Glassy disordered systems beyond Mean Field: Hierarchical models

Silvio Franz

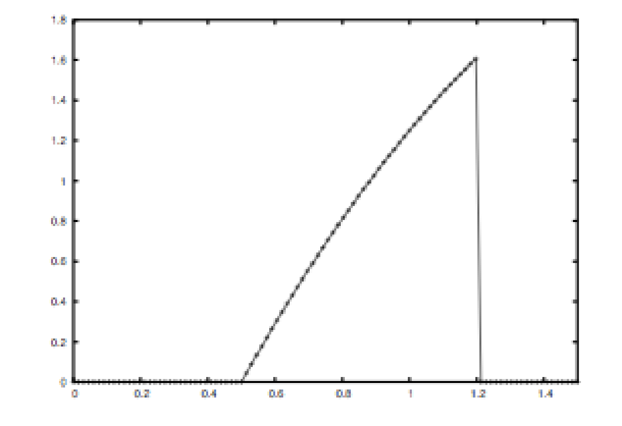
LPTMS Orsay

M. Castellana, A.Decelle, M. Mezard, G. Parisi

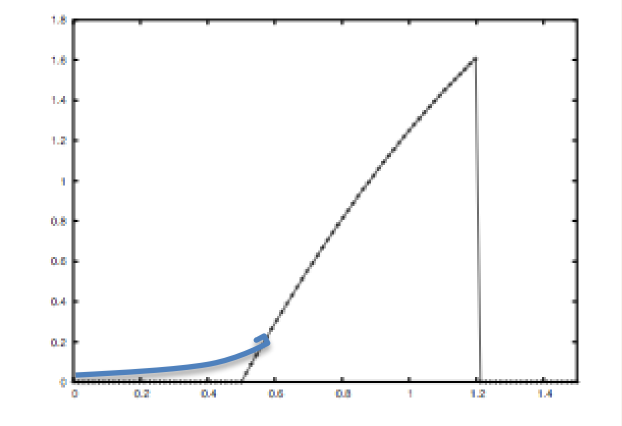
Mean Field models have ideal glass transitions

Are ideal glass transitions possible beyond MF ?

This …..



…. or that ?



Asymptotic expansions around MF do not suggest a mechanism to avoid them.

Reasonable arguments on nature of excitations are against it (Fisher-Huse droplets, Stillinger)

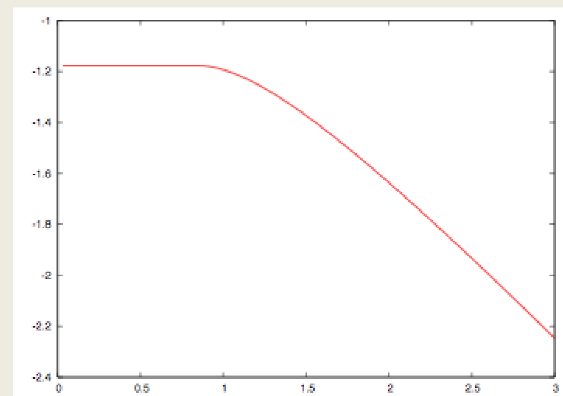
The REM Paradigm: The simplest glass transition

$$\sigma \in \{1, ..., 2^N\} \qquad E(\sigma) \qquad \text{i.i.d. Gaussian var.}$$

$$\overline{E(\sigma)^2} = N/2$$

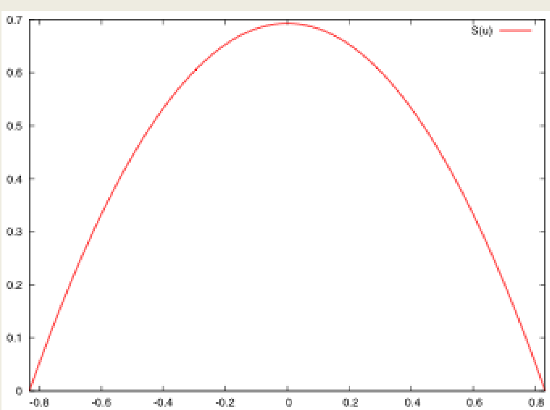
$$Z = \sum_\sigma e^{-\beta E(\sigma)} = \text{"}\sum_E\text{"} \mathcal{N}(E) e^{-\beta E}$$

Exact solution



$$\mathcal{N}(E) \sim e^{NS(E/N)}$$

$$S(u) = \begin{cases} log(2) - u^2 & |u| < \sqrt{log(2)} \\ 0 & |u| \geq \sqrt{log(2)} \end{cases}$$

