From Frustrated Heisenberg Quantum Antiferromagnets to Quantitative Generalized Quantum Dimer Models

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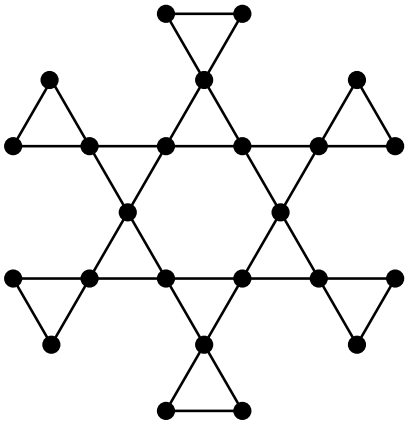
ICMP conference
Dec. 2009 - Mahabaleshwar

A. Läuchli, F. Mila, D. Poilblanc, A. Ralko, D. Schwandt

Phys. Rev. B **80**, 184427 (2009) & ArXiv 0912.0724 + ArXiv (early 2010) + D. Poilblanc's talk (next week)

$$\mathcal{H} = \sum_{(i,j)} J_{ij} \vec{S}_i . \vec{S}_j$$



2d frustrated antiferromagnets

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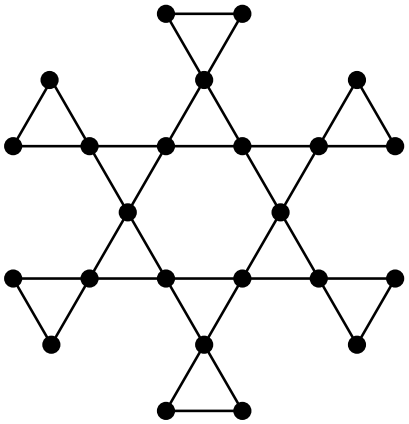

Low energy $\longrightarrow$?

2d frustrated antiferromagnets

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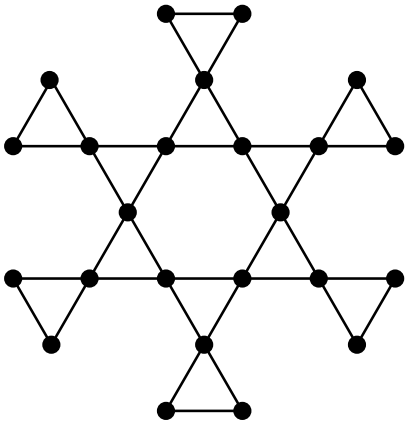
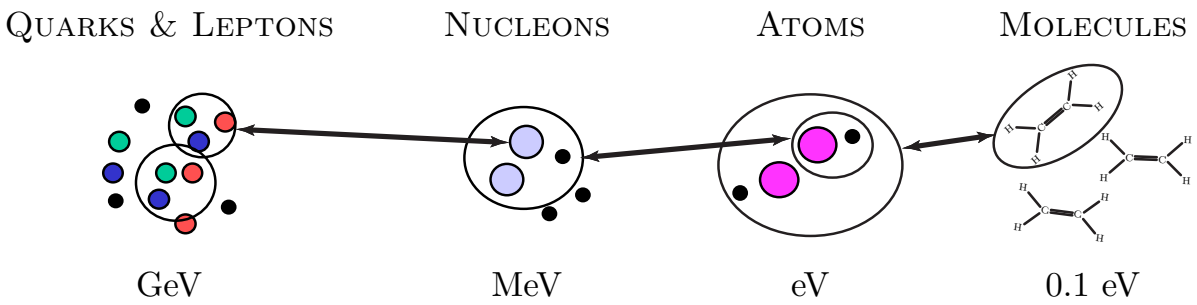


Low energy →

?

2d frustrated antiferromagnets



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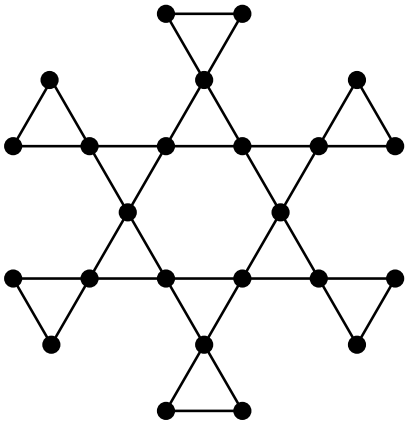


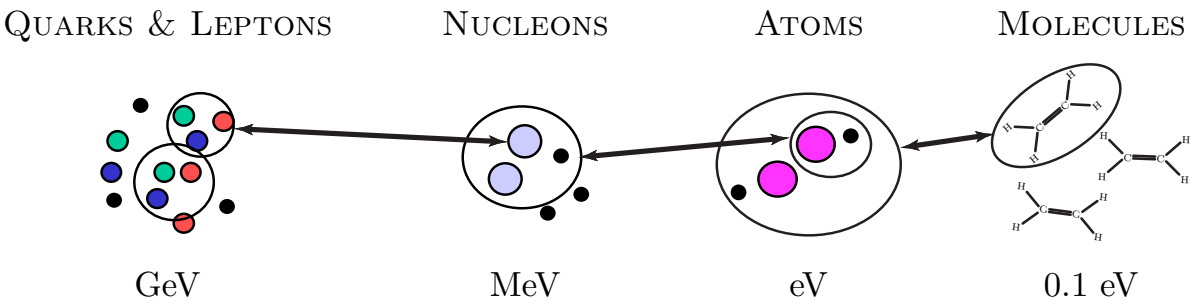
Low energy →

Which objects ?

Which interaction ?

Effective Theory

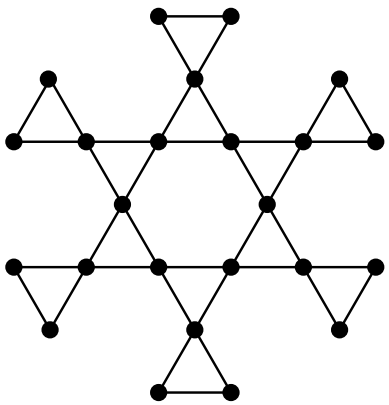
2d frustrated antiferromagnets

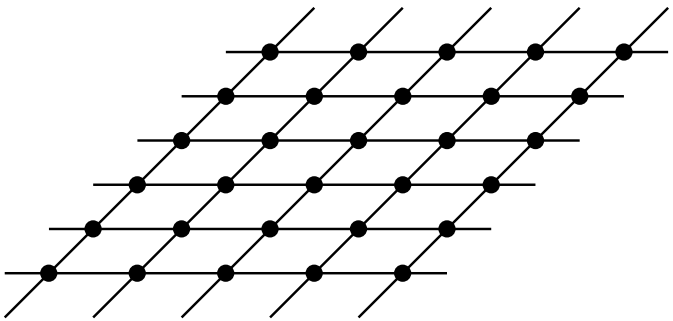


1. Phenomenology of 2d frustrated antiferromagnets
2. Effective hamiltonian : the $J_1 - J_2 - (J_3)$ AF case
3. Linked cluster theorem
4. Kagome antiferromagnet
5. Conclusion

Mott insulator $\rightsquigarrow$ Localized spins Heisenberg model

$$\mathcal{H} = \sum_{(i,j)} J_{ij} \vec{S}_i . \vec{S}_j$$





Mott insulator $\rightsquigarrow$ Localized spins Heisenberg model

$$\mathcal{H} = \sum_{(i,j)} J_{ij} \vec{S}_i . \vec{S}_j$$

Classical case

$\vec{S}$: vector

Néel state

$\ldots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \ldots$

Quantum case

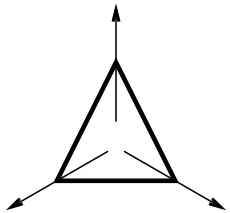
$\vec{S}$: Spin operator

Fluctuations:

$$\vec{S}_i . \vec{S}_j = S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)$$

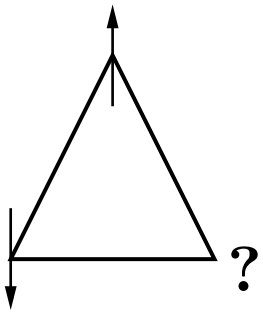
Relevance of Quantum fluctuations ?

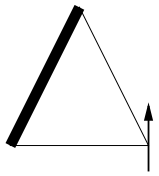
FRUSTRATION



Global compromise &

Classical degeneracy





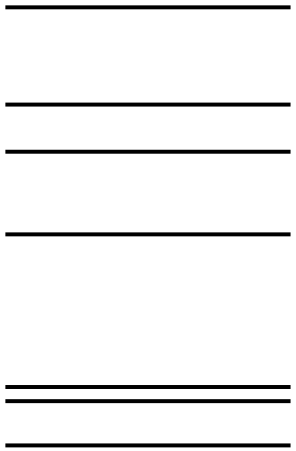
Ground state ?

Degeneracy ?

Density of states ?

Spectrum

$$\mathcal{H} = J \sum_{(i,j)} \vec{S}_i . \vec{S}_j$$



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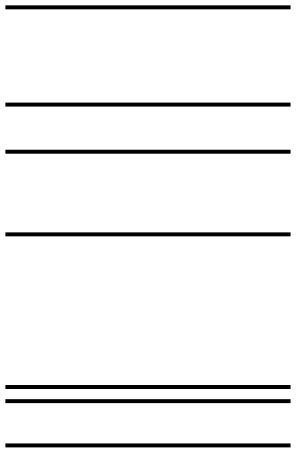

$$\left[\vec{S}^2_{\text{tot}} . \mathcal{H}\right] = 0 \quad \vec{S}^2_{\text{tot}} = \left(\sum_i \vec{S}_i\right)^2$$

SU(2) symmetry

$$\vec{S}^2_{\text{tot}} |\psi\rangle = S(S+1)|\psi\rangle \quad \text{with} \quad S = 0, 1, \ldots, NS.$$

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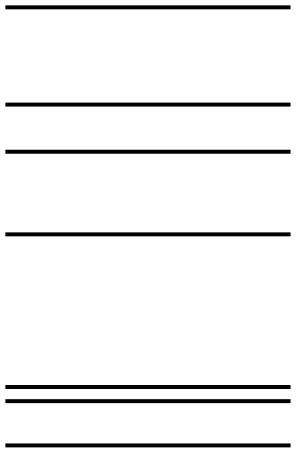

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broken SU(2)

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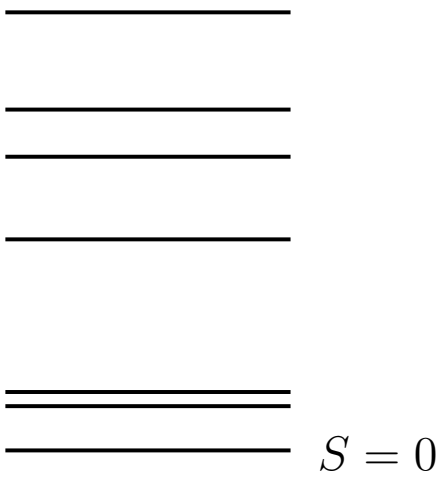

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broken SU(2)

Magnetically disordered

$$\langle \vec{S}_i . \vec{S}_j \rangle \sim e^{-|i-j|/\xi}$$

restored SU(2)

Groundstate $S = 0$

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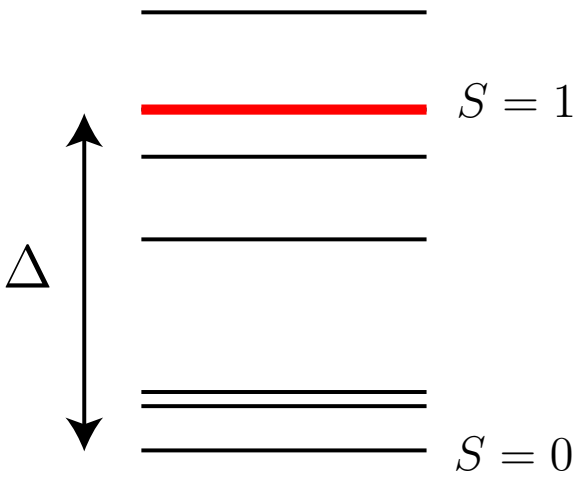

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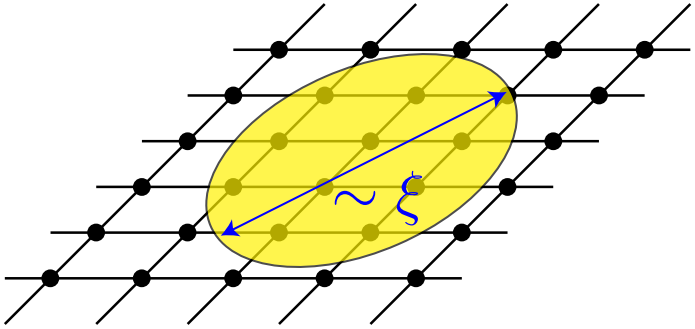

Spin gap

$\Delta \sim J$

Spectrum

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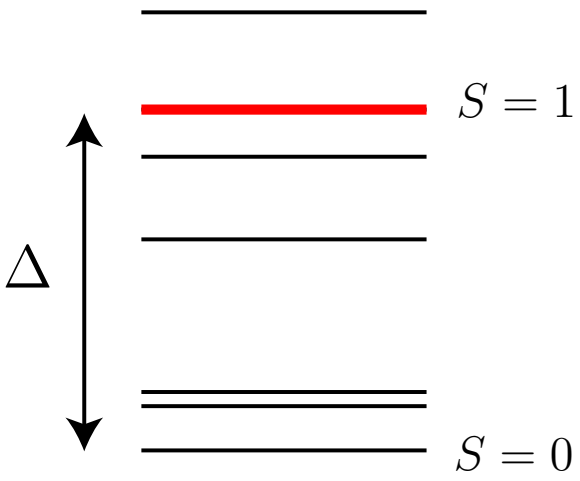

Néel state

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broken SU(2)

Magnetically disordered

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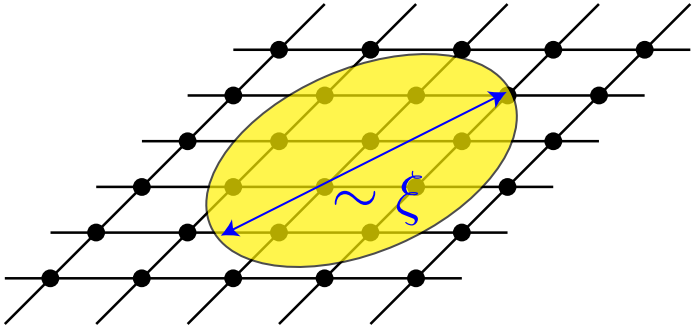
restored SU(2)

Groundstate $S = 0$

Low energy Spin singlets



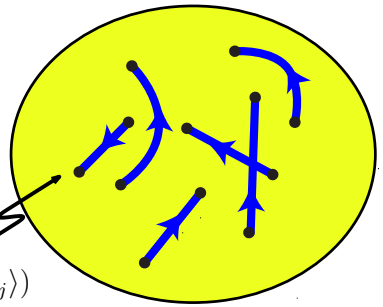
Spin gap

$\Delta \sim J$

Singlet States $S = 0$

Valence Bond State

$$|\varphi_{\mathcal{D}}\rangle = \bigotimes_{(i,j)\in\mathcal{D}} [i,j]$$



$$[i,j] = \frac{1}{\sqrt{2}}\left(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle\right)$$

$$|\psi\rangle = \sum_{\mathcal{D}} \lambda_{\mathcal{D}} |\varphi_{\mathcal{D}}\rangle$$

L. Hulthén, Ark. Mat. Astron. Fys. A **26** (1938)

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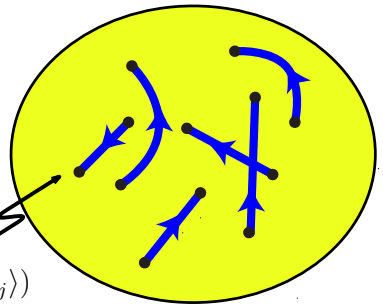

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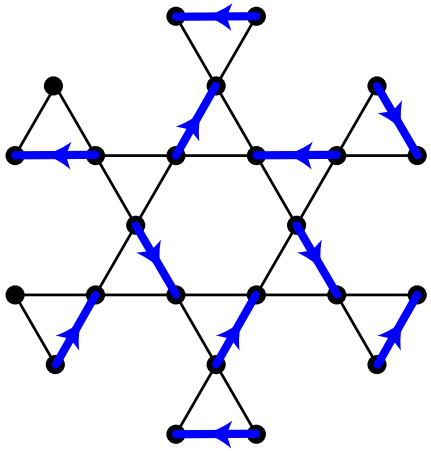
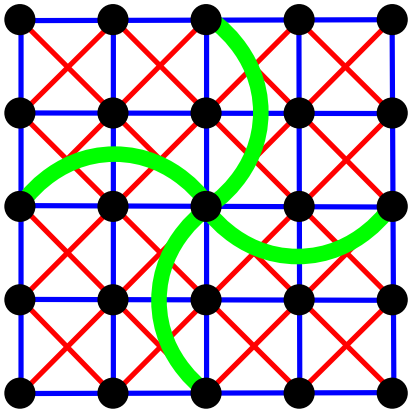
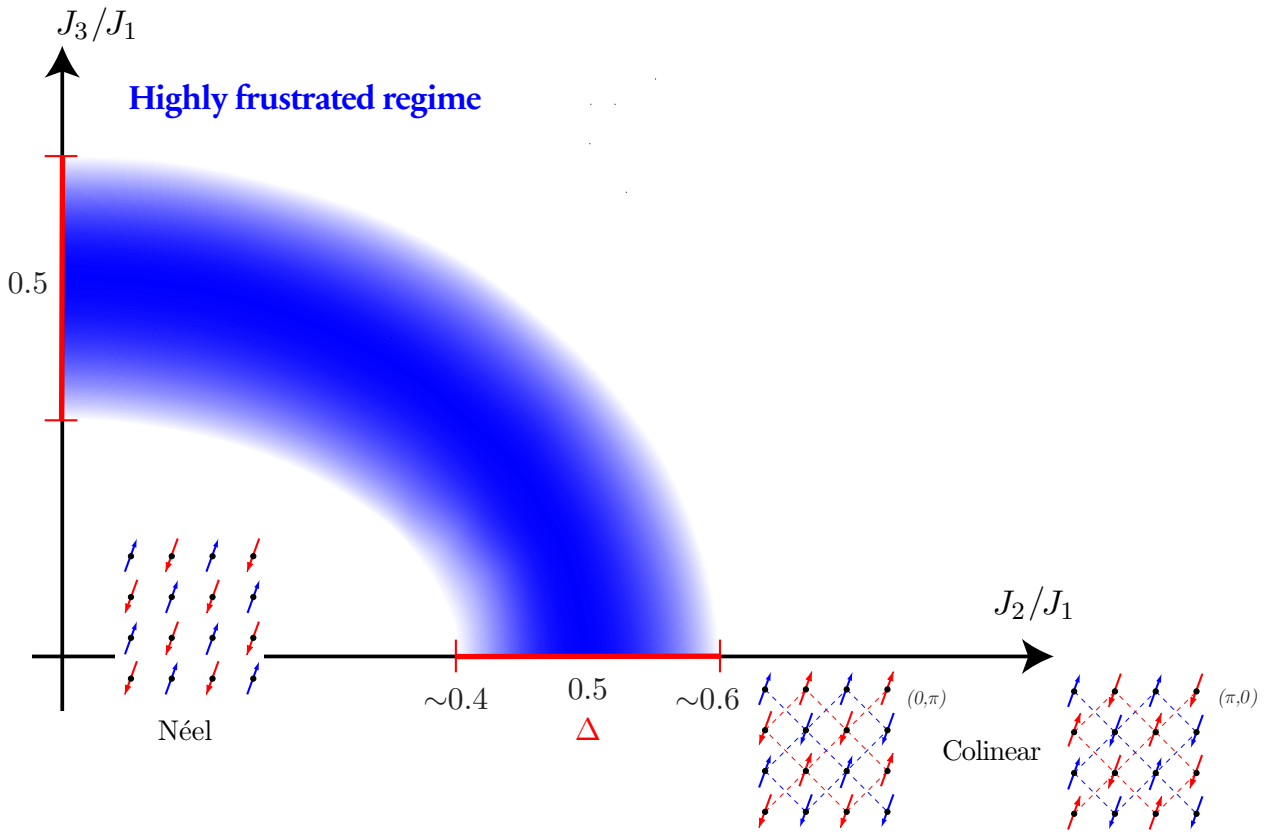


