vershik and collaborators]

- $p = 2$ (minimal difference 2 partitions):

$$\mathrm{e}^{b(2)X} = \frac{1}{2}\left[1 + \sqrt{1 + 4\mathrm{e}^{-b(2)Y}}\right], \quad b(2) = \frac{\pi}{\sqrt{15}}$$

[Romik (2003)]

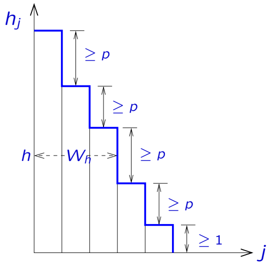
$W_h =$ number of columns whose heights $\geq h$.



$\mathcal{Z}_h(\beta, z) :=$ the restricted grand partition function which counts the columns whose heights $\geq h$.

$\mathcal{Z}(\beta, z) = \mathcal{Z}_1(\beta, z)$ is the full grand partition function which counts all the columns.

$W_h =$ number of columns whose heights $\geq h$.



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A $$\langle W_h \rangle = z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z) \Big|_{z=1}$$

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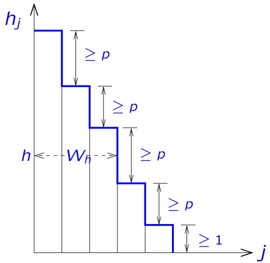

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$$\langle W_h \rangle = z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z)\Big|_{z=1}$$

$$E = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}(\beta, 1)$$

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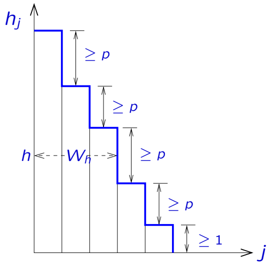

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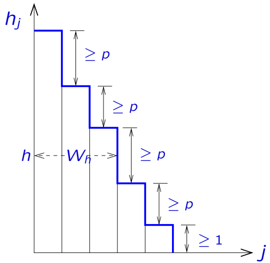
B $$\left[\langle W_h^2 \rangle - \langle W_h \rangle^2\right] = z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z) \Big|_{z=1}$$

Derivation of the limit shapes

$W_h =$ number of columns whose heights $\geq h$.



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A $$\langle W_h \rangle = z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z)\Big|_{z=1} \quad \Longrightarrow \quad \beta \langle W_h \rangle = \ln y_p \left(\mathrm{e}^{-\beta h}\right)$$

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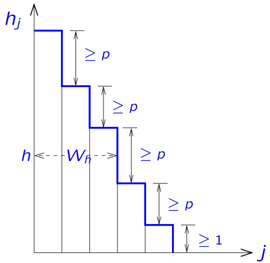

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A $$\langle W_h \rangle = z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z) \Big|_{z=1} \longrightarrow \beta \langle W_h \rangle = \ln y_p \left(\mathrm{e}^{-\beta h} \right)$$

$$E = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}(\beta, 1) \longrightarrow \beta = \frac{b(p)}{\sqrt{E}}$$

B $$\left[\langle W_h^2 \rangle - \langle W_h \rangle^2 \right] = z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z) \Big|_{z=1}$$

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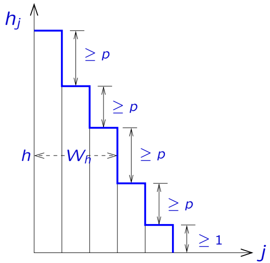

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$E = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}(\beta, 1)$ ⟶ $\beta = \frac{b(p)}{\sqrt{E}}$

B $$\beta^2 \left[\langle W_h^2 \rangle - \langle W_h \rangle^2\right] = \beta^2 z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \ln \mathcal{Z}_h(\beta, z) \Big|_{z=1} = \beta \mathrm{e}^{-\beta h} \left[\frac{y_p'\left(\mathrm{e}^{-\beta h}\right)}{y_p\left(\mathrm{e}^{-\beta h}\right)}\right]$$

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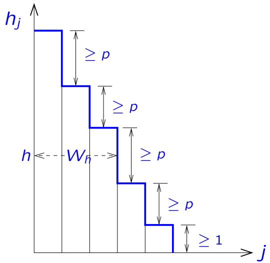

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$\beta \langle W_h \rangle \xrightarrow{\beta \to 0} \beta W_h$ ⟶ $\frac{W_h}{\sqrt{E}}$ vs $\frac{h}{\sqrt{E}} \xrightarrow{E \to \infty}$ Limit shape.