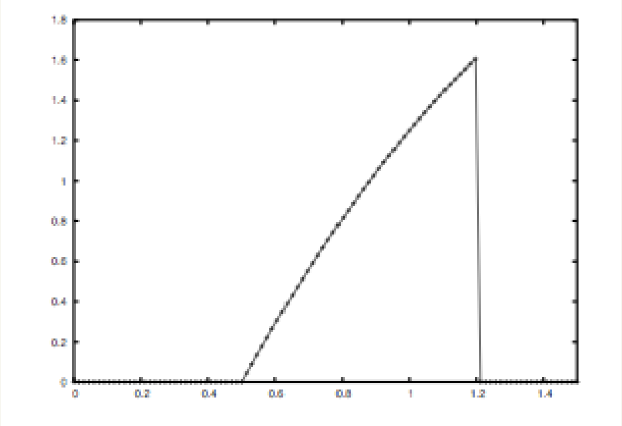


$$F(\beta) = -\frac{T}{N} \log Z = \begin{cases} \beta/2 - \frac{log(2)}{\beta} & \beta < \beta_c = 2\sqrt{log(2)} \\ -\sqrt{2} & \beta > \beta_c \end{cases}$$

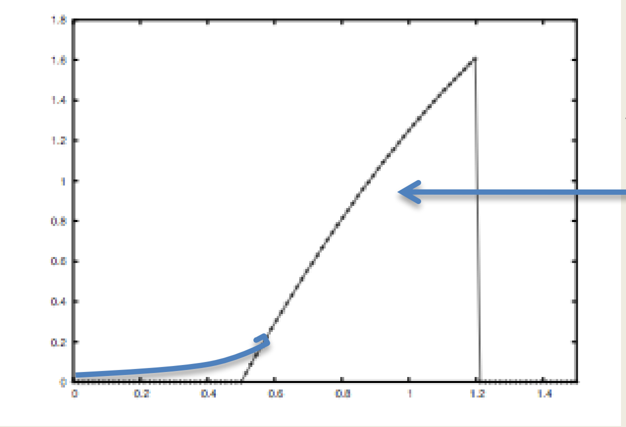
Mean Field models have ideal glass transitions

Are ideal glass transitions possible beyond MF ?

This …..



…. or that ?



Irrelevant for the physics here

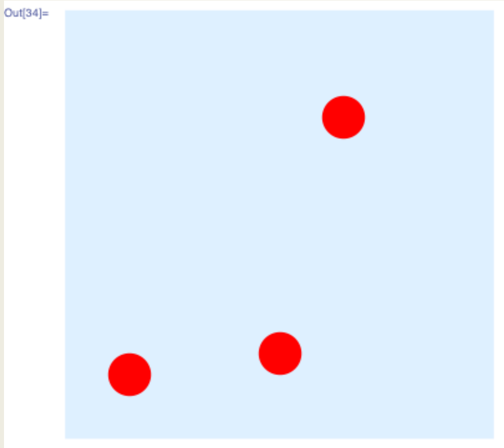
Asymptotic expansions around MF do not suggest a mechanism to avoid them.

Reasonable arguments on nature of excitations are against it (Fisher-Huse droplets, Stillinger)

Effect of finite interaction range

Stillinger argument entropy of excitations prevents an ideal glass transition

Identification of inherent structures with states



From the Ground State one can construct low lying excited state by localized rearrangement

Contribute of local excitations is additive

Stillinger Argument (Inherent Structures)