Short Range Valence Bond States

Working with NNRVB states

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i.\vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i.\vec{S}_j + J_3 \sum_{\langle\langle\langle i,j \rangle\rangle\rangle} \vec{S}_i.\vec{S}_j$$





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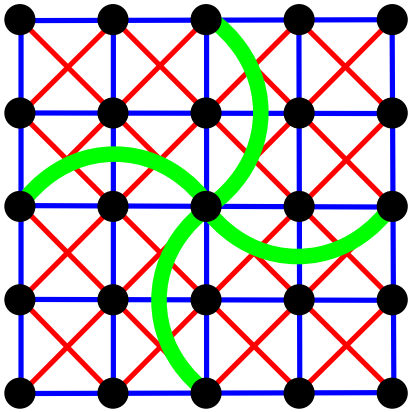
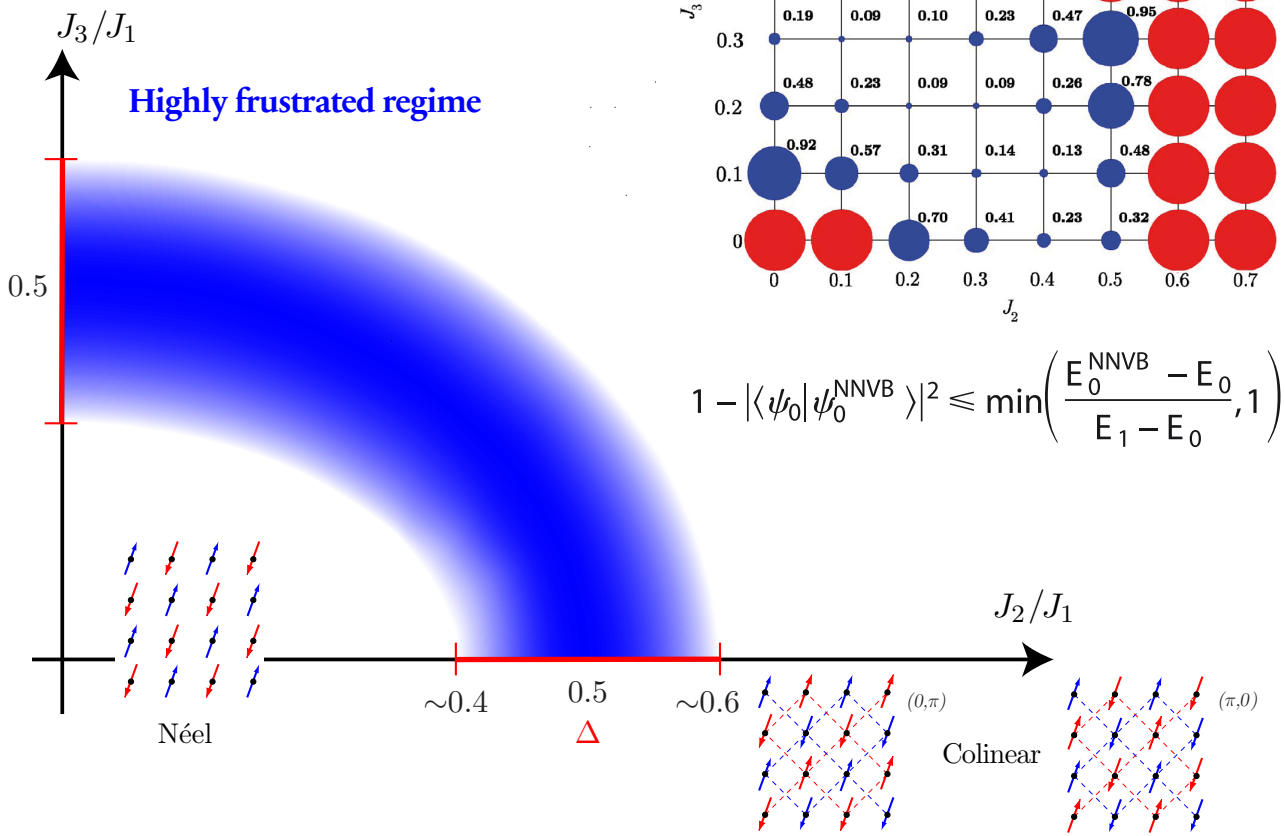
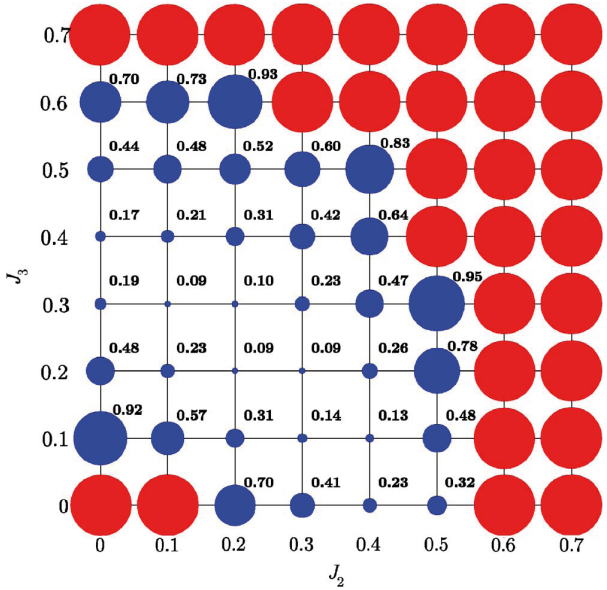




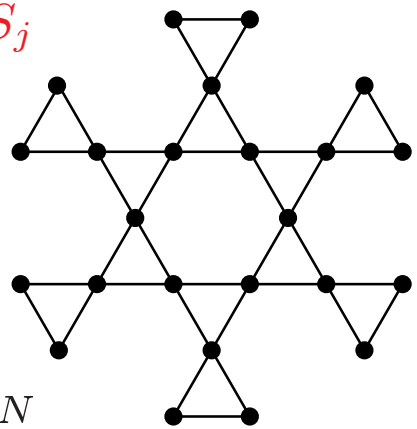
M. Mambrini, A. Laüchli, D. Poilblanc & F. Mila, PRB **74** (2006)



$$1 - |\langle \psi_0 | \psi_0^{\mathrm{NNVB}} \rangle|^2 \leqslant \min\left(\frac{E_0^{\mathrm{NNVB}} - E_0}{E_1 - E_0}, 1 \right)$$

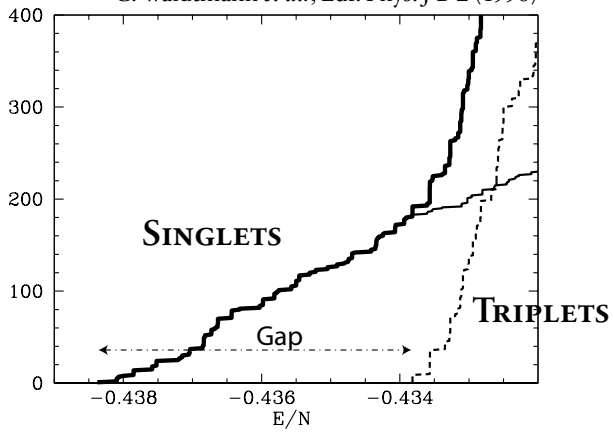
Working with NNRVB states

$$\mathcal{H} = \sum_{(i,j)} J_{ij} \vec{S}_i . \vec{S}_j$$



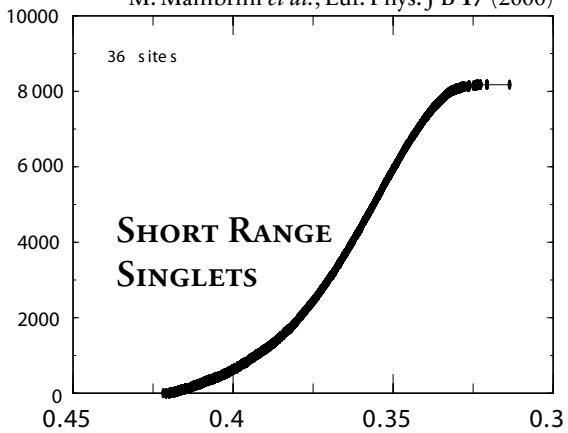
Exact diagonalizations

C. Waldtmann *et al.*, Eur. Phys. J B **2** (1998)

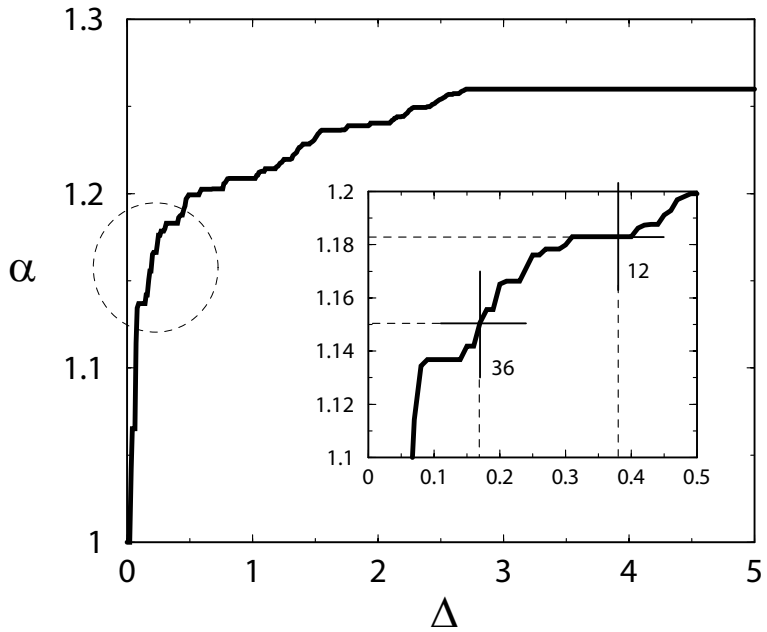


Short Range RVB

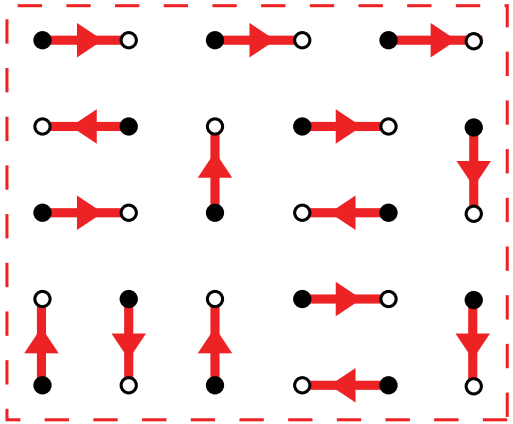
M. Mambrini *et al.*, Eur. Phys. J B **17** (2000)



$$\mathcal{N}_N(\Delta) = A(\Delta)\alpha(\Delta)^N$$



$$\frac{1}{\sqrt{2}}\left(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle\right)$$



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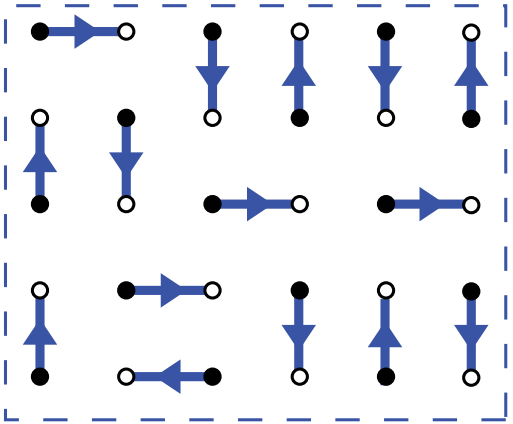

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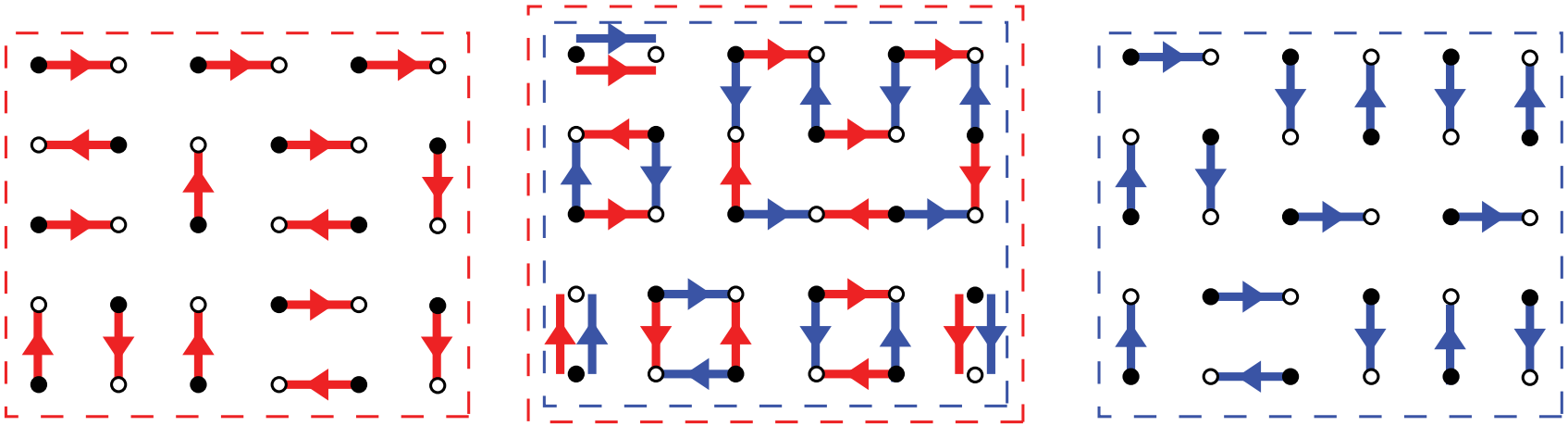

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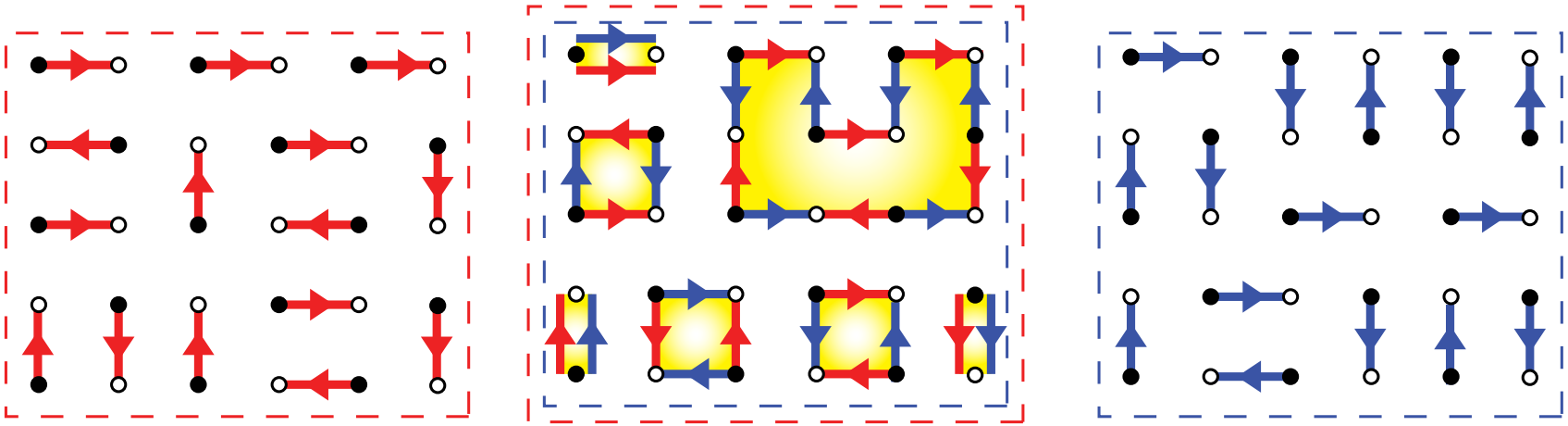

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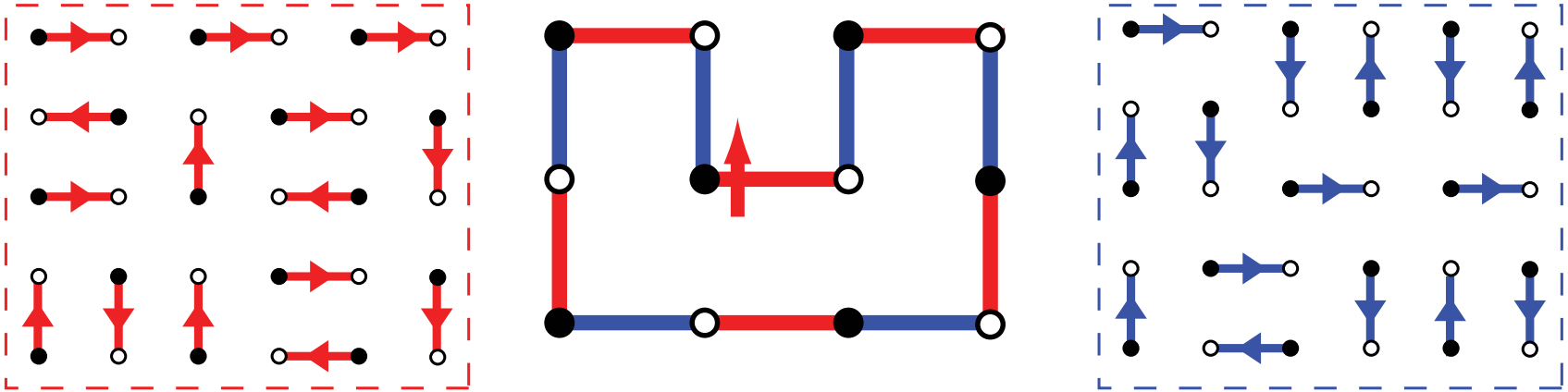