Ⓐ $X = \frac{1}{b(p)} \ln y_p\left(\mathrm{e}^{-b(p)Y}\right)$

or

Ⓑ $Y = -\frac{1}{b(p)} \ln\left(1 - \mathrm{e}^{-b(p)X}\right) - pX$

in which

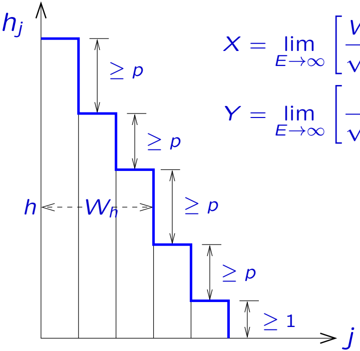
$$y_p(w) - w\, y_p^{1-p}(w) = 1$$

and

$$b^2(p) = \int_0^\infty \ln y_p\left(\mathrm{e}^{-\epsilon}\right)\, \mathrm{d}\epsilon = \frac{\pi^2}{6} - \mathrm{Li}_2(1/y^*) - \frac{p}{2}(\ln y^*)^2$$

where $y^* = y_p(1)$ i.e., $y^* - y^{*1-p} = 1$

and $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ is the dilogarithm function.



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$$b(1) = \frac{\pi}{\sqrt{12}}$$

$$b(2) = \frac{\pi}{\sqrt{15}}$$

$$b(3) = 0.752617\ldots$$

A $$X = \frac{1}{b(p)} \ln y_p\left(\mathrm{e}^{-b(p)Y}\right)$$

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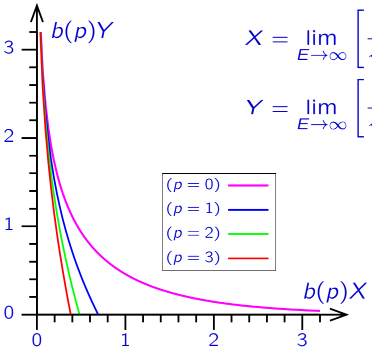

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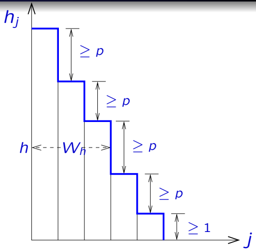
$$\beta W_h = \ln y_p\left(e^{-\beta h}\right) \quad \text{where} \quad y_p(x) - x\, y_p^{1-p}(x) = 1.$$

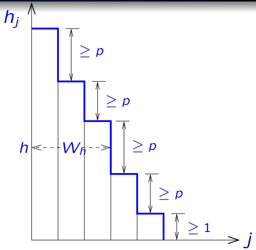


Physical interpretation of limit shape

$$\beta W_h = \ln y_p\left(e^{-\beta h}\right) \quad \text{where} \quad y_p(x) - x\, y_p^{1-p}(x) = 1.$$

Expressing h in terms of W_h yields

$$h = -\frac{1}{\beta}\ln\left(1 - e^{-\beta W_h}\right) - p W_h.$$



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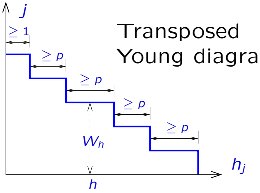

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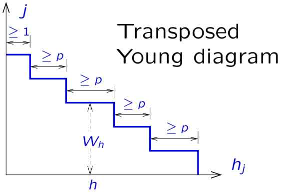