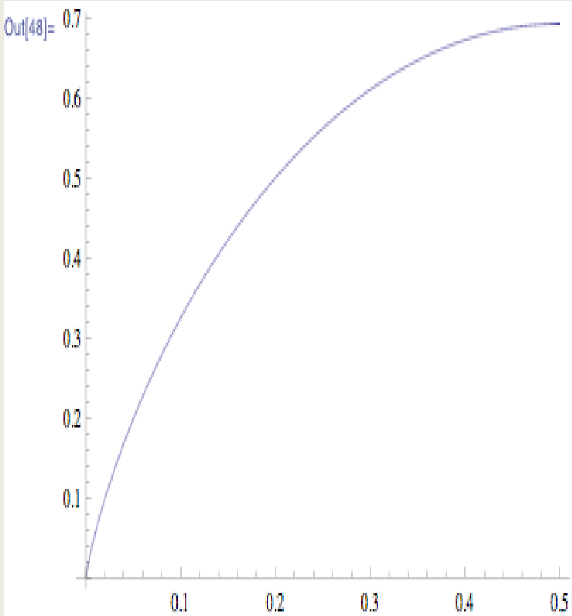
Excitation with energy δE extension δv

$n = xV/\delta v$ excitations $E = n\delta E$ distance from GS

Can be placed in $\binom{V/\delta v}{n} \sim e^{-VE\log E}$

$$\Sigma(E) = -E\log(E)$$

$$\frac{\partial \Sigma}{\partial E} \sim -\log(E) \to \infty \quad (E \to 0)$$



No Ideal transition

Usual objection IS are not states

Non Mean Field Behavior:

Low D systems with short range interactions

1D Systems with power law interactions

$$H = -\sum_{1,j}^{1,L} \frac{1}{|i-j|^{\sigma}} S_i S_j$$

Well defined Thermodynamics

$$\sigma > 1$$

Phase transition for $\sigma < 2$

Convex Free-Energy ;
Interface Cost $DE \sim L^{2-\sigma}$
Nucleation

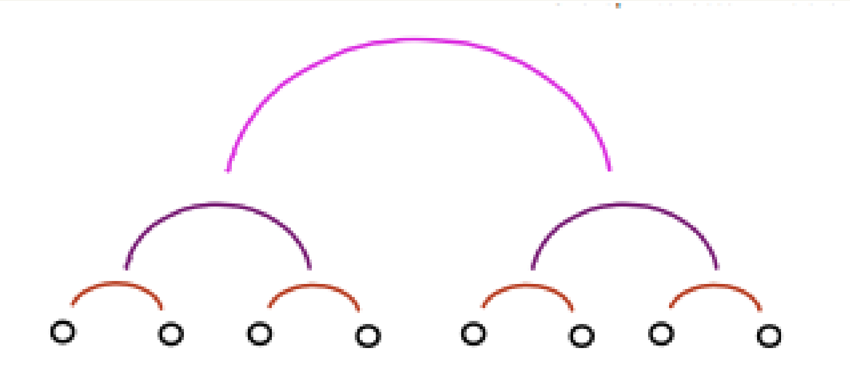
Non trivial exponents for $3/2 < \sigma < 2$

Simplified models of power law interactions systems

Spin models on hierarchical lattices

Dyson Lattices

Power law interactions
hierarchical distance
among spins



Same phenomenology as systems with Euclidean distance

Much simpler to study

$$H_{k+1}[S_1, ..., S_{2^{k+1}}] = H_k[S_1, ..., S_{2^k}] + H_k[S_{2^k+1}, ..., S_{2^{k+1}}] - 2^{-\sigma(k+1)} \sum_{i<j}^{1,2^{k+1}} S_i S_j$$

Ferromagnetic Dyson Hierarchical models

Important in comprehension of the RG Transformation

$$H_{k+1}[S_1, ..., S_{2^{k+1}}] = H_k[S_1, ..., S_{2^k}] + H_k[S_{2^k+1}, ..., S_{2^{k+1}}] - 2^{-\sigma(k+1)} \sum_{i<j}^{1,2^{k+1}} S_i S_j$$

Exact recursion eq. for Z[m]

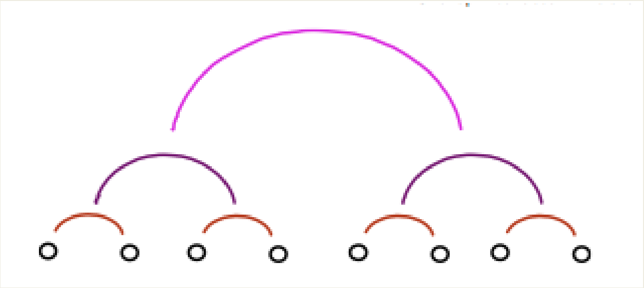
$$Z_{k+1}[m] = e^{\beta 2^{k(2-\sigma)} m^2} \int d\mu Z_k[m+\mu] Z_k[m-\mu]$$

Critical phenomena : non MF for $3/2 < \sigma < 2$

$F(m)$ convex function of m
No infinite-time metastability

Rigorous epsilon expansion $\epsilon = \sigma - 3/2$

Hierarchical Spin-Glasses



$$H_{k+1}^{(J)}[S_1, ..., S_{2^{k+1}}] = \quad H_k^{(J_1)}[S_1, ..., S_{2^k}] + H_k^{(J_2)}[S_{2^k+1}, ..., S_{2^{k+1}}]$$
$$+1/2^{\sigma(k+1)} \sum_{1<j}^{1,2^{k+1}} J_{ij} S_i S_j$$