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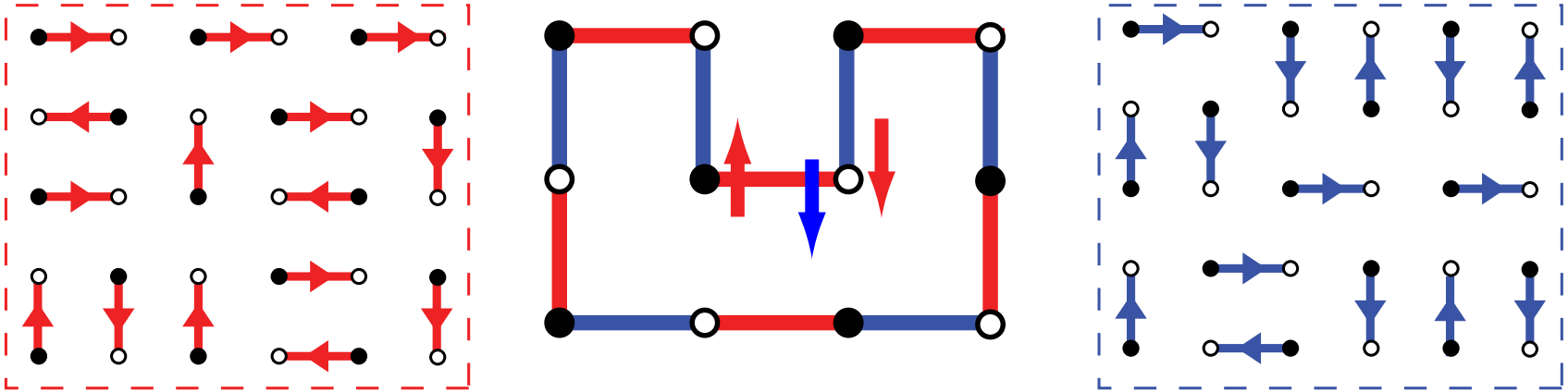

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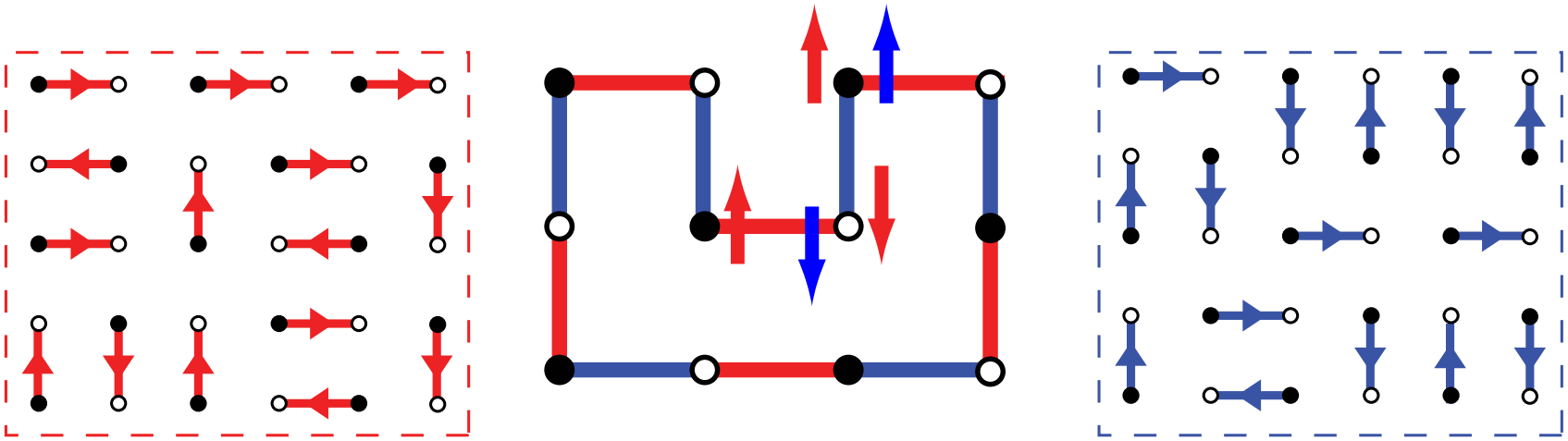
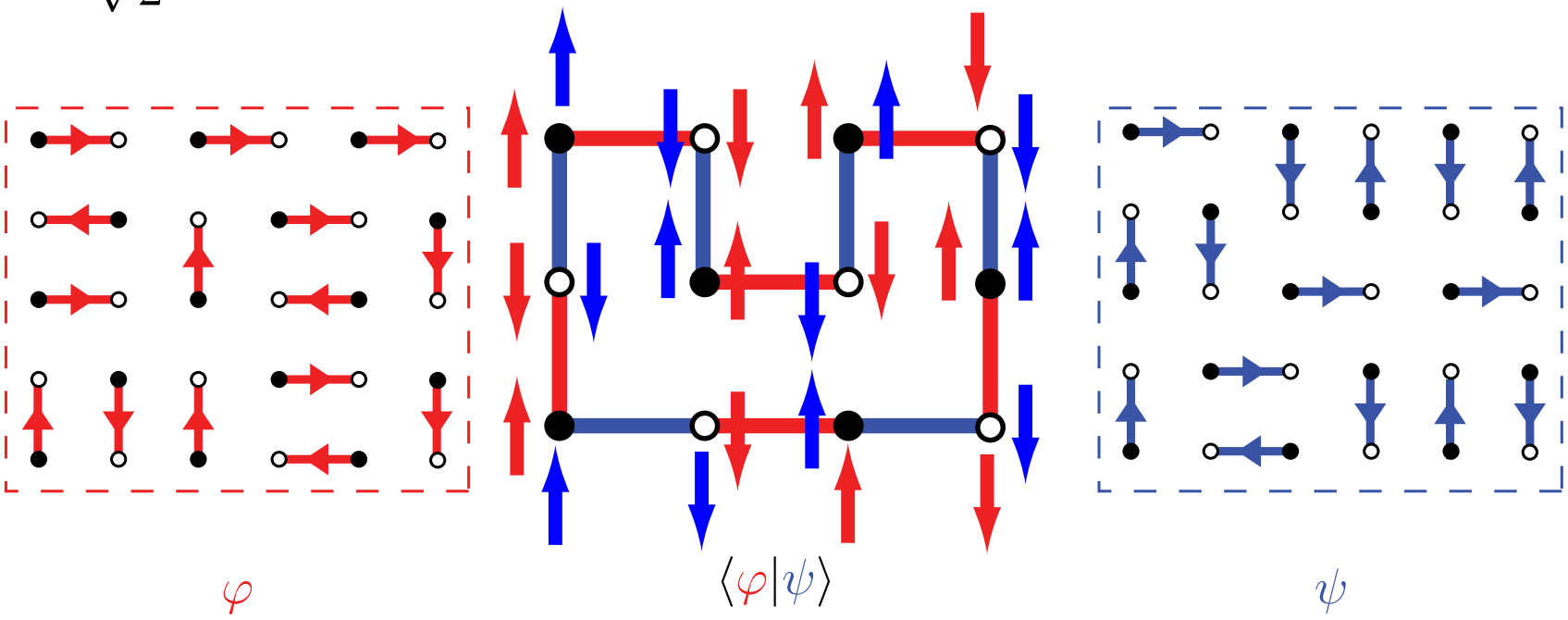
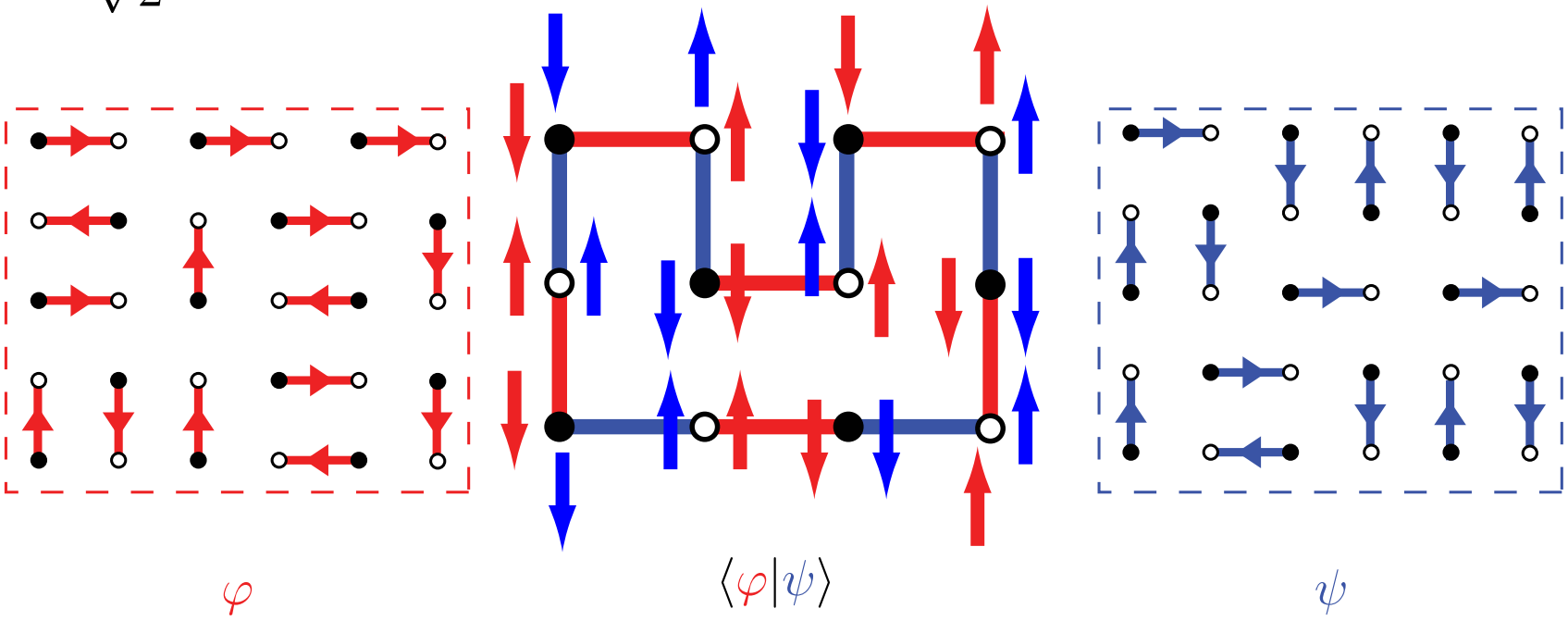


φ $\langle\varphi|\psi\rangle$ ψ

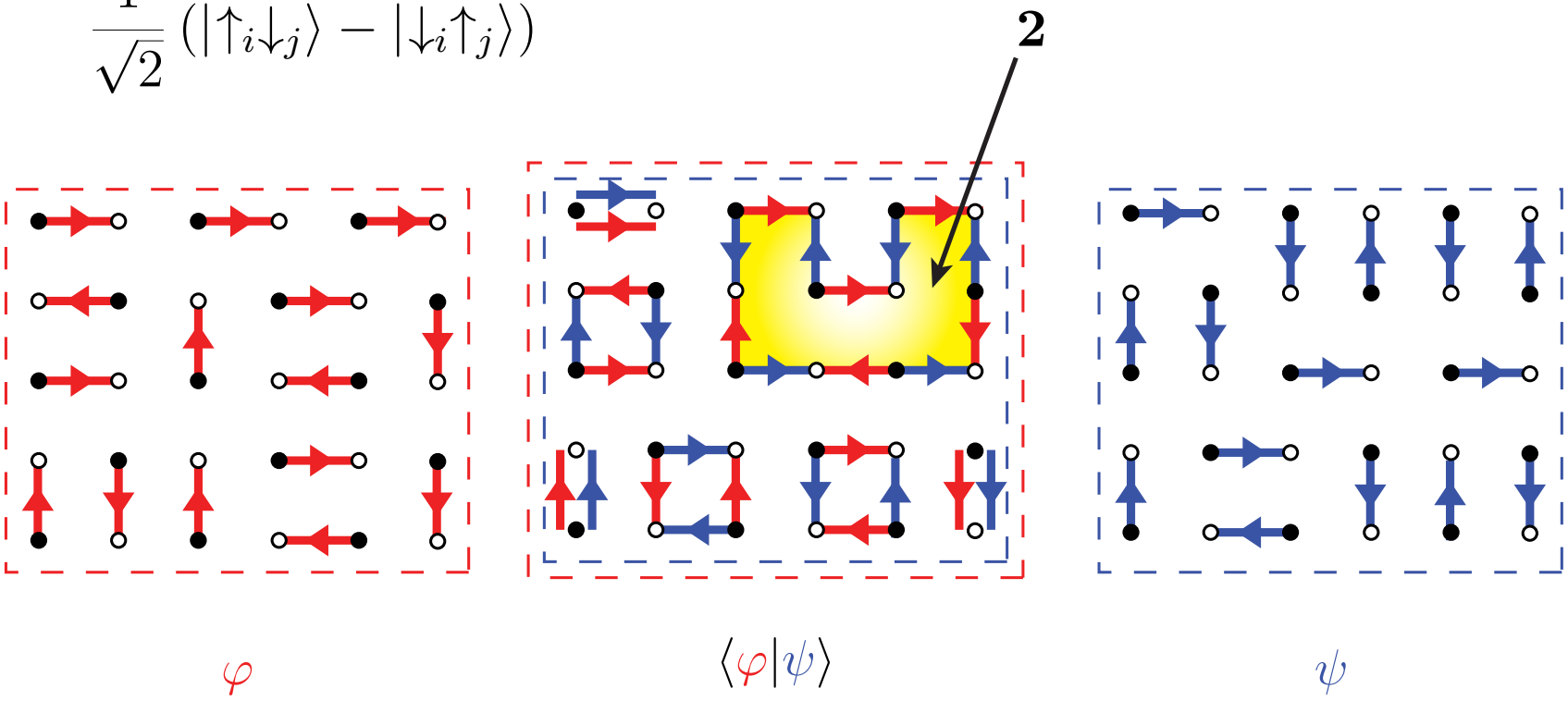
$$\frac{1}{\sqrt{2}}\left(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle\right)$$



$$\frac{1}{\sqrt{2}}\left(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle\right)$$



$$\frac{1}{\sqrt{2}}\left(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle\right)$$



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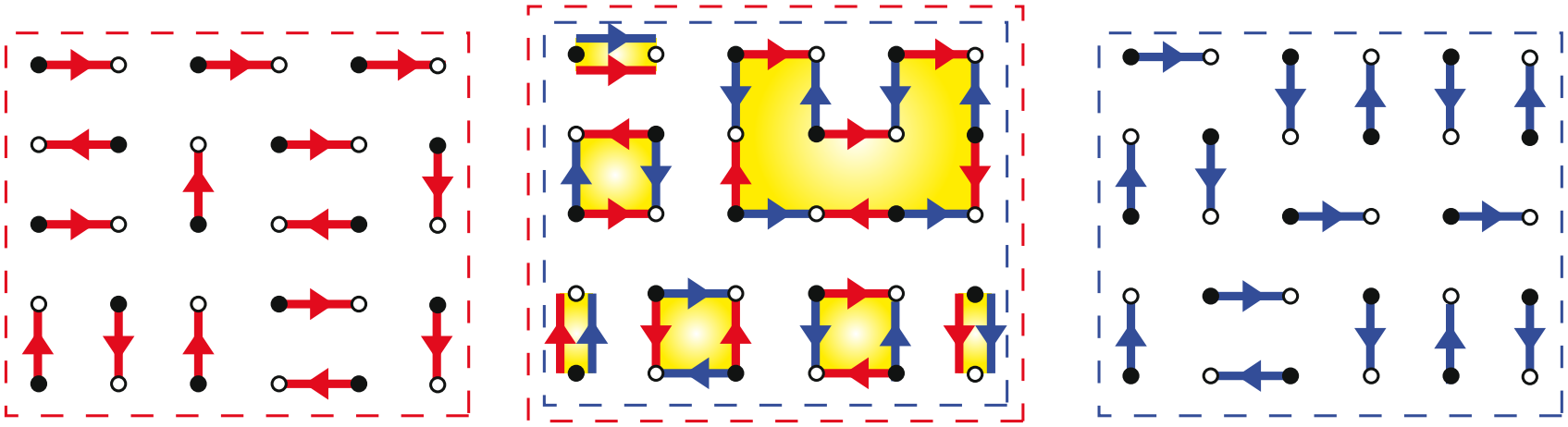
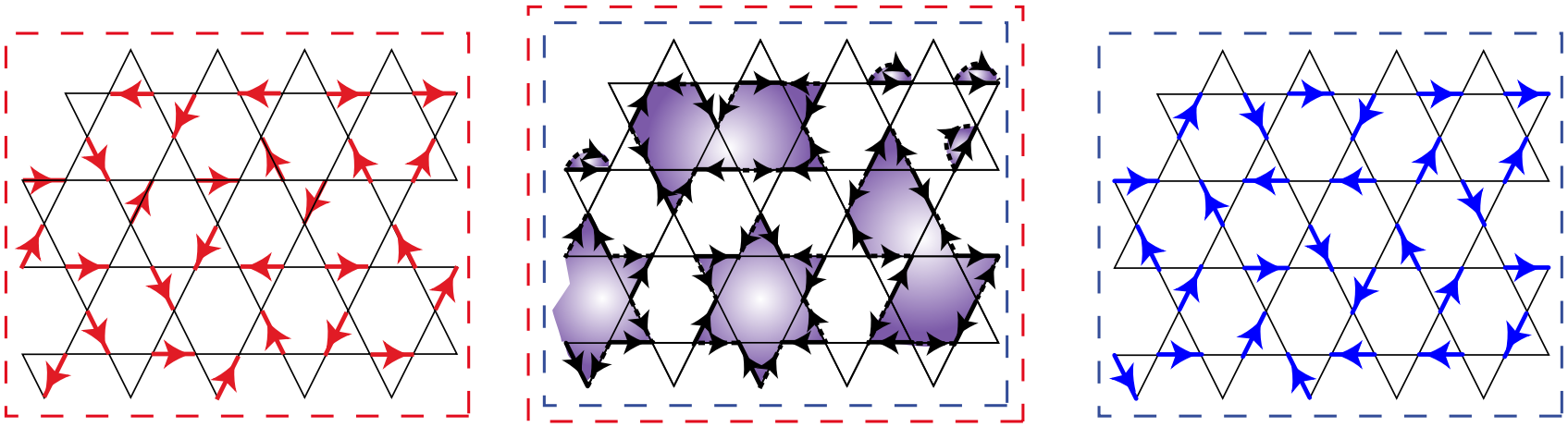


φ $\langle\varphi|\psi\rangle$ ψ

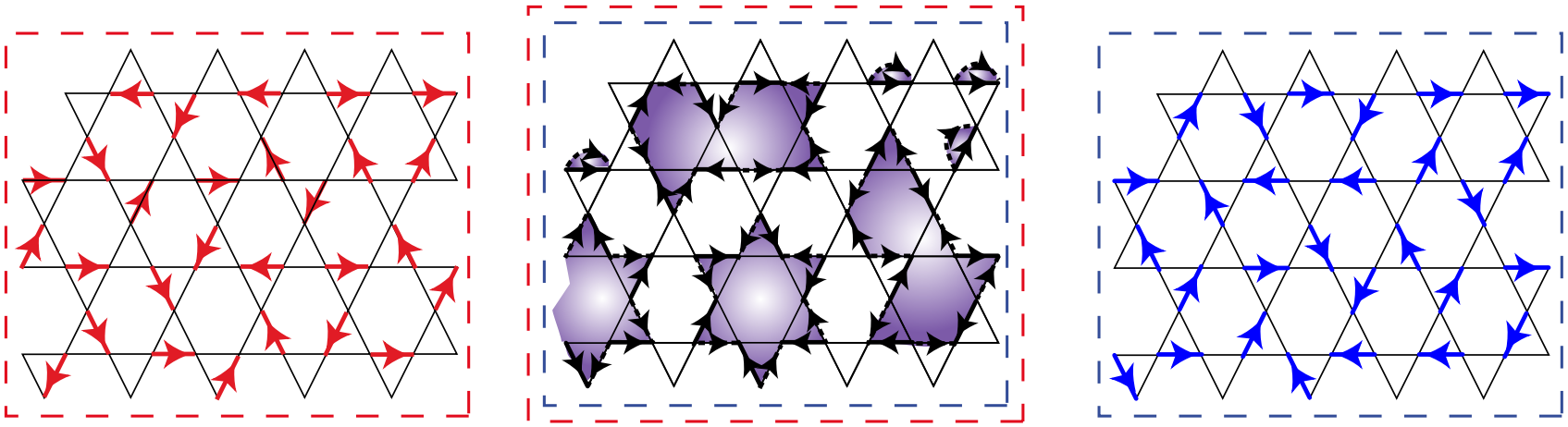
$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)} \quad \text{with} \quad \alpha = \frac{1}{\sqrt{2}}$$

