Center or Limit Cycle: Renormalization Group as a Probe

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Introduction

- Consider a theory with a set of coupling constants g_0 and a natural cutoff. l.
- Field theory defined for $l \to 0$ but perturbation theory diverges.
- Physical quantity $= f(g_0, l)$; Take arbitrary scale μ.
- Let $l \to \infty$ and write physical quantity $f(g_0, \mu)$; μ is arbitrary.
- RG expresses the fact that the physical quantity is independent of μ.

Introduction

L. Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. E **54**, *376 (1996)*

- Asymptotic solutions of differential equations.

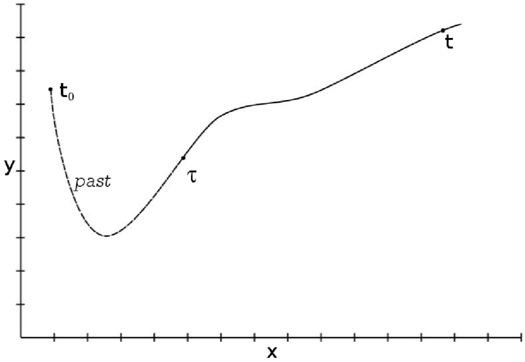
$$\ddot{x} + k\dot{x}(x^2 - 1) + \omega^2 x = 0$$

- Special periodic orbit called *limit cycle*.
- For $k > 0$, trajectory settles on a periodic orbit of fixed radius(independent of initial conditions).
- RG applied to find size and frequency.
- Advantage: straightforward perturbation theory.

Introduction

- Trajectory characterized by amplitude A, phase Θ.
- Chose initial condition at $t = t_0$.
- Answer in terms of perturbation theory.
- Divergent series

Introduction



Initial condition could be anywhere on the path

- Place initial condition at τ
- $x(t)$ independent of τ gives RG flow

Introduction

Amplitude A and phase Θ flow

$\frac{dA}{d\tau} = f(A,\Theta)$: $\frac{d\Theta}{d\tau} = g(A,\Theta)$

Two dimensional autonomous: function of A alone

Centre

$\frac{dA}{d\tau} = 0$ initial condition sets amplitude which cannot change

Isochronous centre

$\frac{d\Theta}{d\tau} = 0$ as well

Limit Cycle: Isolated trajectory

$\frac{dA}{d\tau} = f(A)$ Fixed point gives size of orbit **or** if $A^* = 0$ implies *focus*

Duffing's Equation

$$\ddot{x} + \omega^2 x + \lambda x^3 = 0$$

Expanding $x = x_0 + \lambda x_1 + \lambda^2 x_2 + \dots$, we have at different orders of λ,

$O(\lambda^0)$: $\ddot{x}_0 + \omega^2 x_0 = 0$
$O(\lambda^1)$: $\ddot{x}_1 + \omega^2 x_1 = -x_0^3$
$O(\lambda^2)$: $\ddot{x}_2 + \omega^2 x_2 = -3x_0^2 x_1$

Solving, we get

- $x_0 = A_0 \cos(\omega t + \Theta_0)$; $\quad t = -\Theta_0/\omega \quad x = A_0, \ \dot{x} = 0$
- $\ddot{x}_1 + \omega^2 x_1 = -\frac{A_0^3}{4}\left(3\cos(\omega t + \Theta_0) + \cos 3(\omega t + \Theta_0)\right)$
  $\Rightarrow x_1 = B_1 \cos \omega t + B_2 \sin \omega t + \frac{A_0^3}{32\omega^2} \cos 3(\omega t + \Theta_0)$
  $-\frac{3A_0^3}{8\omega} t \sin(\omega t + \Theta_0)$;
  for initial conditions $x_1 = \dot{x}_1 = 0$ at $t = -\Theta_0/\omega$