Similar to a 1D model with power law interactions

Spin glass transition expected for $\sigma > 1/2$

P-spin model version for Glass phenomenology

Possibility of Kauzmann transition

Hierarchical REM

2^{2^k} States at level k (spin configurations)

$$E_{k+1}[\mathbf{S_1}, \mathbf{S_2}] = E_k^{(1)}[\mathbf{S_1}] + E_k^{(2)}[\mathbf{S_2}] + \epsilon_k(\mathbf{S_1}, \mathbf{S_2})$$

$\epsilon_k(\mathbf{S_1}, \mathbf{S_2})$ i.i.d.r.v. e.g. $N(0, 2^{k(1-\sigma)})$

Or Binomial in $[0, 1, ..., 2^{k(1-\sigma)}]$ (discrete levels)

Notation $\mu_k(\epsilon)$ prob.

Question: How much of the REM physics survives in the Hierarchical case ?

Is there a Rem like Condensation Transition for finite Sigma ?

MF behavior expected for $\sigma = 0$

$$\overline{\epsilon_k^2} = 2^{k(1-\sigma)}$$

No interaction for $\sigma = 1$

Interesting Behavior $0 < \sigma < 1$

Are the of correlations of energy levels important ?

What is their role ?

- Numerical (exact) solution of the model
- Small Coupling Constant Expansion

Solution of the Model (1)

Exploit the tree-structure to write a recursive relation
Thermodynamic limit

A Microcanonical approach $$Z_k(\beta) = \sum_E \mathcal{N}_k^{(dis)}(E)\, e^{-\beta E}$$

$\mathcal{N}_k^{(dis)}(E)$ number of states at level k with energy E (disorder dependent)

"Population Dynamic" approach for computing D.O.S.

Statistical Errors

Algorithm to

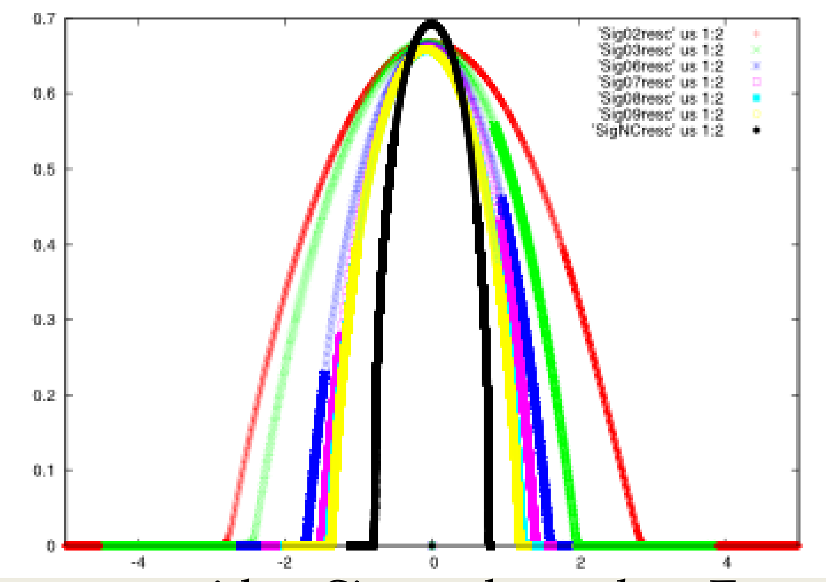
Generate $\mathcal{N}_{k+1}^{(dis)}(E)$ knowing all the $\mathcal{N}_{k}^{(dis)}(E)$

$$\mathcal{N}_k(E) \approx e^{2^k S(E)}$$

(Discrete distribution of interaction energies)

Results

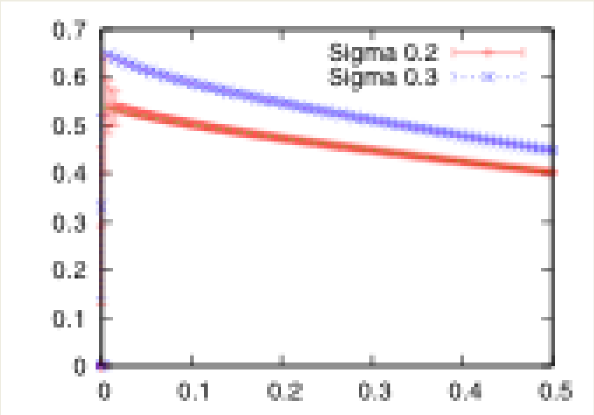
Average Entropy for various values of sigma= 0.2, 0.3, 0.6, 0.7, 0.8, 0.9, 1