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→ # particles above energy level W_h

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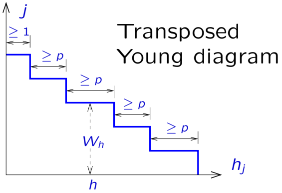
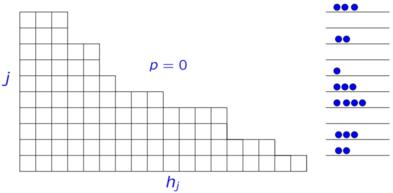


Expressing h in terms of W_h yields

$$h = \underbrace{-\frac{1}{\beta}\ln\left(1 - e^{-\beta W_h}\right)}_{\text{bosonic } (p=0)} - pW_h.$$

h → # particles above energy level W_h



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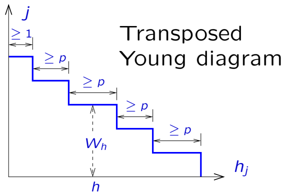

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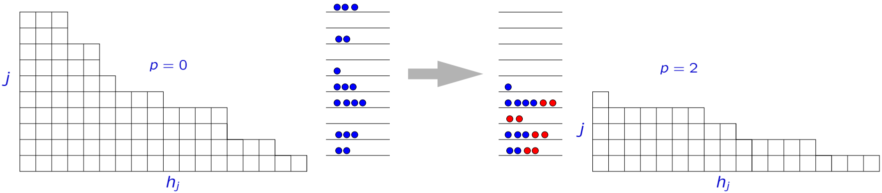
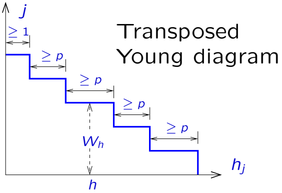


Physical interpretation of limit shape

$$\beta W_h = \ln y_p\left(e^{-\beta h}\right) \quad \text{where} \quad y_p(x) - x\, y_p^{1-p}(x) = 1.$$



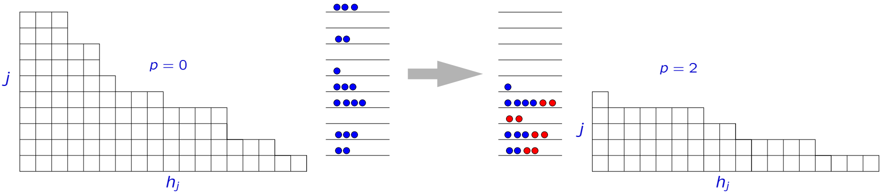
Expressing h in terms of W_h yields

$$h = -\frac{1}{\beta}\ln\left(1 - e^{-\beta W_h}\right) - pW_h.$$

bosonic ($p = 0$)

→ # particles transferred from above W_h to below

→ # particles above energy level W_h



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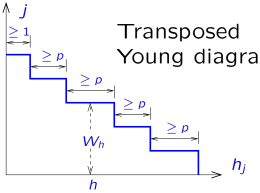


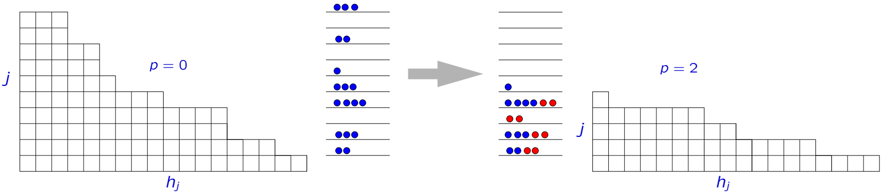
Transposed Young diagram

Expressing h in terms of W_h yields

$$h = \underbrace{-\frac{1}{\beta}\ln\left(1 - e^{-\beta W_h}\right)}_{\text{bosonic } (p=0)} - pW_h.$$

h → # particles above energy level W_h

pW_h → # particles transferred from above W_h to below



$$\int_0^{W_h^*} h(W_h)\, dW_h = E \;\Longrightarrow\; \beta = \frac{b(p)}{\sqrt{E}}, \qquad b^2(p) = \frac{\pi^2}{6} - \mathrm{Li}_2(1/y^*) - \frac{p}{2}(\ln y^*)^2.$$