Non bipartite lattices



φ $\langle\varphi|\psi\rangle$ ψ

Non bipartite lattices



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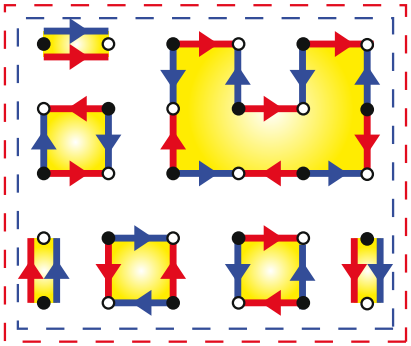
$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)} \quad \text{with} \quad \alpha = \frac{i}{\sqrt{2}}$$

fermionic representation

N. Read & B. Chakraborty, Phys. Rev. B **40** (1989)

Bipartite lattices



$\langle\varphi|\psi\rangle$

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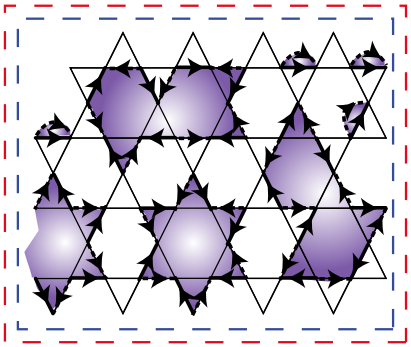
bosonic representation

Non bipartite lattices



$\langle\varphi|\psi\rangle$

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fermionic representation

Low energy singlets

$$\mathcal{H}|\psi\rangle = E\ \mathcal{O}|\psi\rangle$$

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Hermitic Hamiltonien
Orthogonal basis

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

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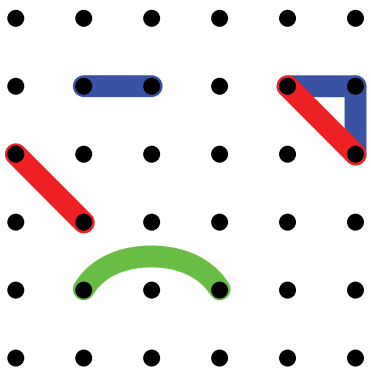
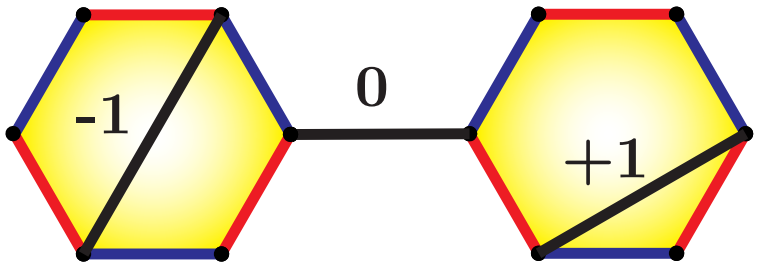
$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\mathcal{H} = J_1 \sum_{\langle i,j\rangle} \vec{S}_i.\vec{S}_j + J_2 \sum_{\langle\langle i,j\rangle\rangle} \vec{S}_i.\vec{S}_j + J_3 \sum_{\langle\langle\langle i,j\rangle\rangle\rangle} \vec{S}_i.\vec{S}_j$$



Matrix elements



$$\langle\varphi|\frac{4}{3}\vec{S}_i.\vec{S}_j|\psi\rangle = f_{ij}\langle\varphi|\psi\rangle$$

Low energy singlets

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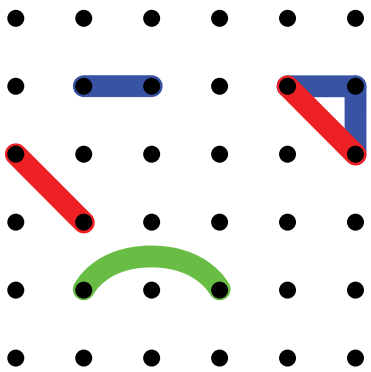
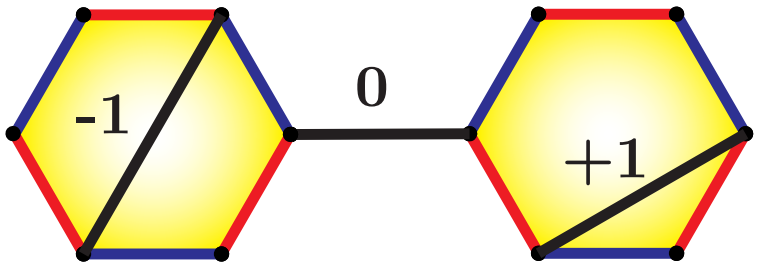


Matrix elements



$$\langle\varphi|\frac{4}{3}\vec{S}_i.\vec{S}_j|\psi\rangle = f_{ij}\langle\varphi|\psi\rangle$$

$$\mathcal{H} \to (4/3)\mathcal{H} + J_1 N/2$$

$$\langle\varphi|\mathcal{H}|\psi\rangle = h_{\varphi,\psi}\langle\varphi|\psi\rangle$$

with $h_{\varphi,\varphi} = 0$

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)}$$

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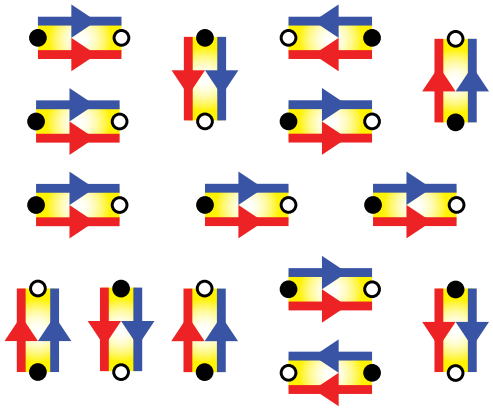


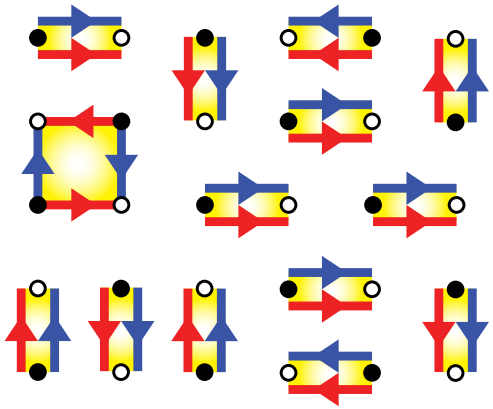
$\langle\varphi|\psi\rangle$

α^0

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)}$$

$$\langle\varphi|\mathcal{H}|\psi\rangle = h_{\varphi,\psi}\langle\varphi|\psi\rangle$$



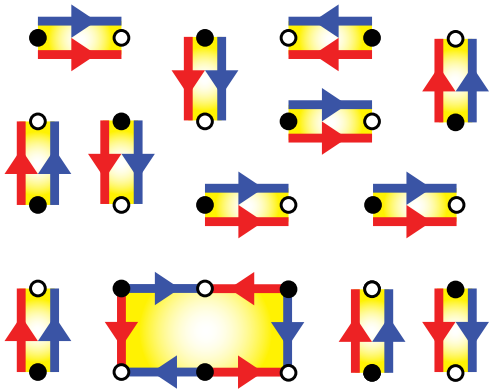
$\langle\varphi|\psi\rangle$

α^2

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

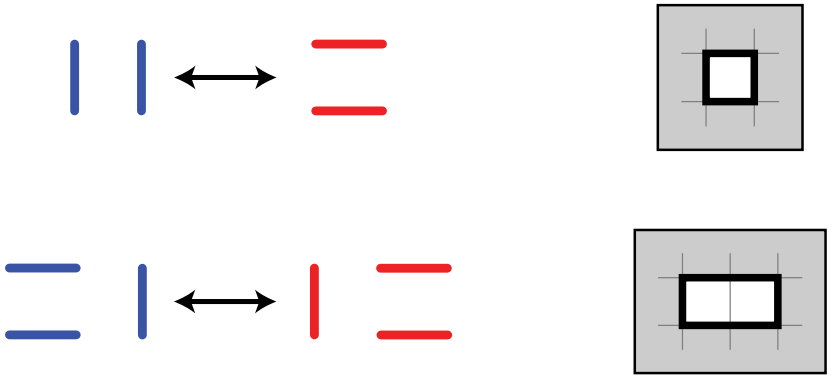
$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)}$$

$$\langle\varphi|\mathcal{H}|\psi\rangle = h_{\varphi,\psi}\langle\varphi|\psi\rangle$$



$\langle\varphi|\psi\rangle$

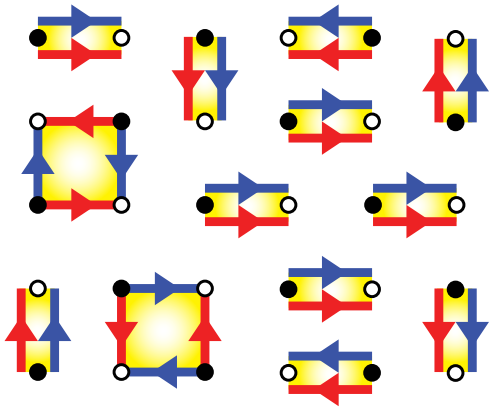
α^4



$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

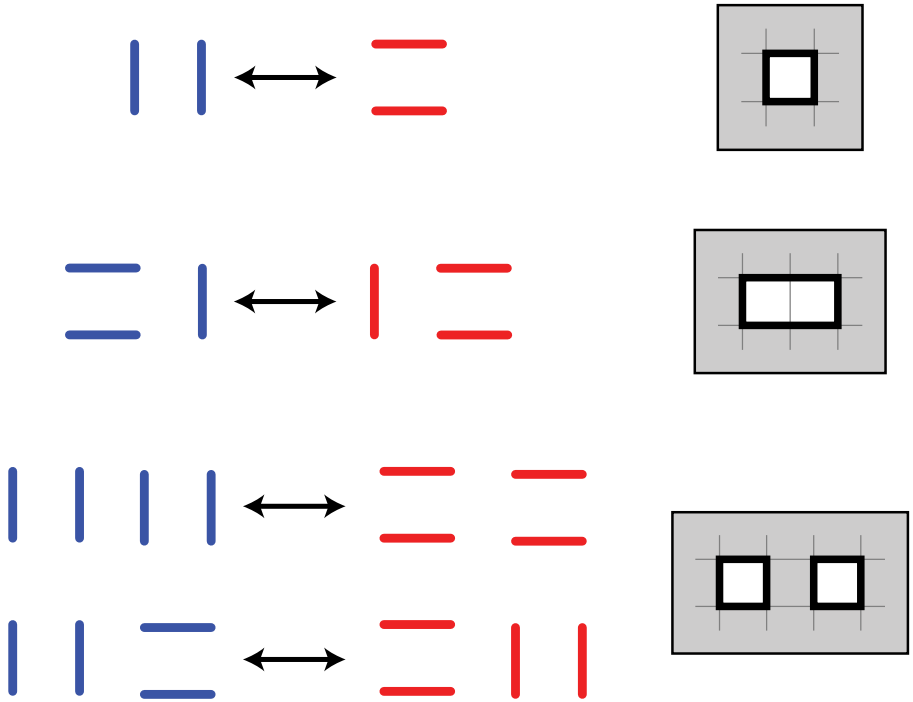
$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)}$$

$$\langle\varphi|\mathcal{H}|\psi\rangle = h_{\varphi,\psi}\langle\varphi|\psi\rangle$$



$\langle\varphi|\psi\rangle$

α^4



$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\langle\varphi|\psi\rangle = \alpha^{N-2n_l(g)}$$

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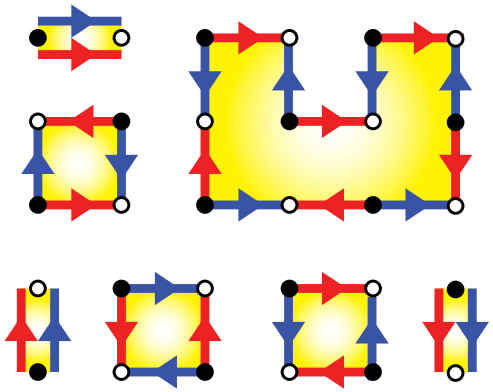
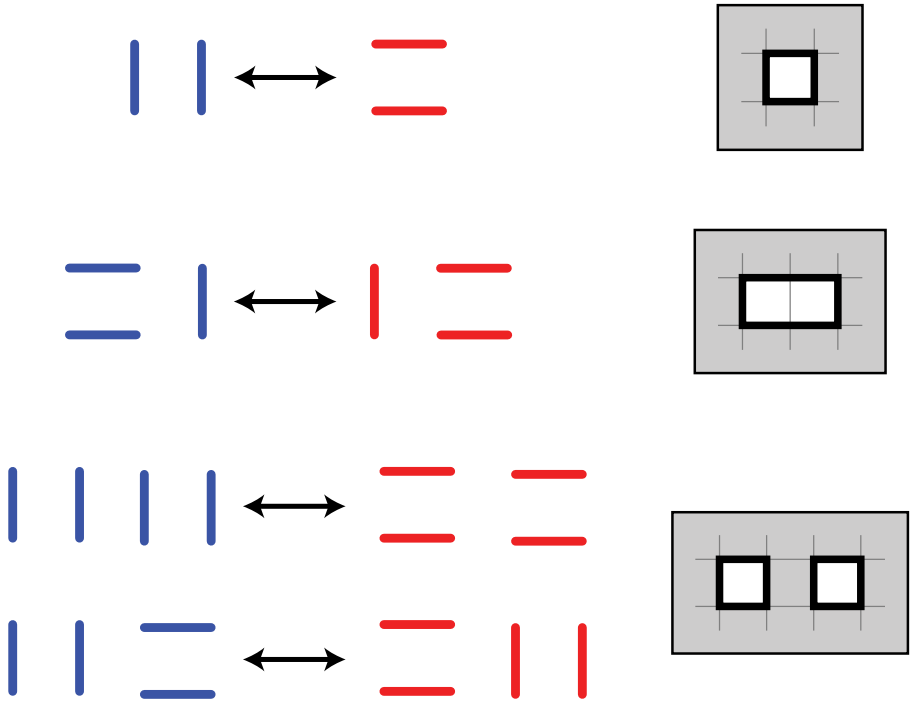


$\langle\varphi|\psi\rangle$

α^{16}

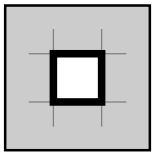


$$\sum_l \mathcal{L}_l = N$$

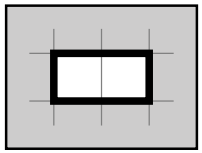
$$\langle \varphi | \psi \rangle = \alpha^{N - 2n_l(g)}$$

$$\langle \varphi | \psi \rangle = \prod_l \alpha^{\mathcal{L}_l - 2} = \prod_l 2\alpha^{\mathcal{L}_l}$$

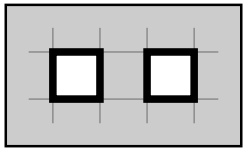
D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61** (1988)
C. Zeng and V. Elser, Phys. Rev. B **51** (1995)



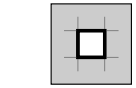
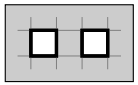
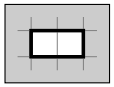
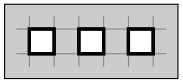
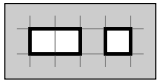
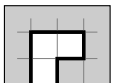
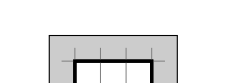
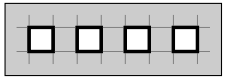
α^2 $\qquad$ $2\alpha^4$

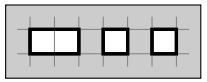
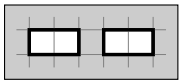
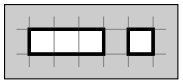
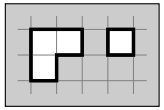
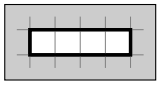
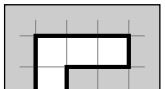
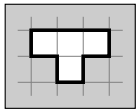
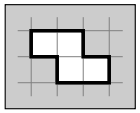


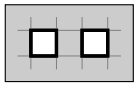
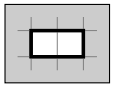
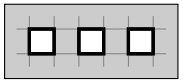
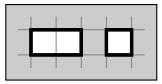
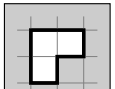
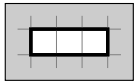
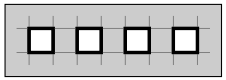
α^4 $\qquad$ $2\alpha^6$

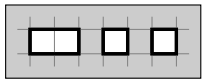
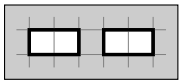
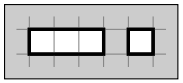
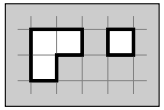
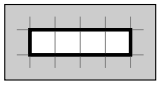
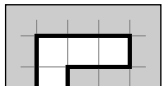
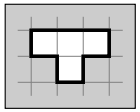
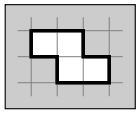


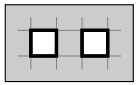
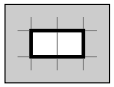
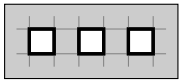
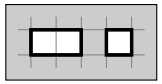
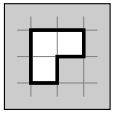
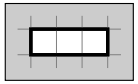
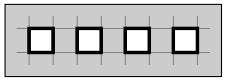
α^4 $\qquad$ $4\alpha^8$

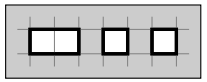
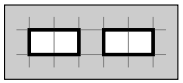
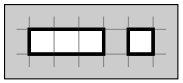
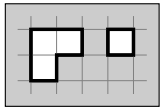
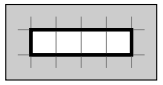
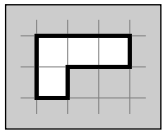
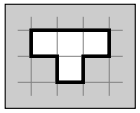
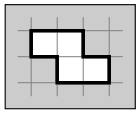
Processes	Overlap	Hamiltonian
		
		
		
		
		
		
		
		

Processes	Overlap	Hamiltonian
		
		
		
		
		
		
		
		