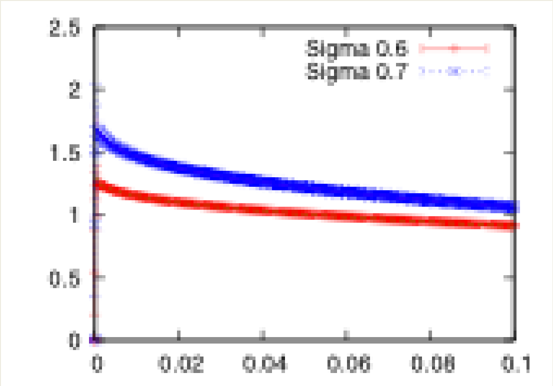


Entropy becomes zero outside a Sigma dependent Energy range.

Condensation (entropy crisis) Transition

$$\beta(u) = \frac{dS(u)}{du} \qquad u = 2^{-k}(E - E_o)$$

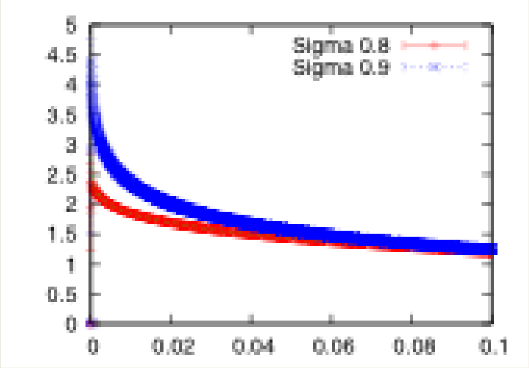


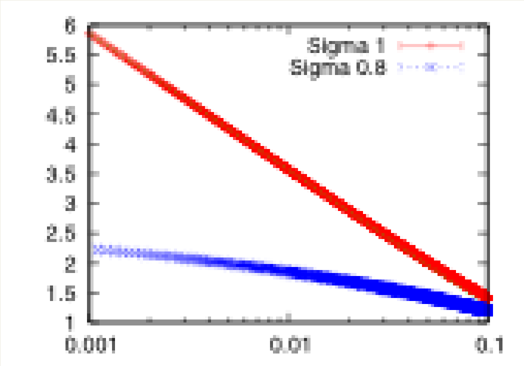


Fitting function

$$\beta(u) = \beta_c + a \times u^{\alpha}$$

$$\alpha \approx 1 - \sigma$$





$$\sigma = 1 \quad : \quad \beta(u) = -\log(u)$$

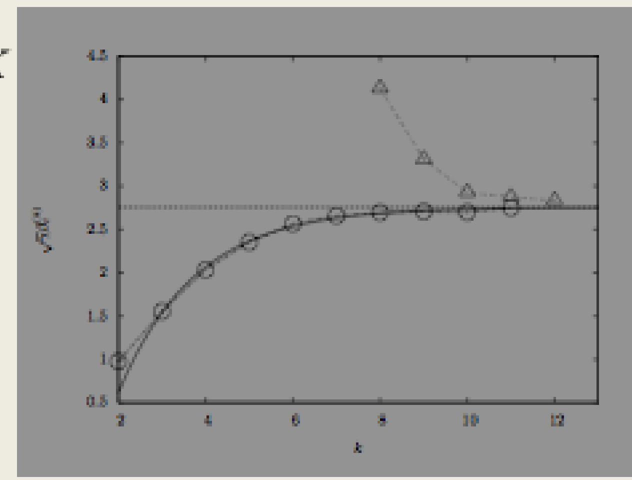
Distribution of Low lying states:

Random System : The occupation number of Ground State depends on the sample.