W_h^* → $h(W_h^*) = 0$

$$y^* - y^{*1-p} = 1.$$

Outline

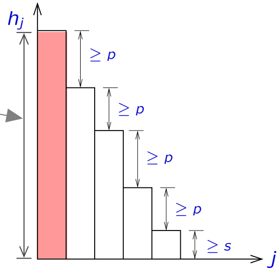
1. **Integer partitions and exclusion statistics**
   1. Integer partitions
   2. Integer partitions ⟷ ideal bosons/fermions
   3. Exclusion statistics
   4. Exclusion statistics ⟷ minimal difference partitions

2. **On minimal difference partitions**
   1. Grand partition function
   2. Limit shapes of Young diagrams
      1. Earlier results
      2. Our results
      3. Physical interpretation
   3. Largest part of Young diagrams

3. **Summary**

- $\rho(E,\ell) := \#\left[\begin{array}{l}\text{minimal difference partitions of } E\\ \text{such that the largest part } h_1 \le \ell\end{array}\right]$

  $\rho(E,\ell\to\infty) = \rho(E) \equiv \#$ partitions



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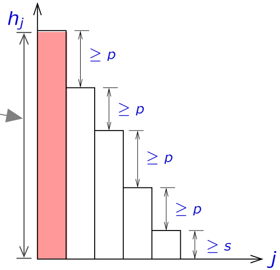

- Consider uniform measure: $1/\rho(E)$

$$C(\ell|E) := \frac{\rho(E,\ell)}{\rho(E)} = \mathrm{Prob}[h_1 \leq \ell]$$

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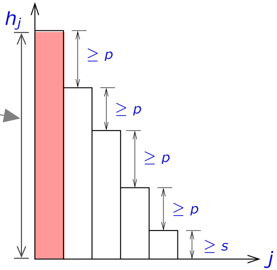

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- We show that (for all p):

$$C(\ell|E) \xrightarrow[\ell \gg \sqrt{E}]{E\to\infty} F\left(\frac{\ell-\ell^*(E)}{\sigma(E)}\right)$$

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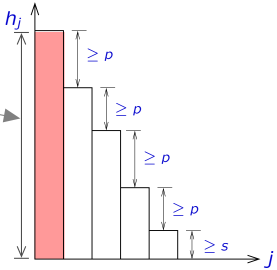

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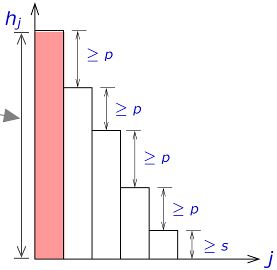

- Consider uniform measure: $1/\rho(E)$

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- We show that (for all p):

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where $\ell^*(E) = \frac{\sqrt{E}}{b(p)} \ln \frac{\sqrt{E}}{b(p)}$ and $\sigma(E) = \frac{\sqrt{E}}{b(p)}$

$b^2(p) = \frac{\pi^2}{6} - \mathrm{Li}_2(1/y^*) - \frac{p}{2}(\ln y^*)^2$ where $y^* - y^{*1-p} = 1$

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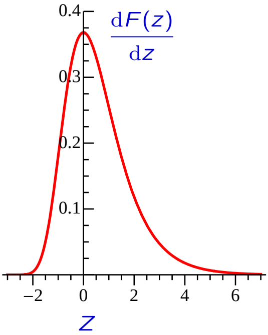