Processes	Overlap	Hamiltonian
	α^2	
	α^4	
	α^4	
	α^6	
	α^6	
	α^6	
	α^6	
	α^8	

Processes	Overlap	Hamiltonian
	α^8	
	α^8	
	α^8	
	α^8	
	α^8	
	α^8	
	α^8	
	α^8	

Processes	Overlap	Hamiltonian
	α^2	$2\,(J_2 - J_1)\,\alpha^2$
	α^4	$4\,(J_2 - J_1)\,\alpha^4$
	α^4	$2\,(-2J_1 + 2J_2 + J_3)\,\alpha^4$
	α^6	$6\,(J_2 - J_1)\,\alpha^6$
	α^6	$2\,(-3J_1 + 3J_2 + J_3)\,\alpha^6$
	α^6	$(-6J_1 + 7J_2 + 4J_3)\,\alpha^6$
	α^6	$2\,(-3J_1 + 3J_2 + 2J_3)\,\alpha^6$
	α^8	$8\,(J_2 - J_1)\,\alpha^8$

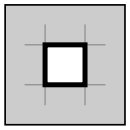
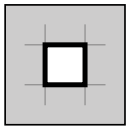
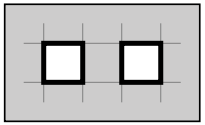
Processes	Overlap	Hamiltonian
	α^8	$2\,(-4J_1 + 4J_2 + J_3)\,\alpha^8$
	α^8	$4\,(-2J_1 + 2J_2 + J_3)\,\alpha^8$
	α^8	$4\,(-2J_1 + 2J_2 + J_3)\,\alpha^8$
	α^8	$(-8J_1 + 9J_2 + 4J_3)\,\alpha^8$
	α^8	$2\,(-4J_1 + 4J_2 + 3J_3)\,\alpha^8$
	α^8	$(-8J_1 + 9J_2 + 6J_3)\,\alpha^8$
	α^8	$2\,(-4J_1 + 5J_2 + 3J_3)\,\alpha^8$
	α^8	$2\,(-4J_1 + 5J_2 + 3J_3)\,\alpha^8$

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\mathcal{O}^{\tau} = \sum_{k} \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^{k}$$

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$

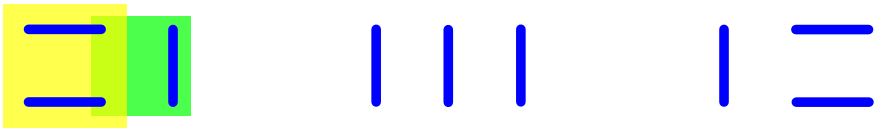
$\frac{1}{2}\Big\{$  ,  $\Big\} =$ $+ 2$ 

$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$

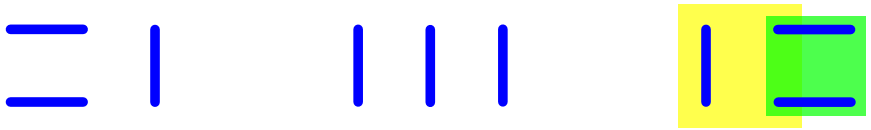
$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$



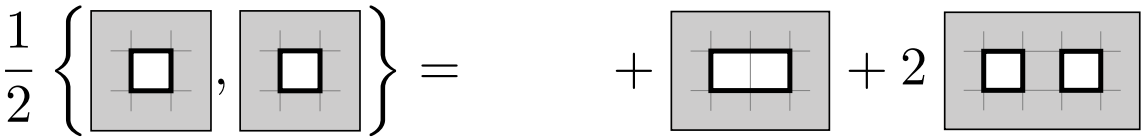
$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$



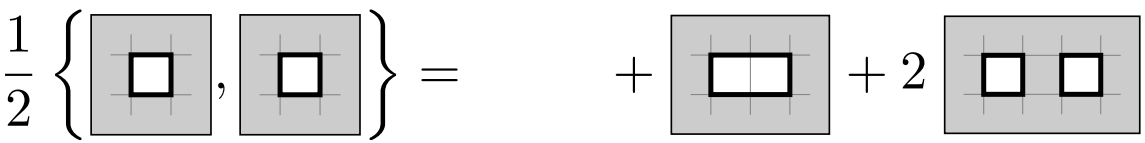
$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

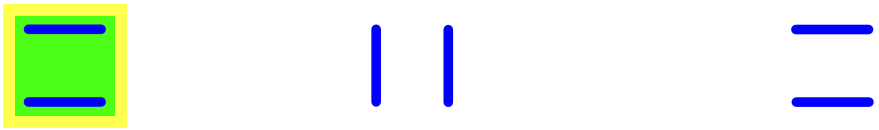
$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$



$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

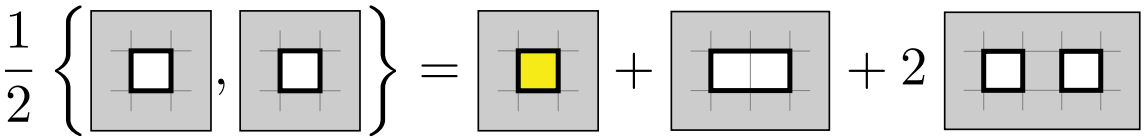
$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$





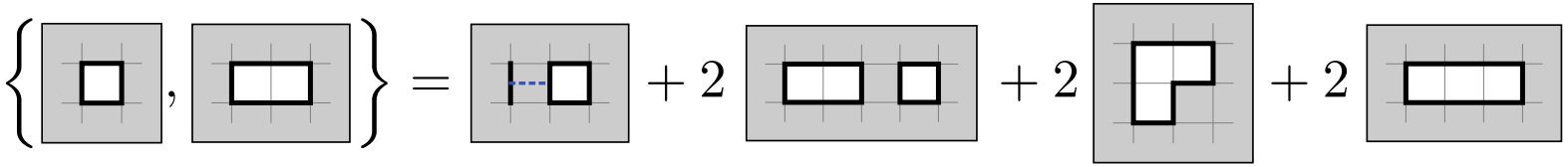
$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

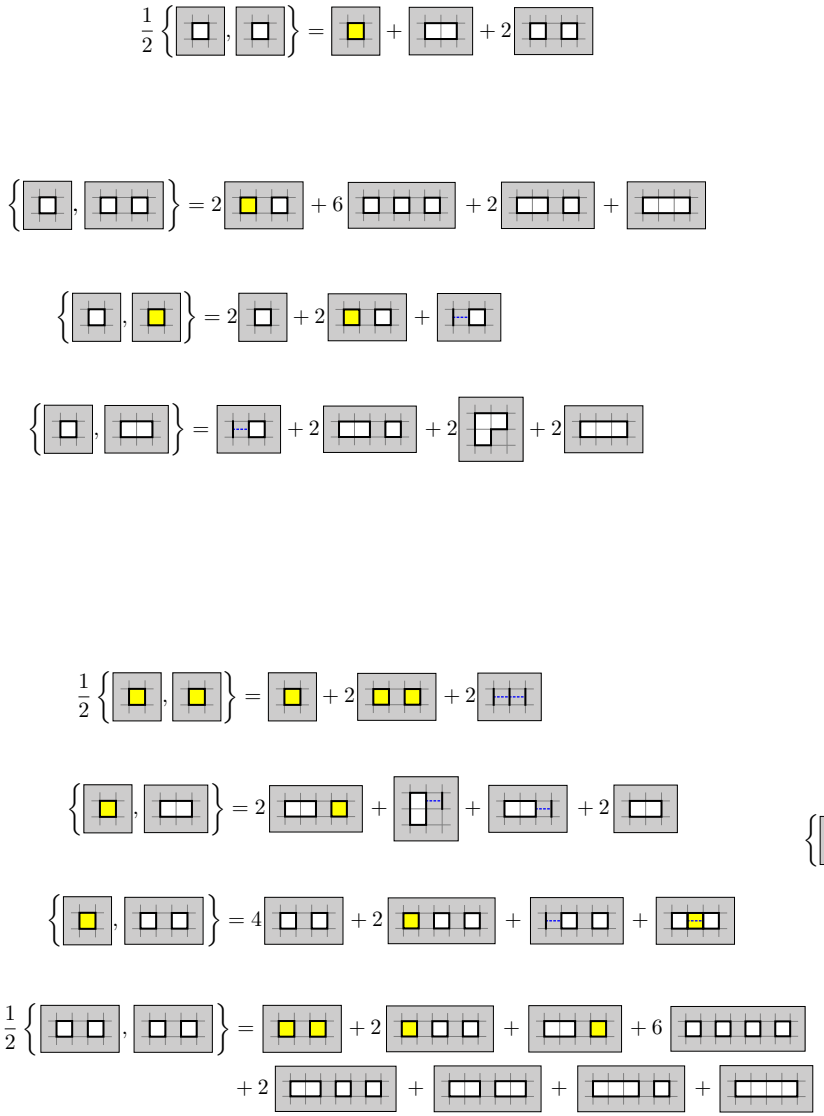
$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$

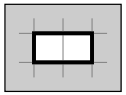
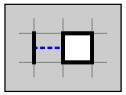
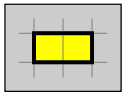
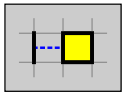
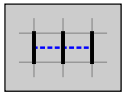
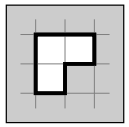
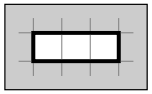


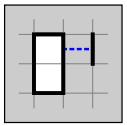
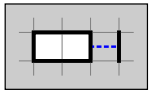
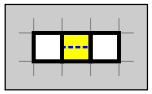
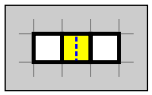
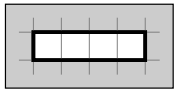
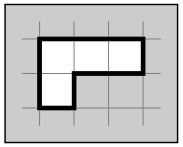
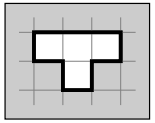
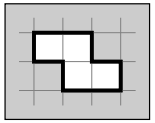
$$\mathcal{H}^{\text{eff}} = \mathcal{O}^{-1/2}\mathcal{H}\mathcal{O}^{-1/2}$$

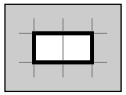
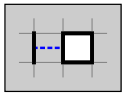
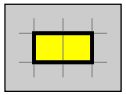
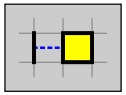
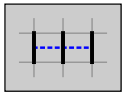
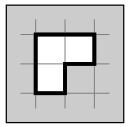
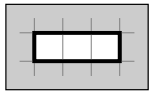
$$\mathcal{O}^{\tau} = \sum_k \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)\Gamma(1+k)} (\mathcal{O}-1)^k$$



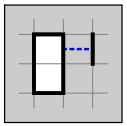
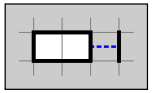
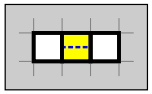
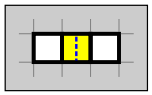
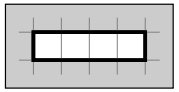
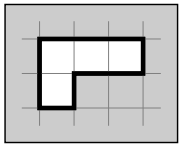
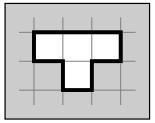
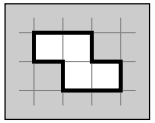


Processes	Weight
	$-2\left(J_1 - J_2\right)\alpha^2\left(\alpha^4 + 1\right)$
	$2\left(J_1 - J_2\right)\alpha^4\left(\alpha^4 + 1\right)$
	$\frac{1}{4}\alpha^4\left(J_3\left(5\alpha^4 + 8\right) - \left(J_1 - J_2\right)\left(\alpha^4 + 8\right)\right)$
	$\left(J_1 - J_2 - J_3\right)\alpha^6$
	$\left(3J_1 - 3J_2 - J_3\right)\alpha^8$
	$\frac{1}{2}\left(-5J_1 + 5J_2 + J_3\right)\alpha^8$
	$4\left(J_1 - J_2\right)\alpha^8$
	$\left(-2J_1 + 3J_2 + 2J_3\right)\alpha^6$
	$\left(-J_1 + J_2 + 2J_3\right)\alpha^6$

Processes	Weight
	$\frac{1}{4}\left(3J_1 - 5J_2 - 3J_3\right)\alpha^8$
	$\frac{1}{4}\left(2J_1 - 2J_2 - 3J_3\right)\alpha^8$
	$\frac{1}{4}\left(6J_1 - 6J_2 - 5J_3\right)\alpha^8$
	$\frac{1}{4}\left(7J_1 - 7J_2 - 3J_3\right)\alpha^8$
	$\left(3J_1 - 3J_2 + J_3\right)\alpha^8$
	$\frac{1}{4}\left(-3J_1 + 5J_2 + 6J_3\right)\alpha^8$
	$\frac{1}{4}\left(-5J_1 + 9J_2 + 6J_3\right)\alpha^8$
	$\frac{1}{2}\left(3J_1 - J_2 + 4J_3\right)\alpha^8$

Processes	Weight
	$-2\left(J_1 - J_2\right)\alpha^2\left(\alpha^4 + 1\right)$
	$2\left(J_1 - J_2\right)\alpha^4\left(\alpha^4 + 1\right)$
	$\frac{1}{4}\alpha^4\left(J_3\left(5\alpha^4 + 8\right) - \left(J_1 - J_2\right)\left(\alpha^4 + 8\right)\right)$
	$\left(J_1 - J_2 - J_3\right)\alpha^6$
	
	
	$4\left(J_1 - J_2\right)\alpha^8$
	$\left(-2J_1 + 3J_2 + 2J_3\right)\alpha^6$
	$\left(-J_1 + J_2 + 2J_3\right)\alpha^6$

Only connected terms !

Processes	Weight
	$\frac{1}{4}\left(3J_1 - 5J_2 - 3J_3\right)\alpha^8$
	$\frac{1}{4}\left(2J_1 - 2J_2 - 3J_3\right)\alpha^8$
	$\frac{1}{4}\left(6J_1 - 6J_2 - 5J_3\right)\alpha^8$
	$\frac{1}{4}\left(7J_1 - 7J_2 - 3J_3\right)\alpha^8$
	$\left(3J_1 - 3J_2 + J_3\right)\alpha^8$
	$\frac{1}{4}\left(-3J_1 + 5J_2 + 6J_3\right)\alpha^8$
	$\frac{1}{4}\left(-5J_1 + 9J_2 + 6J_3\right)\alpha^8$
	$\frac{1}{2}\left(3J_1 - J_2 + 4J_3\right)\alpha^8$

Processes	**Weight**
	$-2\left(J_1 - J_2\right)\alpha^2\left(\alpha^4 + 1\right)$
	$2\left(J_1 - J_2\right)\alpha^4\left(\alpha^4 + 1\right)$

Processes	Weight
	$-2\left(J_1 - J_2\right)\alpha^2\left(\alpha^4 + 1\right)$
	$2\left(J_1 - J_2\right)\alpha^4\left(\alpha^4 + 1\right)$

$$\frac{V}{|t|} = \alpha^2 = 1/2$$

Recover RK at lowest orders

Processes	Weight
	$-2\left(J_1 - J_2\right)\alpha^2\left(\alpha^4 + 1\right)$
	$2\left(J_1 - J_2\right)\alpha^4\left(\alpha^4 + 1\right)$

Recover RK at lowest orders

$$\frac{V}{|t|} = \alpha^2 = 1/2$$

does not depend on the couplings...

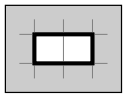
Processes	Weight
	$-2\left(J_1 - J_2\right)\alpha^2\left(\alpha^4+1\right)$
	$2\left(J_1 - J_2\right)\alpha^4\left(\alpha^4+1\right)$

Recover RK at lowest orders

$$\frac{V}{|t|} = \alpha^2 = 1/2$$

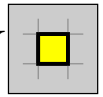
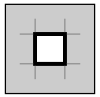
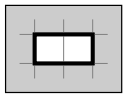
does not depend on the couplings...

First non trivial term

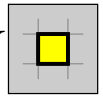
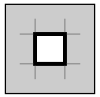
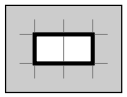


$$\tfrac{1}{4}\alpha^4\left(J_3\left(5\alpha^4+8\right) - \left(J_1 - J_2\right)\left(\alpha^4+8\right)\right)$$

Effective dimer model

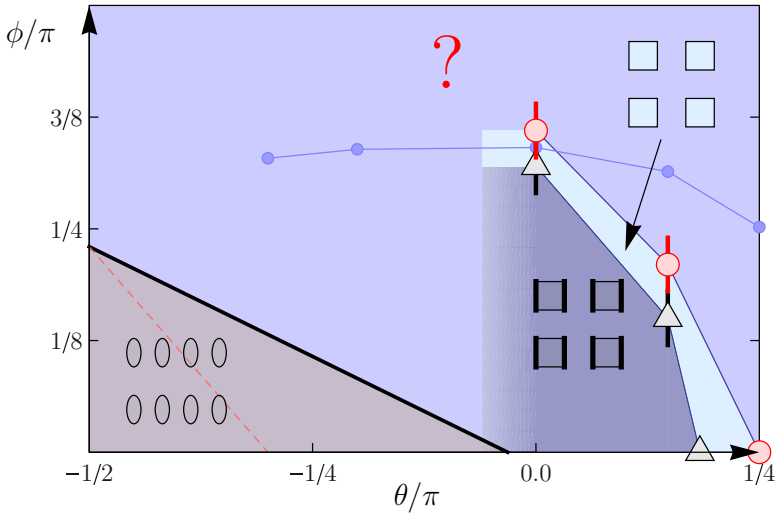
$\mathcal{H}_{\text{eff}} = V$  $- t_4$  $- t_6$ 

Effective dimer model

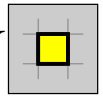
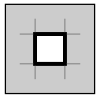
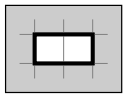
$\mathcal{H}_{\text{eff}} = V$  $- t_4$  $- t_6$ 

$$V = \cos(\phi)\sin(\theta)$$
$$t_4 = \cos(\phi)\cos(\theta)$$
$$t_6 = \sin(\theta)$$

A. Ralko, M. Mambrini & D. Poilblanc, PRB **80** (2009)



Effective dimer model

$\mathcal{H}_{\text{eff}} = V$  $- t_4$  $- t_6$ 

$$V = \cos(\phi)\sin(\theta)$$
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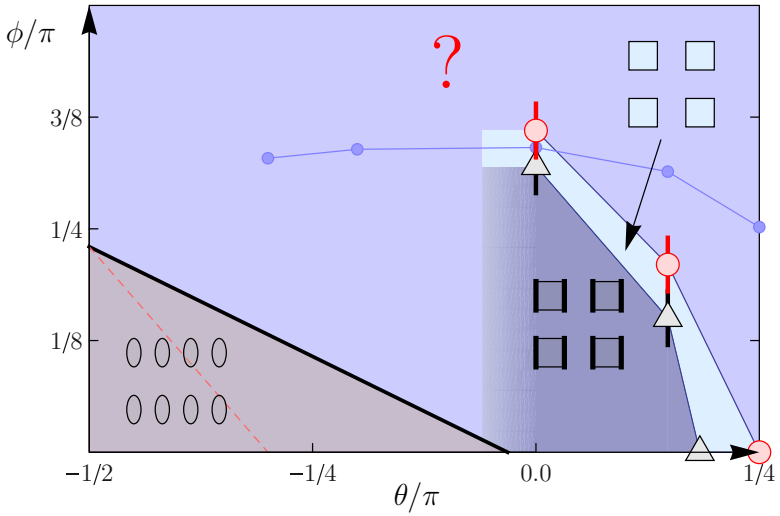
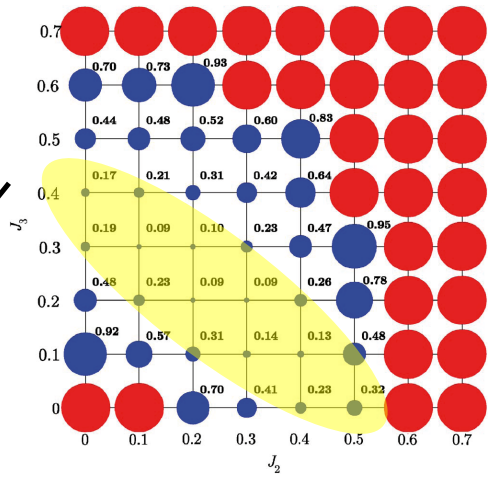
A. Ralko, M. Mambrini & D. Poilblanc, PRB **80** (2009)