Assuming REM statistics:

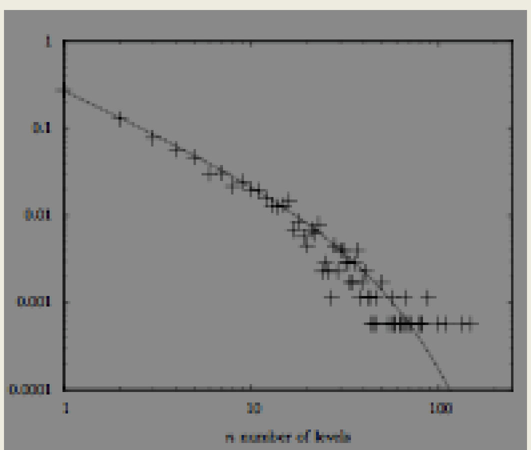
$$\nu(n) = \frac{(1 - e^{\beta_K})^n}{n\beta_c}$$

β_K



Log(Vol)

$\nu(n)$

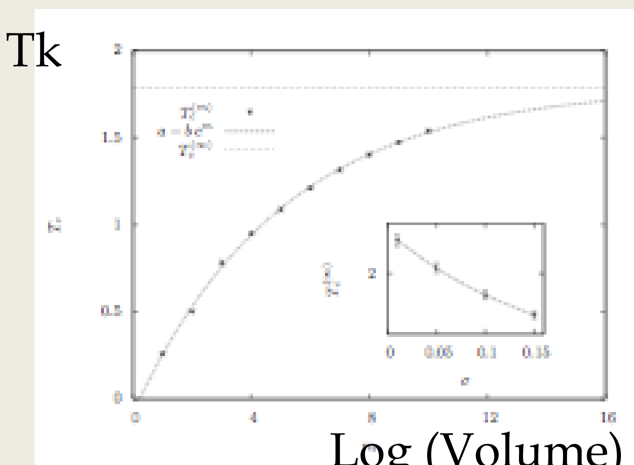


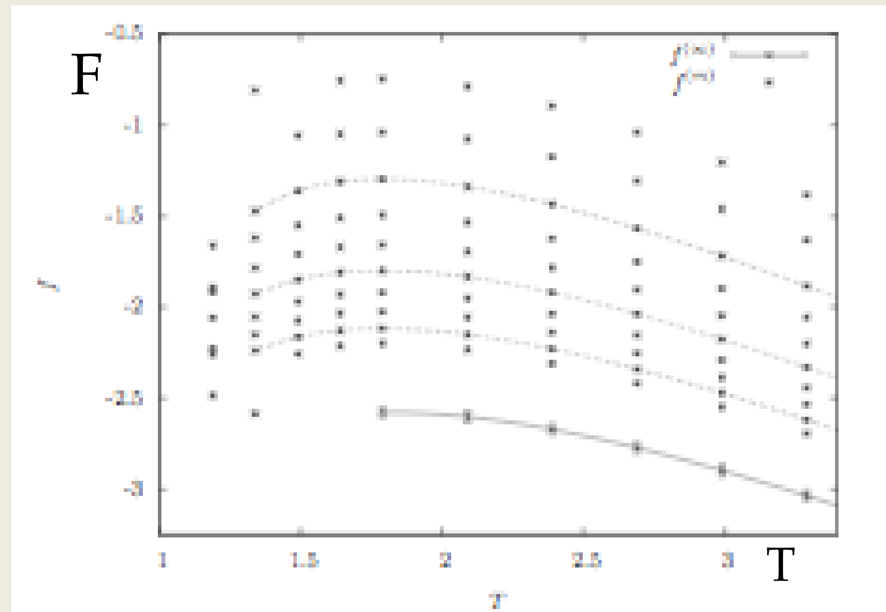
Solution of the model (2)

Series Expansion of Free-Energy in the Coupling Constant

$$g = 2^{1-\sigma}$$

Coherent Results





Confirm the REM phase transition

Is the system identical to a (Mean Field) REM ?

$$S(u) = \beta_K u + C u^a \qquad\qquad a \approx 2 - \sigma$$

Free-Energy non analytic at the transition

$$F(T) = E_{GS} + A \times (T - T_k)^{\frac{2-\sigma}{1-\sigma}}$$

$$\alpha = -\frac{\sigma}{1-\sigma}$$ Specific heat exponent

Summary

Hierarchical REM :

- Interaction among parts scales sub-extensively
  Non Mean-Field

- "Exact" Solution via Population Dynamics

- Low Coupling Expansion

Message :

Existence of an Entropy crisis transition
in a system where interaction between parts
is sub-extensive

Generalization to p-spin ??

References:

S. Franz, T. Jorg and G. Parisi
. stacks.iop.org/JSTAT/2009/P02002

M. Castellana, A. Decelle, S. Franz, M. Mezard and G. Parisi
ArXiv:0912.3634v2