Duffing's Equation

- $x_1 = -\frac{3A^3}{8\omega}\left(t + \frac{\theta_0}{\omega}\right)\sin(\omega t + \theta_0) + \frac{A^3}{32\omega^2}\{\cos 3(\omega t + \theta_0) - \cos(\omega t + \theta_0)\}$
- $x_0 = A_0 \cos(\omega t + \Theta_0)$
- $x = x_0 + \lambda x_1 + \lambda^2 x_2 + \dots$
- Split interval $-\Theta/\omega$ to t as $-\Theta/\omega$ to τ; τ to t for the divergent term.
- Finally

$$\mathrm{x} = \mathrm{A}(\mathrm{t}_0)\cos\left(\omega t + \theta(t_0)\right) - \frac{3\lambda A^3}{8\omega}\left(t + \frac{\theta_0}{\omega}\right)\sin(\omega t + \theta_0) + \left[\frac{\lambda A^3}{32\omega^2}\{\cos 3(\omega t + \theta_0) - \cos(\omega t + \theta_0)\}\right]$$

Duffing's Equation

Define renormalization constants Z_1 and Z_2

$$A(t_0) = Z_1(t_0, \tau) A(\tau)$$

$$\Theta(t_0) = \Theta(\tau) + Z_2(t_0, \tau)$$

Where,

$$Z_1 = \sum_{n=1}^{\infty} a_n \lambda^n \qquad Z_2 = \sum_{n=1}^{\infty} b_n \lambda^n$$

Duffing's Equation

$$\begin{aligned}
x &= A(\tau)\,(1 + a_1\lambda + \dots)\cos(\omega t + \Theta + b_1\lambda) + \dots \\
&= A(\tau)\cos(\omega t + \Theta) + a_1\lambda A(\tau)\cos(\omega t + \Theta) - b_1\lambda A(\tau)\sin(\omega t + \Theta) \\
&\quad -\frac{3\lambda A^3}{8\omega}\left(t - \tau + \tau + \frac{\Theta_0}{\omega}\right)\sin(\omega t + \Theta) + \dots
\end{aligned}$$

Choose $a_1 = 0$ and $b_1 = -\frac{3A^2}{8\omega}\left(\tau + \frac{\Theta_0}{\omega}\right)$

$$\begin{aligned}
\frac{dx}{d\tau} &= 0 \\
0 &= \frac{dA}{d\tau}\cos(\omega t + \Theta) - A\frac{d\Theta}{d\tau}\sin(\omega t + \Theta) + \frac{3\lambda A^3}{8\omega}\sin(\omega t + \Theta) + \dots
\end{aligned}$$

So,

$$\frac{dA}{d\tau} = 0 \qquad ; \qquad \frac{d\Theta}{d\tau} = \frac{3\lambda A^2}{8\omega} + \dots$$

Duffing's Equation

$$x = A(\tau)\cos\left(\omega t + \frac{3\lambda A^2}{8\omega}\tau + \Theta_0\right) - \frac{3\lambda A^3}{8\omega}(t-\tau)\sin(\omega t + \Theta_0) + \dots$$

Set $\tau = t$ to remove the remaining problem

$x = A(t)\cos(\Omega t + \Theta_0) + O(\lambda)$
$\Omega = \omega + \frac{3\lambda A^2}{8\omega}$
$\dot{A} = 0$

$$x = A_0\cos(\Omega t + \Theta_0) + \frac{\lambda A^3}{32\Omega^2}\Big[\cos 3(\Omega t + \Theta_0) - \cos(\Omega t + \Theta_0)\Big] + O(\lambda^2)$$

In agreement with perturbation expansion of exact solution.

Forced Oscillator

Best known limit cycle

$$\ddot{x} + k\dot{x} + \omega^2 x = f\cos\Omega t$$

Calculation strategy: Find a centre about which to perturb

$$\ddot{x} + \Omega^2 x = f\cos\Omega t - k\dot{x} + (\Omega^2 - \omega^2)x$$

- Non-autonomous system: frequency fixed
- Centre for $f = k = \Omega^2 - \omega^2 = 0$; Perturbation theory treats each as small

$$\frac{dA}{d\tau} = -\frac{kA}{2} - \frac{F\sin\Theta}{2\Omega}; \qquad \frac{d\Theta}{d\tau} = -\frac{F\cos\Theta}{2\Omega A} + \Delta\omega$$

where $\Delta\omega \equiv \omega - \Omega$. $\frac{d\Theta}{d\tau} = 0$ and $\frac{dA}{d\tau} = 0$ gives

$$A = F/[k^2 + 4(\Delta\omega)^2]^{1/2} \text{ and } \Theta = \tan^{-1}[-k/2(\Delta\omega)].$$