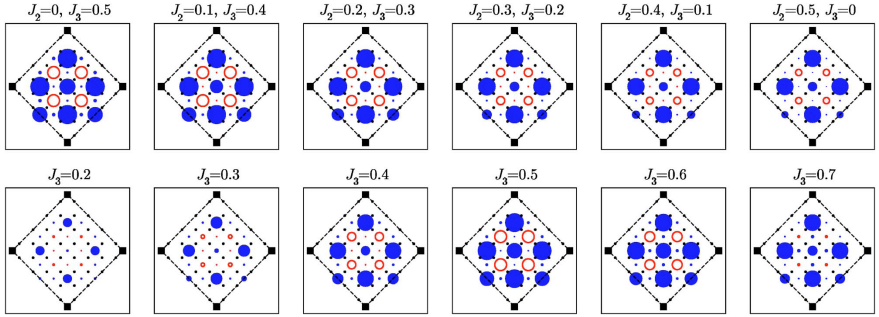


Microscopic model

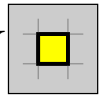
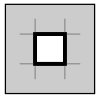
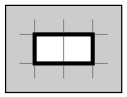
$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \, \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i . \vec{S}_j + J_3 \sum_{\langle\langle\langle i,j \rangle\rangle\rangle} \vec{S}_i . \vec{S}_j$$



M. Mambrini, A. Laüchli, D. Poilblanc & F. Mila, PRB **74** (2006)



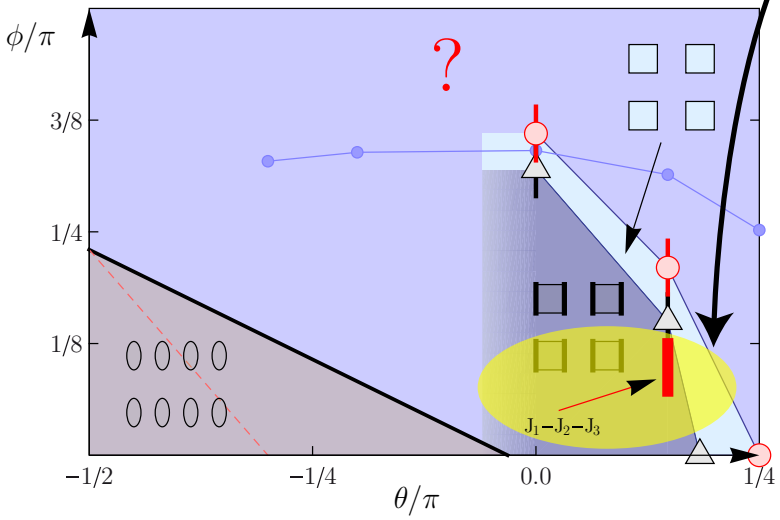
Effective dimer model

$\mathcal{H}_{\text{eff}} = V$  $- t_4$  $- t_6$ 

$$V = \cos(\phi)\sin(\theta)$$
$$t_4 = \cos(\phi)\cos(\theta)$$
$$t_6 = \sin(\theta)$$

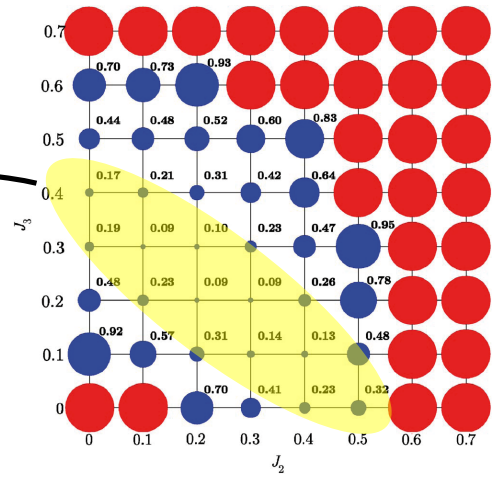
$$V = 2\,(J_1 - J_2)\,\alpha^4\,(\alpha^4 + 1)$$
$$t_4 = 2\,(J_1 - J_2)\,\alpha^2\,(\alpha^4 + 1)$$
$$t_6 = -\tfrac{1}{4}\alpha^4\left(J_3\,(5\alpha^4 + 8) - (J_1 - J_2)\,(\alpha^4 + 8)\right)$$

A. Ralko, M. Mambrini & D. Poilblanc, PRB **80** (2009)

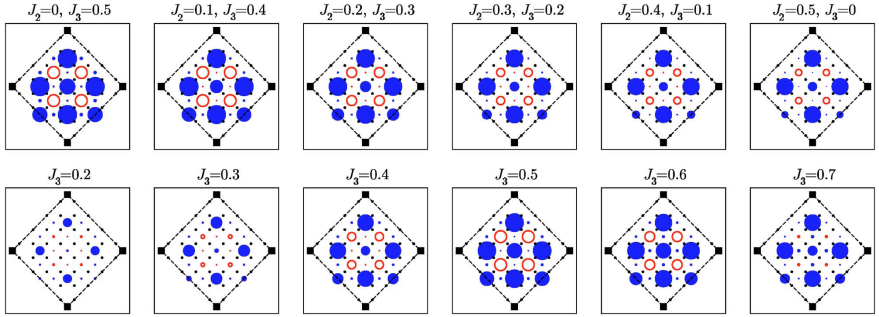


Microscopic model

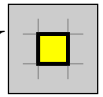
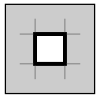
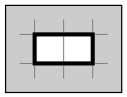
$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i\,\vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i.\vec{S}_j + J_3 \sum_{\langle\langle\langle i,j \rangle\rangle\rangle} \vec{S}_i.\vec{S}_j$$



M. Mambrini, A. Laüchli, D. Poilblanc & F. Mila, PRB **74** (2006)



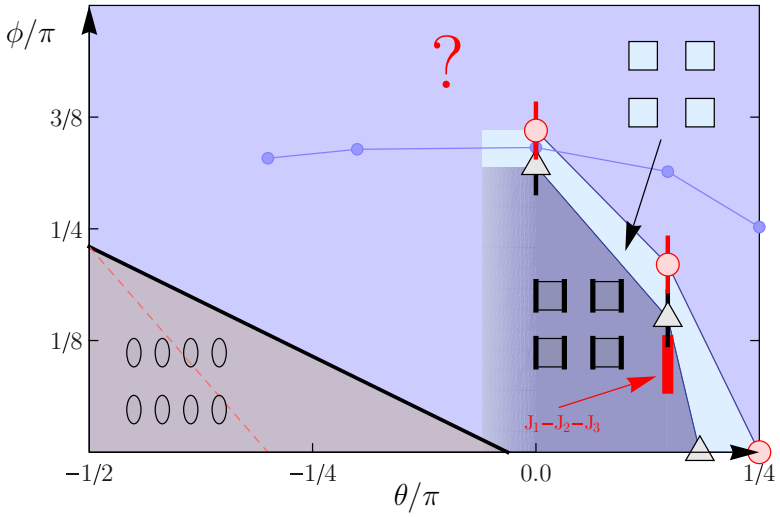
Effective dimer model

$\mathcal{H}_{\text{eff}} = V$  $- t_4$  $- t_6$ 

$$V = \cos(\phi)\sin(\theta)$$
$$t_4 = \cos(\phi)\cos(\theta)$$
$$t_6 = \sin(\theta)$$

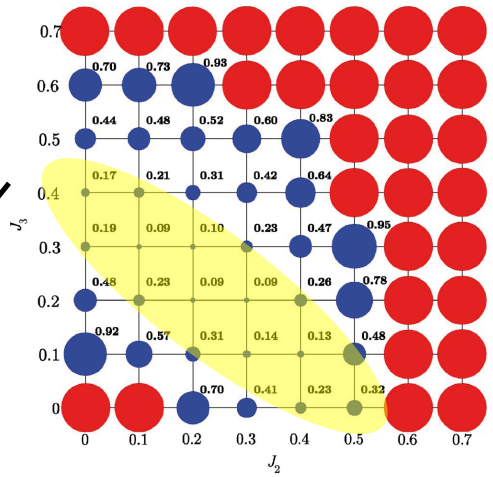
Plaquette phase

A. Ralko, M. Mambrini & D. Poilblanc, PRB **80** (2009)

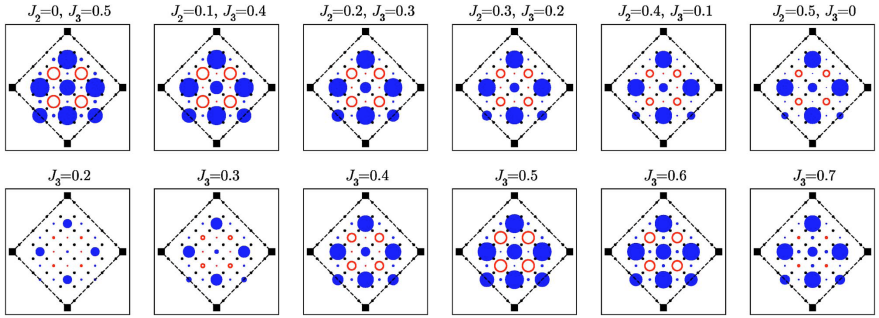


Microscopic model

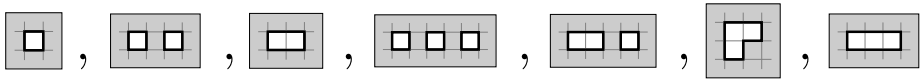
$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \, \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i . \vec{S}_j + J_3 \sum_{\langle\langle\langle i,j \rangle\rangle\rangle} \vec{S}_i . \vec{S}_j$$



M. Mambrini, A. Laüchli, D. Poilblanc & F. Mila, PRB **74** (2006)



$\mathrm{n}(g)$ **1** **2** **2** **3** **3** **3** **3**

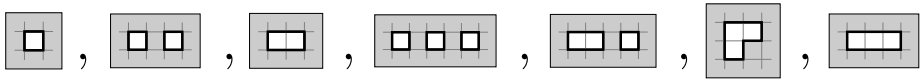
$\hat{\omega}_g =$  $\ldots$

$$\hat{\mathcal{O}} = \sum_g \alpha^{2\mathrm{n}(g)} \hat{\omega}_g$$

$$\hat{\mathcal{H}} = \sum_g \alpha^{2\mathrm{n}(g)} h_g \hat{\omega}_g$$

Order

$\mathrm{n}(g)$ **1** **2** **2** **3** **3** **3** **3**

$\hat{\omega}_g =$  $\ldots$

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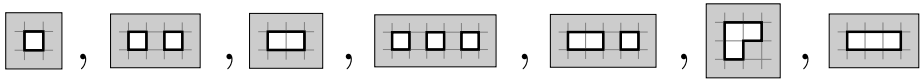
$$\hat{\mathcal{H}} = \sum_g \alpha^{2\mathrm{n}(g)} h_g \hat{\omega}_g$$

Order

$$\hat{X}_g = e^{\mu h_g} \hat{\omega}_g$$

$$\hat{X}_g|_{\mu=0} = \hat{\omega}_g$$

$$\partial_\mu \hat{X}_g|_{\mu=0} = h_g \hat{\omega}_g$$

$\mathrm{n}(g)$	1	2	2	3	3	3	3
$\hat{\omega}_g =$							$\dots$
$\mathrm{dg}(g)$	1	2	1	3	2	1	1

$$\hat{\mathcal{O}} = \sum_g \alpha^{2\mathrm{n}(g)} \hat{\omega}_g$$

$$\hat{\mathcal{H}} = \sum_g \alpha^{2\mathrm{n}(g)} h_g \hat{\omega}_g$$

Order

$$\hat{X}_g = e^{\mu h_g} \hat{\omega}_g$$

$$\hat{X}_g|_{\mu=0} = \hat{\omega}_g$$

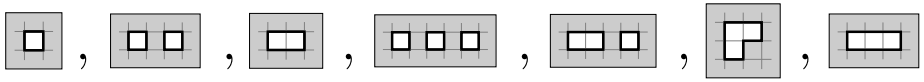
$$\partial_\mu \hat{X}_g|_{\mu=0} = h_g \hat{\omega}_g$$

Degree

$$\hat{Z}(\alpha, \beta, \mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

$$\hat{\mathcal{O}} = \hat{Z}|_{\substack{\mu=0 \\ \beta=1}}$$

$$\hat{\mathcal{H}} = \partial_\mu \hat{Z}|_{\substack{\mu=0 \\ \beta=1}}$$

$\mathrm{n}(g)$	1	2	2	3	3	3	3
$\hat{\omega}_g =$							$\dots$
$\mathrm{dg}(g)$	1	2	1	3	2	1	1

$$\hat{\mathcal{O}} = \sum_g \alpha^{2\mathrm{n}(g)} \hat{\omega}_g$$

$$\hat{\mathcal{H}} = \sum_g \alpha^{2\mathrm{n}(g)} h_g \hat{\omega}_g$$

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$$\hat{\mathcal{H}} = \partial_\mu \hat{Z}|_{\substack{\mu=0 \\ \beta=1}}$$

$$\hat{\mathcal{H}}_{\mathrm{eff}} \underset{\substack{\mu=0 \\ \beta=1}}{=} \hat{Z}^{-1/2} \left(\partial_\mu \hat{Z} \right) \hat{Z}^{-1/2}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0 \\ \beta=1}}{=} \hat{Z}^{-1/2} \left(\partial_\mu \hat{Z} \right) \hat{Z}^{-1/2}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0 \\ \beta=1}}{=} \hat{Z}^{-1/2} \left(\partial_\mu \hat{Z} \right) \hat{Z}^{-1/2}$$

$$\partial \left(e^F \right) = \int_0^1 e^{(1-s)F} \left(\partial F \right) e^{sF} \, \mathrm{d}s$$

$$F = \ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0 \\ \beta=1}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{Z}^{-s} \left(\partial_\mu \ln \hat{Z} \right) \hat{Z}^{s} \, \mathrm{d}s$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \hat{Z}^{-1/2} \left(\partial_\mu \hat{Z}\right) \hat{Z}^{-1/2}$$

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$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{Z}^{-s} \left(\partial_\mu \ln \hat{Z}\right) \hat{Z}^{s} \, \mathrm{d}s$$

$$F = \ln \hat{Z}$$

$$e^A B e^{-A} = e^{[A,} B \equiv B + [A,B] + \frac{1}{2!}\left[A,[A,B]\right] + \ldots$$

$$A = s \ln \hat{Z} \qquad B = \partial_\mu \ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-s\left[\ln \hat{Z},\right.} \, \mathrm{d}s\right) \partial_\mu \ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \hat{Z}^{-1/2}\left(\partial_\mu \hat{Z}\right)\hat{Z}^{-1/2}$$

$$\partial\left(e^F\right) = \int_0^1 e^{(1-s)F}\left(\partial F\right)e^{sF}\,\mathrm{d}s$$

$$F = \ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{Z}^{-s}\left(\partial_\mu \ln \hat{Z}\right)\hat{Z}^{s}\,\mathrm{d}s$$

$$e^A B e^{-A} = e^{[A,}B \equiv B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots$$

$$A = s\ln\hat{Z} \qquad B = \partial_\mu \ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-s[\ln\hat{Z},}\,\mathrm{d}s\right)\partial_\mu \ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \sum_{p=0}^{\infty}\frac{1}{2^{2p}}\frac{\left(\left[\ln\hat{Z},\right)^{2p}}{(2p+1)!}\partial_\mu \ln \hat{Z}$$

$$\hat{Z}(\alpha, \beta, \mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

$$\hat{Z} = \sum_{n=0}^{\infty} \beta^n \hat{z}^{(n)}$$

$$\hat{z}^{(d)} = \sum_{\substack{g \text{ such as} \\ \mathrm{dg}(g)=d}} \alpha^{2\mathrm{n}(g)} \hat{X}_g$$

$$\hat{Z}(\alpha,\beta,\mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

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$$\ln \hat{Z} = \sum_{m=1}^{\infty} \beta^m \hat{z}_c^{(m)}$$

$$\hat{z}_c^{(m)} = -\sum_{n=1}^{m} \frac{(-1)^n}{n\,2^n} \sum_{s \in \mathcal{S}_n^m} \left(\prod_{i=1}^{n} \left\{ \hat{z}^{(s_i)}, \right. \right) \hat{1}$$

$$\hat{Z}(\alpha,\beta,\mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

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$$\begin{aligned}
\hat{z}_c^{(1)} &= \hat{z}^{(1)} \\
\hat{z}_c^{(2)} &= \hat{z}^{(2)} - \frac{1}{4}\left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\} \\
\hat{z}_c^{(3)} &= \hat{z}^{(3)} - \frac{1}{2}\left\{\hat{z}^{(1)}, \hat{z}^{(2)}\right\} + \frac{1}{12}\left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}\right\} \\
\hat{z}_c^{(4)} &= \hat{z}^{(4)} - \frac{1}{2}\left\{\hat{z}^{(1)}, \hat{z}^{(3)}\right\} - \frac{1}{4}\left\{\hat{z}^{(2)}, \hat{z}^{(2)}\right\} \\
&\quad + \frac{1}{6}\left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \hat{z}^{(2)}\right\}\right\} + \frac{1}{12}\left\{\hat{z}^{(2)}, \left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}\right\} \\
&\quad - \frac{1}{32}\left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}\right\}\right\}
\end{aligned}$$

$$\hat{Z}(\alpha,\beta,\mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

$$\hat{Z} = \sum_{n=0}^{\infty} \beta^n \hat{z}^{(n)}$$

$$\hat{z}^{(d)} = \sum_{\substack{g \text{ such as} \\ \mathrm{dg}(g)=d}} \alpha^{2\mathrm{n}(g)} \hat{X}_g$$

$$\ln \hat{Z} = \sum_{m=1}^{\infty} \beta^m \hat{z}_c^{(m)}$$

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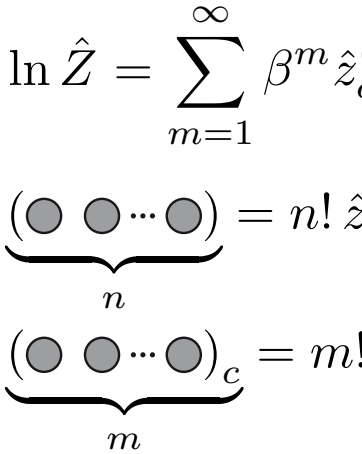

$$= n!\,\hat{z}^{(n)}$$

$$= m!\,\hat{z}_c^{(m)}$$

$$[s] = \frac{1}{m!} \frac{\left(\sum_{p=1}^{m} s_p\right)!}{\prod_{p=1}^{m} s_p!}$$