- We show that (for all p):

$$C(\ell|E) \xrightarrow[\ell \gg \sqrt{E}]{E\to\infty} F\left(\frac{\ell-\ell^*(E)}{\sigma(E)}\right) \quad \text{or} \quad \lim_{E\to\infty} C\big(\ell^*+\sigma z|E\big) = F(z)$$

where $\ell^*(E) = \frac{\sqrt{E}}{b(p)} \ln \frac{\sqrt{E}}{b(p)}$ and $\sigma(E) = \frac{\sqrt{E}}{b(p)}$

$b^2(p) = \frac{\pi^2}{6} - \mathrm{Li}_2(1/y^*) - \frac{p}{2}(\ln y^*)^2$ where $y^* - y^{*1-p} = 1$

The scaling function has the Gumbel form: $F(z) = \exp\big(-\exp(-z)\big)$

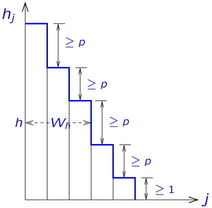
Earlier result existed only for the $p=0$ case [Erdös & Lehner (1951)]

Outline

1. **Minimal difference partitions**

$$E = \sum_j h_j$$

such that $h_j - h_{j+1} \geq p$ for $j = 1, 2, 3 \ldots$



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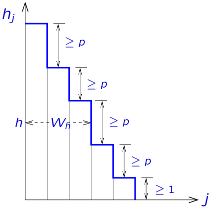

2. **Partitions** ⟷ **exclusion statistics**

$$\ln \mathcal{Z}(\beta, z, \ell, s) \xrightarrow{\beta \to 0} \frac{1}{\beta} \int_{\beta s}^{\beta \ell} \ln y_p \left(z \mathrm{e}^{-\epsilon} \right) \, \mathrm{d}\epsilon \quad \text{where} \quad y_p(x) - x\, y_p^{1-p}(x) = 1.$$

Summary

1. **Minimal difference partitions**

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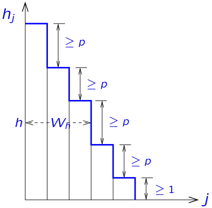

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3. **Limit shapes of Young diagrams** (and physical interpretation)

$$X = \frac{1}{b(p)} \ln y_p \left(\mathrm{e}^{-b(p) Y} \right) \quad \text{or} \quad Y = -\frac{1}{b(p)} \ln \left(1 - \mathrm{e}^{-b(p) X} \right) - pX$$

$$X \leftrightarrow \lim_{E \to \infty} \left[W_h / \sqrt{E} \right] \qquad Y \leftrightarrow \lim_{E \to \infty} \left[h / \sqrt{E} \right] \qquad b^2(p) = \int_0^\infty \ln y_p \left(\mathrm{e}^{-\epsilon} \right) \, \mathrm{d}\epsilon$$

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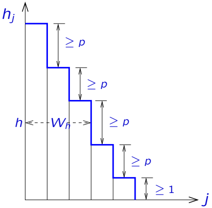

2. **Partitions** ⟷ **exclusion statistics**

$$\ln \mathcal{Z}(\beta, z, \ell, s) \xrightarrow{\beta \to 0} \frac{1}{\beta} \int_{\beta s}^{\beta \ell} \ln y_p\left(z e^{-\epsilon}\right) d\epsilon \quad \text{where} \quad y_p(x) - x\, y_p^{1-p}(x) = 1.$$

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$$\lim_{E\to\infty}\left[W_h/\sqrt{E}\right] \qquad \lim_{E\to\infty}\left[h/\sqrt{E}\right] \qquad b^2(p) = \int_0^\infty \ln y_p\left(e^{-\epsilon}\right) d\epsilon$$

4. **Asymptotic distribution of the largest part of the Young diagram**

Distribution of $z = \frac{b(p)}{\sqrt{E}}\left[\ell - \frac{\sqrt{E}}{b(p)} \ln \frac{\sqrt{E}}{b(p)}\right]$ $\xrightarrow{E \to \infty}$ Gumbel distribution $F(z) = \exp\left(-\exp(-z)\right)$

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