Lotka-Volterra equation

Population dynamics: Lotka Volterra system

$$\frac{dx}{dt} = x - xy$$
$$\frac{dy}{dt} = -y + xy$$

Fixed points: $(0,0)$: Saddle and (1,1)
Shift origin to $x = y = 1$. Resulting equations:

$$\dot{X} = -Y - XY$$
$$\dot{Y} = X + XY$$

$X = Y = 0$ is Centre

$$\frac{dA}{d\tau} = 0; \qquad \frac{d\theta}{d\tau} = -\frac{A^2}{12}$$

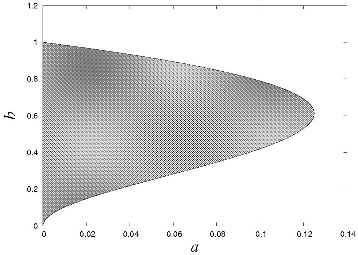
Oscillations with frequency $\omega = 1 - \frac{A^2}{12}$

Glycolytic Oscillator

Biological Oscillator: glycolysis

$$\begin{aligned}\dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y\end{aligned}$$

x: ADP (adenosine diphosphate) ; y: F6P (fructose-6-phosphate)
fixed point $x = b$, $y = \frac{b}{a+b^2}$



The shaded region of parameter space (a, b), we get limit cycle solution; fixed point solution outside this region;

nonlinear centre at any point of the bounding curve

System with no linear term

Consider the system,
$\ddot{x} + \lambda x^3 = 0$
Introduce $\Omega^2 = \lambda a\langle x^2\rangle$ to write

$$\ddot{x} + \lambda a\langle x^2\rangle + \lambda\left[x^3 - a\langle x^2\rangle x\right] = 0$$
$$\Rightarrow \ddot{x} + \Omega^2 x = -\lambda\left[x^3 - a\langle x^2\rangle x\right]$$

At different orders of λ we get,

$$\begin{aligned} \ddot{x}_0 + \Omega^2 x_0 &= 0 \\ \ddot{x}_1 + \Omega^2 x_1 &= -x_0^3 + a\langle x_0^2\rangle x_0 \end{aligned}$$

Now, $\Omega^2 = \lambda a\left[\langle x_0^2\rangle + 2\lambda\langle x_0 x_1\rangle + \ldots\right]$
Flow equations upto $O(\lambda)$, $\frac{dA}{d\tau} = 0$
$\frac{d\Theta}{d\tau} = \frac{A^2}{2\Omega}\left(a - \frac{3}{2}\right)$ Fix a to keep frequency at Ω, i.e. $\frac{d\Theta}{d\tau} = 0$

Riccati Equation

Riccati equation of second kind

$$\ddot{x} + 3\dot{x}x + x^3 = 0$$

Exact solution where $x, \dot{x} \to 0$ as $t \to \infty$

$$\ddot{x} + \lambda k \dot{x}x + \lambda^2 x^3 = 0$$

Jordan and Smith: numerically periodic solution for $k = 0.1$
Consider the general system with arbitrary k: for what value of k does system become aperiodic?

Riccati Equation

Flow equations

$$\frac{dA}{d\tau} = 0$$
$$\frac{d\Theta}{d\tau} = \frac{\lambda A^2}{2\Omega}\left(9 - k^2 - 6a\right)$$

- For periodic orbit $\lambda > 0$; possible only if $k < 3$
- "Two loop": $k < 2.61$; $T \propto (k - k_c)^{-1/2}$
- Numerics: No periodic orbit for $k > 2.80$

Isochronous oscillation

$$\ddot{x} + x = \frac{1}{x^3}; \qquad V(x) = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$$

- Checked to sixth order in amplitude
- Cherkas System
- RG immediately yields constraints on the parameters.