$$\left\langle \prod_{i=1}^{n} \hat{o}_i \right\rangle = \frac{1}{2^n} \left(\prod_{i=1}^{n} \{\hat{o}_i, \right. \hat{1}$$

$$\hat{z}_c^{(1)} = \hat{z}^{(1)}$$

$$\hat{z}_c^{(2)} = \hat{z}^{(2)} - \frac{1}{4}\left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}$$

$$\hat{z}_c^{(3)} = \hat{z}^{(3)} - \frac{1}{2}\left\{\hat{z}^{(1)}, \hat{z}^{(2)}\right\} + \frac{1}{12}\left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}\right\}$$

$$\begin{aligned}\hat{z}_c^{(4)} = \hat{z}^{(4)} &- \frac{1}{2}\left\{\hat{z}^{(1)}, \hat{z}^{(3)}\right\} - \frac{1}{4}\left\{\hat{z}^{(2)}, \hat{z}^{(2)}\right\} \\ &+ \frac{1}{6}\left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \hat{z}^{(2)}\right\}\right\} + \frac{1}{12}\left\{\hat{z}^{(2)}, \left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}\right\} \\ &- \frac{1}{32}\left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \left\{\hat{z}^{(1)}, \hat{z}^{(1)}\right\}\right\}\right\}\end{aligned}$$

$$\hat{Z}(\alpha,\beta,\mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

$$\hat{Z} = \sum_{n=0}^{\infty} \beta^n \hat{z}^{(n)}$$

$$\hat{z}^{(d)} = \sum_{\substack{g \text{ such as} \\ \mathrm{dg}(g)=d}} \alpha^{2\mathrm{n}(g)} \hat{X}_g$$

$$\ln \hat{Z} = \sum_{m=1}^{\infty} \beta^m \hat{z}_c^{(m)}$$

$$\underbrace{(\bullet\ \bullet \cdots \bullet)}_{n} = n!\, \hat{z}^{(n)}$$

$$\underbrace{(\bullet\ \bullet \cdots \bullet)_c}_{m} = m!\, \hat{z}_c^{(m)}$$

$$[s] = \frac{1}{m!} \frac{\left(\sum_{p=1}^m s_p\right)!}{\prod_{p=1}^m s_p!}$$

$$\left\langle \prod_{i=1}^n \hat{o}_i \right\rangle = \frac{1}{2^n} \left(\prod_{i=1}^n \{\hat{o}_i, \right) \hat{1}$$

$$\langle \underbrace{(\bullet\ \bullet \cdots \bullet)_c}_{m} \rangle =$$

$$- \sum_{n=1}^{m} (-1)^n (n-1)! \sum_{s \in \mathcal{S}_n^m} [s] \langle \underbrace{(\bullet\ \bullet \cdots \bullet)}_{s_1} \times \ldots \times \underbrace{(\bullet\ \bullet \cdots \bullet)}_{s_n} \rangle$$

$$\hat{z}_c^{(1)} = \hat{z}^{(1)}$$

$$\hat{z}_c^{(2)} = \hat{z}^{(2)} - \frac{1}{4} \left\{ \hat{z}^{(1)}, \hat{z}^{(1)} \right\}$$

$$\hat{z}_c^{(3)} = \hat{z}^{(3)} - \frac{1}{2} \left\{ \hat{z}^{(1)}, \hat{z}^{(2)} \right\} + \frac{1}{12} \left\{ \hat{z}^{(1)}, \left\{ \hat{z}^{(1)}, \hat{z}^{(1)} \right\} \right\}$$

$$\begin{aligned} \hat{z}_c^{(4)} = \hat{z}^{(4)} &- \frac{1}{2} \left\{ \hat{z}^{(1)}, \hat{z}^{(3)} \right\} - \frac{1}{4} \left\{ \hat{z}^{(2)}, \hat{z}^{(2)} \right\} \\ &+ \frac{1}{6} \left\{ \hat{z}^{(1)}, \left\{ \hat{z}^{(1)}, \hat{z}^{(2)} \right\} \right\} + \frac{1}{12} \left\{ \hat{z}^{(2)}, \left\{ \hat{z}^{(1)}, \hat{z}^{(1)} \right\} \right\} \\ &- \frac{1}{32} \left\{ \hat{z}^{(1)}, \left\{ \hat{z}^{(1)}, \left\{ \hat{z}^{(1)}, \hat{z}^{(1)} \right\} \right\} \right\} \end{aligned}$$

$$\hat{Z}(\alpha,\beta,\mu) = \sum_g \alpha^{2\mathrm{n}(g)} \beta^{\mathrm{dg}(g)} \hat{X}_g$$

$$\hat{Z} = \sum_{n=0}^{\infty} \beta^n \hat{z}^{(n)}$$

$$\hat{z}^{(d)} = \sum_{\substack{g \text{ such as} \\ \mathrm{dg}(g)=d}} \alpha^{2\mathrm{n}(g)} \hat{X}_g$$

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$$\underbrace{(\bullet\ \bullet \cdots \bullet)}_{n} = n!\, \hat{z}^{(n)}$$

$$\underbrace{(\bullet\ \bullet \cdots \bullet)_c}_{m} = m!\, \hat{z}_c^{(m)}$$

$$[s] = \frac{1}{m!} \frac{\left(\sum_{p=1}^{m} s_p\right)!}{\prod_{p=1}^{m} s_p!}$$

$$\left\langle \prod_{i=1}^{n} \hat{o}_i \right\rangle = \frac{1}{2^n} \left(\prod_{i=1}^{n} \{\hat{o}_i, \right) \hat{1}$$

$$\langle \underbrace{(\bullet\ \bullet \cdots \bullet)_c}_{m} \rangle =$$
$$- \sum_{n=1}^{m} (-1)^n (n-1)! \sum_{s \in \mathcal{S}_n^m} [s] \langle \underbrace{(\bullet\ \bullet \cdots \bullet)}_{s_1} \times \ldots \times \underbrace{(\bullet\ \bullet \cdots \bullet)}_{s_n} \rangle$$

$$\begin{aligned}
\langle (\bullet)_c \rangle &= 0!\,[1] \langle (\bullet) \rangle \\
\langle (\bullet\ \bullet)_c \rangle &= 0!\,[2] \langle (\bullet\ \bullet) \rangle \\
&\quad - 1!\,[1,1] \langle (\bullet) \times (\bullet) \rangle \\
\langle (\bullet\ \bullet\ \bullet)_c \rangle &= 0!\,[3] \langle (\bullet\ \bullet\ \bullet) \rangle \\
&\quad - 1!\,[1,2] \langle (\bullet) \times (\bullet\ \bullet) \rangle \\
&\quad - 1!\,[2,1] \langle (\bullet\ \bullet) \times (\bullet) \rangle \\
&\quad + 2!\,[1,1,1] \langle (\bullet) \times (\bullet) \times (\bullet) \rangle
\end{aligned}$$

Fusion rules

$$\langle (\bullet\ \bullet\ \bullet)_c \rangle = 0!\,[3]\langle (\bullet\ \bullet\ \bullet) \rangle - 1!\,[1,2]\langle (\bullet) \times (\bullet\ \bullet) \rangle - 1!\,[2,1]\langle (\bullet\ \bullet) \times (\bullet) \rangle + 2!\,[1,1,1]\langle (\bullet) \times (\bullet) \times (\bullet) \rangle$$

Fusion rules

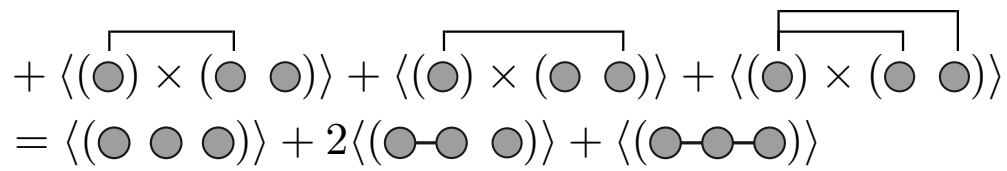
- All possible contractions.
- No internal contraction !

$$\langle (\bullet\ \bullet\ \bullet)_c \rangle = 0!\,[3]\langle (\bullet\ \bullet\ \bullet) \rangle - 1!\,[1,2]\langle (\bullet) \times (\bullet\ \bullet) \rangle - 1!\,[2,1]\langle (\bullet\ \bullet) \times (\bullet) \rangle + 2!\,[1,1,1]\langle (\bullet) \times (\bullet) \times (\bullet) \rangle$$

Fusion rules

- All possible contractions.
- No internal contraction !

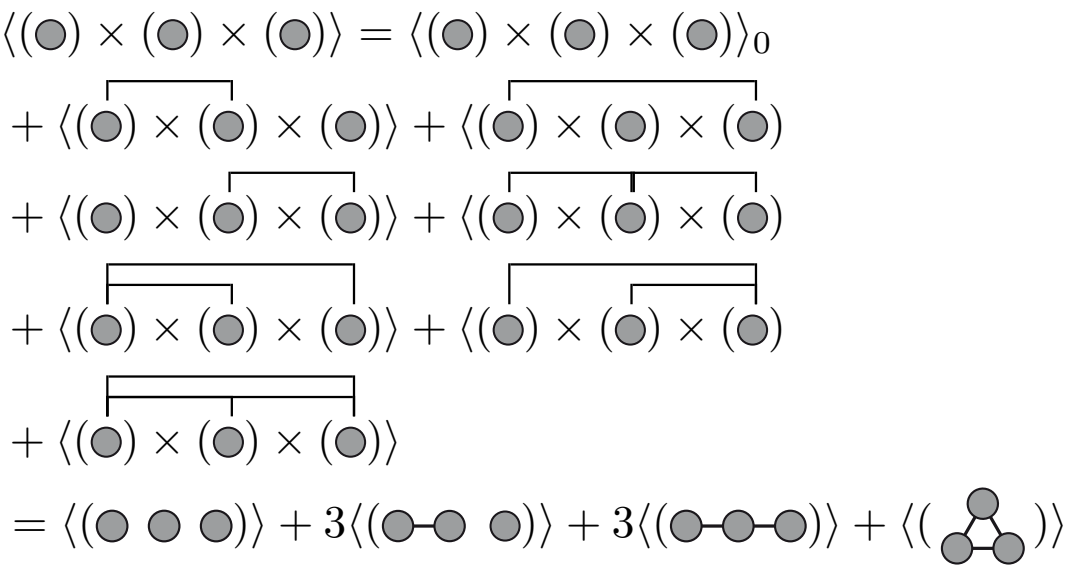
$$\langle (\bullet\ \bullet\ \bullet)_c \rangle = 0!\,[3]\langle (\bullet\ \bullet\ \bullet)\rangle - 1!\,[1,2]\langle (\bullet) \times (\bullet\ \bullet)\rangle - 1!\,[2,1]\langle (\bullet\ \bullet) \times (\bullet)\rangle + 2!\,[1,1,1]\langle (\bullet) \times (\bullet) \times (\bullet)\rangle$$



Fusion rules

- All possible contractions.
- No internal contraction !

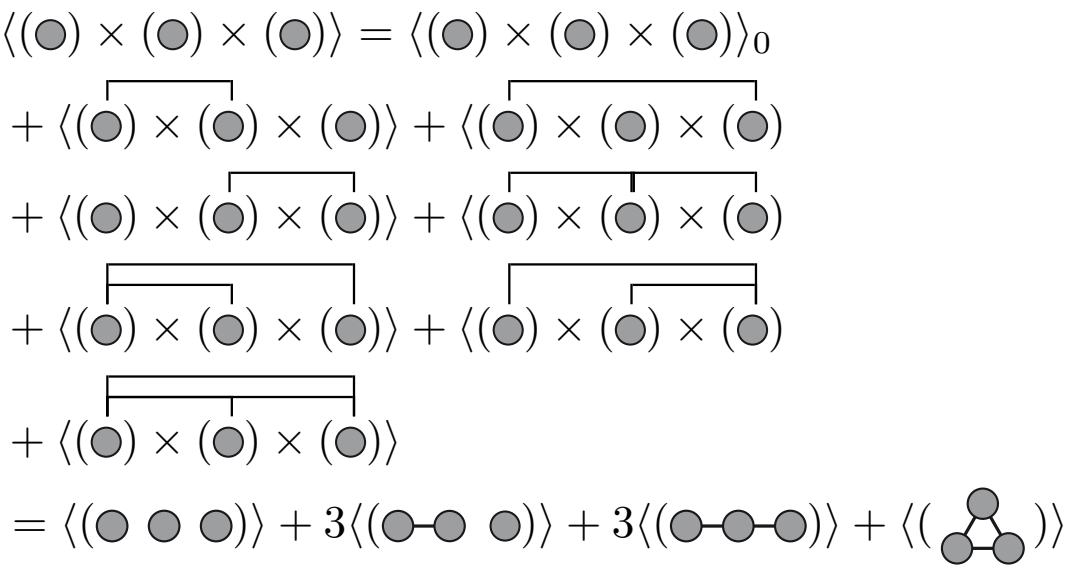
$$\langle (\bullet\ \bullet\ \bullet)_c \rangle = 0!\,[3]\langle (\bullet\ \bullet\ \bullet)\rangle - 1!\,[1,2]\langle (\bullet) \times (\bullet\ \bullet)\rangle - 1!\,[2,1]\langle (\bullet\ \bullet) \times (\bullet)\rangle + 2!\,[1,1,1]\langle (\bullet) \times (\bullet) \times (\bullet)\rangle$$



Fusion rules

- All possible contractions.
- No internal contraction !

$$\langle(\bullet\ \bullet\ \bullet)_c\rangle = 0!\,[3]\langle(\bullet\ \bullet\ \bullet)\rangle - 1!\,[1,2]\langle(\bullet)\times(\bullet\ \bullet)\rangle - 1!\,[2,1]\langle(\bullet\ \bullet)\times(\bullet)\rangle + 2!\,[1,1,1]\langle(\bullet)\times(\bullet)\times(\bullet)\rangle$$



$$\langle(\bullet)\times(\bullet)\times(\bullet)\rangle = \langle(\bullet)\times(\bullet)\times(\bullet)\rangle_0$$
$$+\langle(\bullet)\times(\bullet)\times(\bullet)\rangle + \langle(\bullet)\times(\bullet)\times(\bullet)$$
$$+\langle(\bullet)\times(\bullet)\times(\bullet)\rangle + \langle(\bullet)\times(\bullet)\times(\bullet)$$
$$+\langle(\bullet)\times(\bullet)\times(\bullet)\rangle + \langle(\bullet)\times(\bullet)\times(\bullet)$$
$$+\langle(\bullet)\times(\bullet)\times(\bullet)\rangle$$
$$= \langle(\bullet\ \bullet\ \bullet)\rangle + 3\langle(\bullet\!-\!\bullet\ \bullet)\rangle + 3\langle(\bullet\!-\!\bullet\!-\!\bullet)\rangle + \langle(\triangle)\rangle$$

$$\langle(\bullet\ \bullet\ \bullet)_c\rangle = 3\langle(\bullet\!-\!\bullet\!-\!\bullet)\rangle + 2\langle(\triangle)\rangle$$

Cumulant order deflation

D. Schwandt, M. Mambrini & D. Poilblanc, ArXiv early 2010

$$\underbrace{\langle(\bullet\ \bullet \cdots \bullet)_c\rangle}_{m} = -\sum_{G=1}^{m-1}\sum_{\mu\in\mathcal{S}_G^m}[\mu]\underbrace{\langle(\overset{\mu_1}{\bullet}\ \overset{\mu_2}{\bullet}\cdots\overset{\mu_G}{\bullet})_c\rangle}_{G}$$

$\overset{\mu_p}{\bullet}$: all possible *connected* contractions of μ_p operators

Cumulant order deflation

D. Schwandt, M. Mambrini & D. Poilblanc, ArXiv early 2010

$$\langle \underbrace{(\bullet\ \bullet \cdots \bullet)_c}_{m} \rangle = -\sum_{G=1}^{m-1} \sum_{\mu \in \mathcal{S}_G^m} [\mu] \langle \underbrace{(\overset{\mu_1}{\bullet}\ \overset{\mu_2}{\bullet} \cdots \overset{\mu_G}{\bullet})_c}_{G} \rangle$$

$\overset{\mu_p}{\bullet}$: all possible *connected* contractions of μ_p operators

$$\langle (\bullet)_c \rangle = \langle (\bullet) \rangle$$

$$\langle (\bullet\ \bullet)_c \rangle = -[2] \langle (\overset{2}{\bullet})_c \rangle = -\langle (\bullet\!-\!\bullet) \rangle$$

$$\langle (\bullet\ \bullet\ \bullet)_c \rangle = -[3] \langle (\overset{3}{\bullet})_c \rangle$$

$$- [1,2] \langle (\overset{1}{\bullet}\ \overset{2}{\bullet})_c \rangle - [2,1] \langle (\overset{2}{\bullet}\ \overset{1}{\bullet})_c \rangle$$

$$= -\left(\langle (\triangle) \rangle + 3 \langle (\bullet\!-\!\bullet\!-\!\bullet) \rangle \right)$$

$$+ 2 \times \frac{3}{2} \left(2 \langle (\bullet\!-\!\bullet\!-\!\bullet) \rangle + \langle (\triangle) \rangle \right)$$

$$= 3 \langle (\bullet\!-\!\bullet\!-\!\bullet) \rangle + 2 \langle (\triangle) \rangle$$

Cumulant order deflation

D. Schwandt, M. Mambrini & D. Poilblanc, ArXiv early 2010

$$\underbrace{\langle(\bullet\ \bullet \cdots \bullet)_c\rangle}_{m} = -\sum_{G=1}^{m-1}\sum_{\mu\in\mathcal{S}_G^m}[\mu]\underbrace{\langle(\overset{\mu_1}{\bullet}\ \overset{\mu_2}{\bullet}\cdots\overset{\mu_G}{\bullet})_c\rangle}_{G}$$

$\overset{\mu_p}{\bullet}$: all possible *connected* contractions of μ_p operators

$$\langle(\bullet)_c\rangle = \langle(\bullet)\rangle$$

$$\langle(\bullet\ \bullet)_c\rangle = -[2]\langle(\overset{2}{\bullet})_c\rangle = -\langle(\bullet\!-\!\bullet)\rangle$$

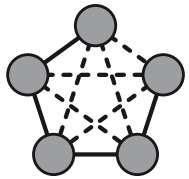
$$\langle(\bullet\ \bullet\ \bullet)_c\rangle = -[3]\langle(\overset{3}{\bullet})_c\rangle$$

$$-[1,2]\langle(\overset{1}{\bullet}\ \overset{2}{\bullet})_c\rangle - [2,1]\langle(\overset{2}{\bullet}\ \overset{1}{\bullet})_c\rangle$$

$$= -\left(\langle(\triangle)\rangle + 3\langle(\bullet\!-\!\bullet\!-\!\bullet)\rangle\right)$$

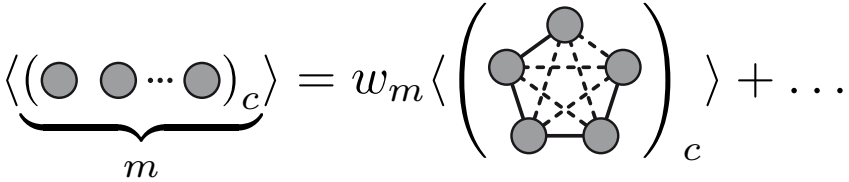
$$+2\times\frac{3}{2}\left(2\langle(\bullet\!-\!\bullet\!-\!\bullet)\rangle + \langle(\triangle)\rangle\right)$$

$$= 3\langle(\bullet\!-\!\bullet\!-\!\bullet)\rangle + 2\langle(\triangle)\rangle$$



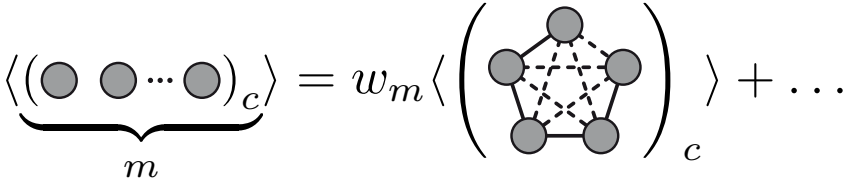
Fully connected diagrams

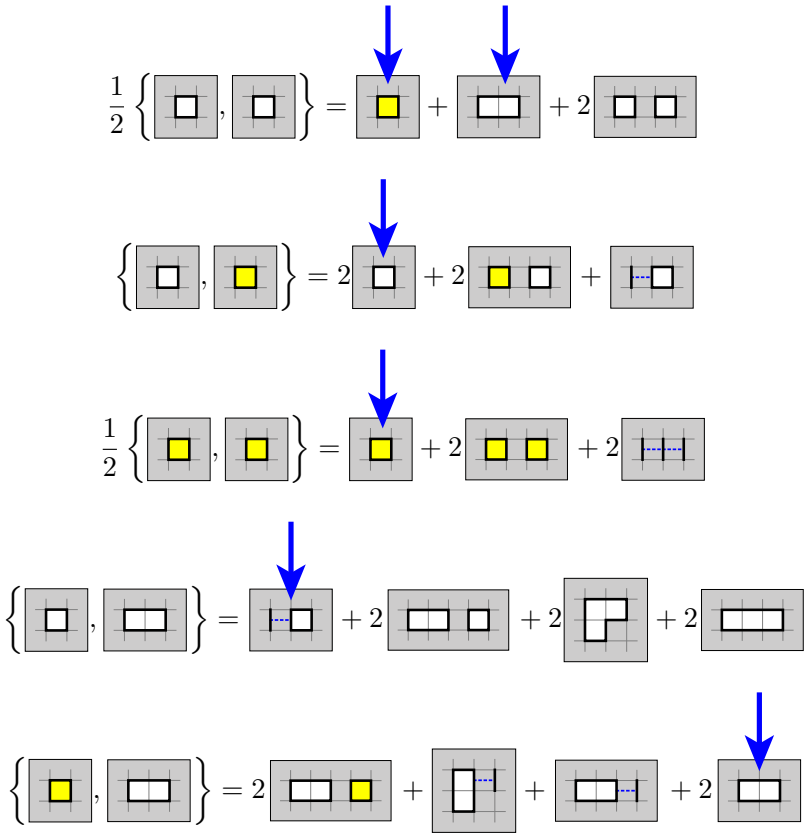
Fully connected diagrams



Fully connected diagrams

- Most compact terms
- Leading order





Fully connected diagrams

- Most compact terms
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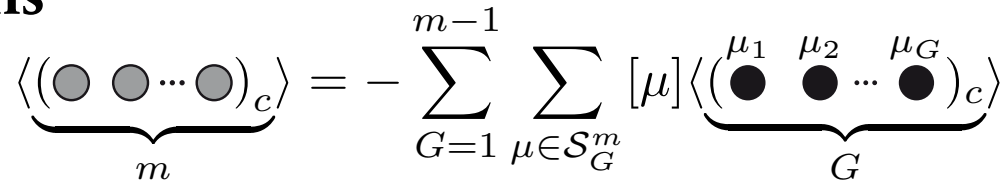
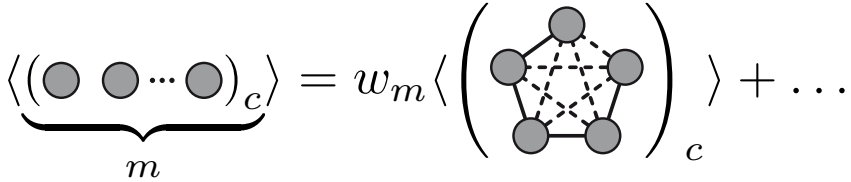
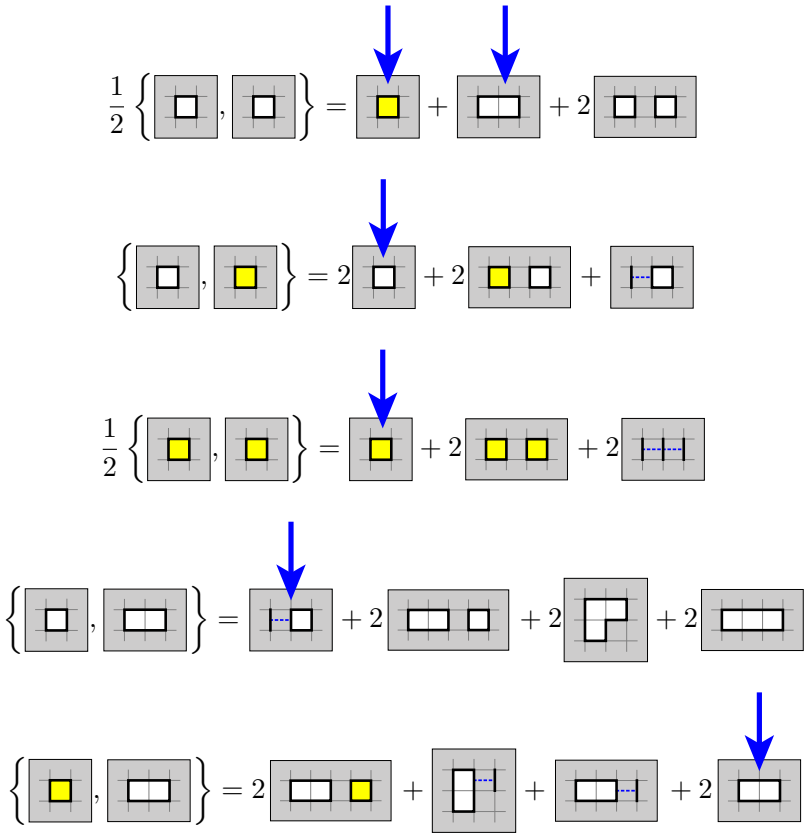




$$w_m = -\sum_{p=1}^{m-1} w_p \sum_{\mu \in \mathcal{S}_p^m} [\mu]$$



Fully connected diagrams

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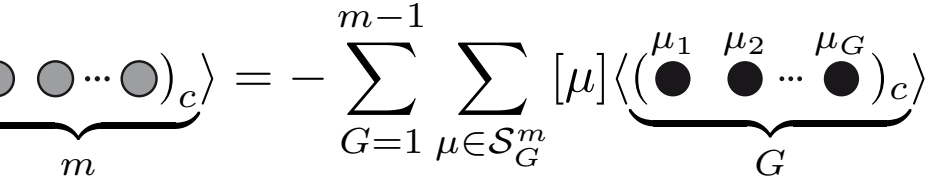
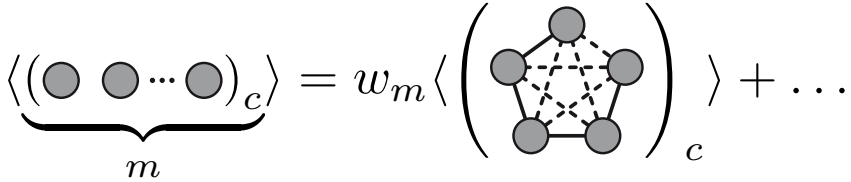
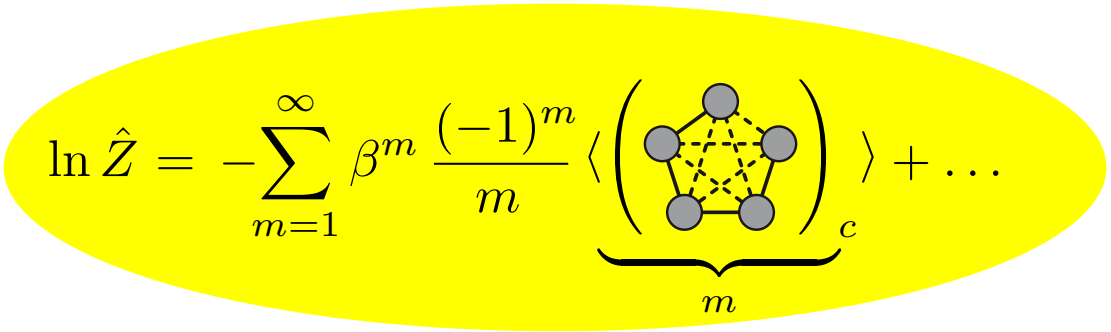


$$\langle (\bullet\ \bullet \cdots \bullet)_c \rangle = -\sum_{G=1}^{m-1} \sum_{\mu \in \mathcal{S}_G^m} [\mu] \langle (\overset{\mu_1}{\bullet}\ \overset{\mu_2}{\bullet} \cdots \overset{\mu_G}{\bullet})_c \rangle$$



$$\langle (\bullet\ \bullet \cdots \bullet)_c \rangle = w_m \langle \left(\quad \right)_c \rangle + \dots$$

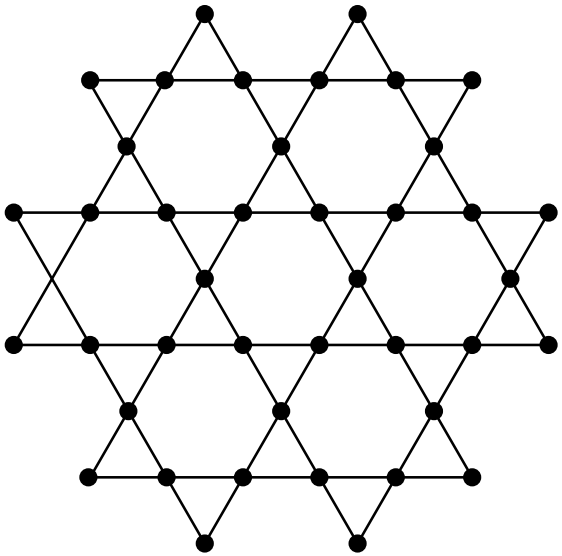
$$w_m = -\sum_{p=1}^{m-1} w_p \sum_{\mu \in \mathcal{S}_p^m} [\mu] \qquad w_p = (-1)^{p-1}(p-1)!$$



$$\ln \hat{Z} = -\sum_{m=1}^{\infty} \beta^m \frac{(-1)^m}{m} \langle \left(\quad \right)_c \rangle + \dots$$

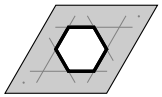
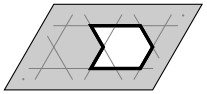
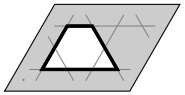
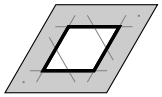
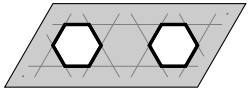
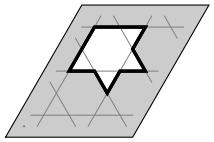
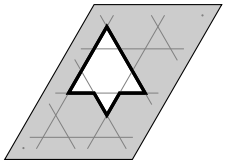
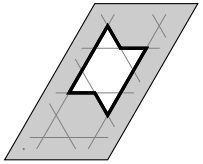
D. Schwandt, M. Mambrini & D. Poilblanc, ArXiv (early 2010)

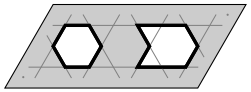
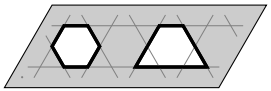
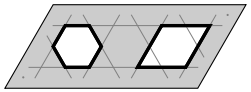
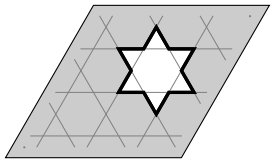
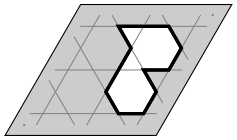
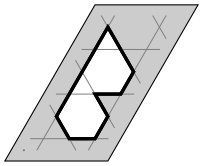
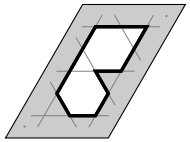
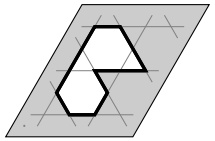
Kagome antiferromagnet



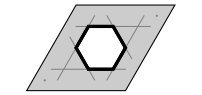
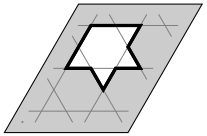
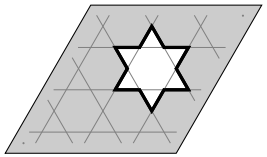
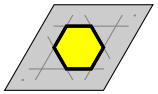
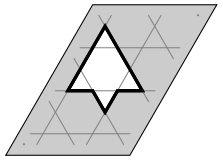
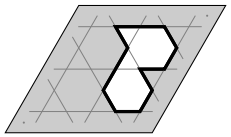
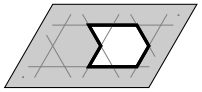
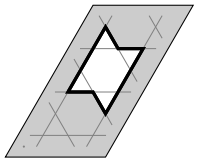
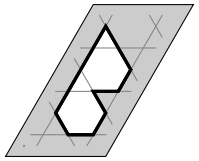
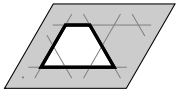
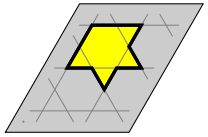
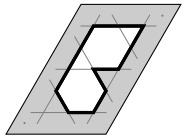
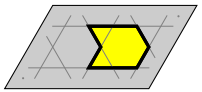
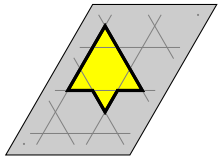
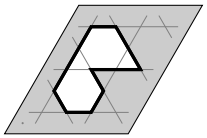
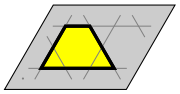
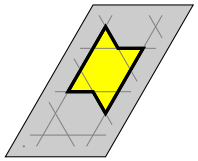
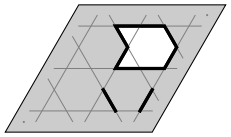
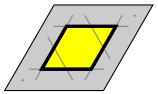
$$\mathcal{H} = \sum_{(i,j)} J\, \vec{S}_i . \vec{S}_j$$

Kagome antiferromagnet

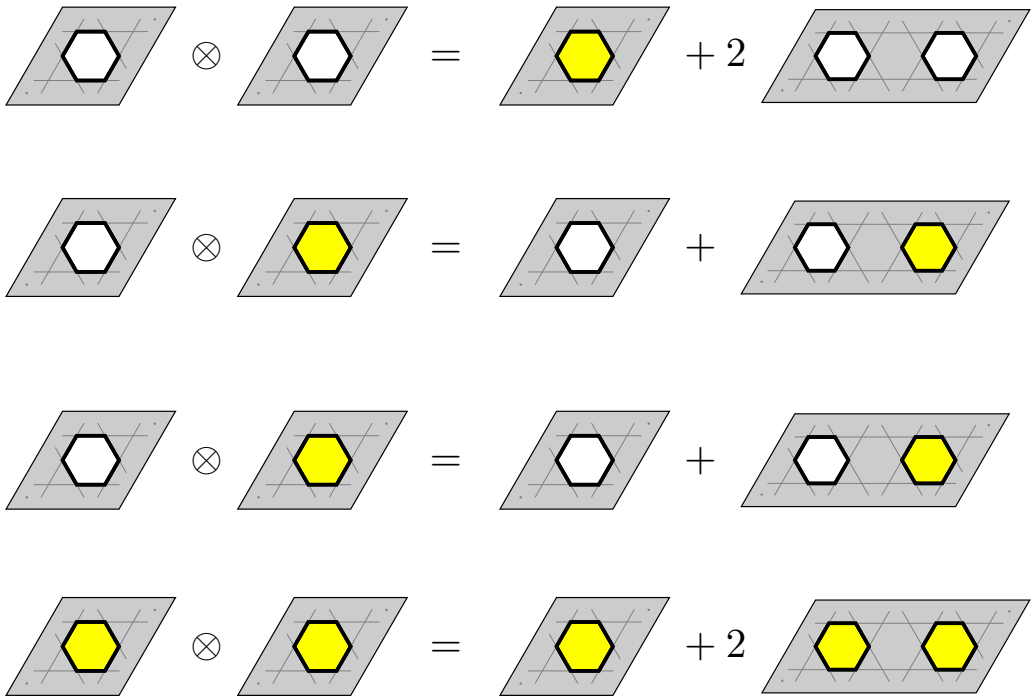
Processes	Overlap	Hamiltonian
	α^4	$-3\alpha^4$
	α^6	$-2\alpha^6$
	α^6	$-2\alpha^6$
	α^6	$-2\alpha^6$
	α^8	$-6\alpha^8$
	α^8	$-\alpha^8$
	α^8	$-\alpha^8$
	α^8	$-\alpha^8$

Processes	Overlap	Hamiltonian
	α^{10}	$-5\alpha^{10}$
	α^{10}	$-5\alpha^{10}$
	α^{10}	$-5\alpha^{10}$
	α^{10}	0
	α^{10}	$-4\alpha^{10}$
	α^{10}	$-4\alpha^{10}$
	α^{10}	$-4\alpha^{10}$
	α^{10}	$-4\alpha^{10}$

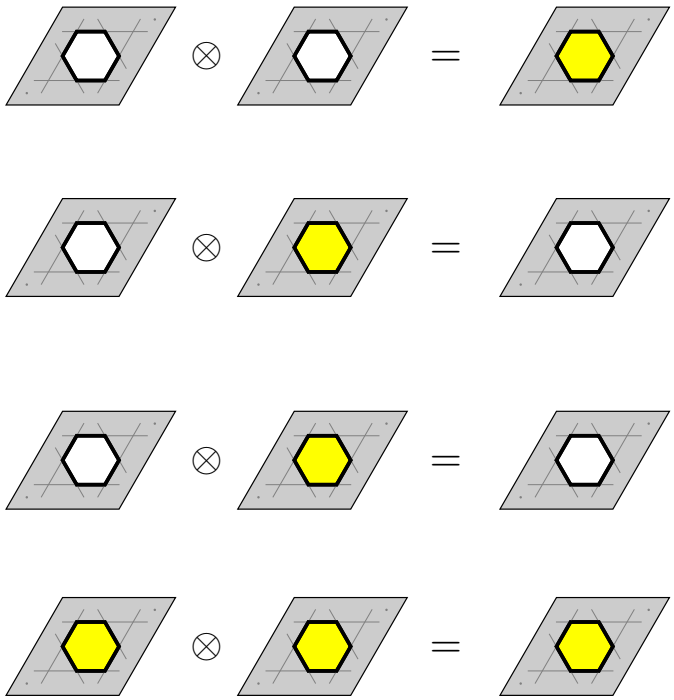
Kagome antiferromagnet

Processes	Weight	Processes	Weight	Processes	Weight
	$-3\alpha^4$		$-\alpha^8$		0
	$3\alpha^8$		$-\alpha^8$		$-\frac{3\alpha^{10}}{2}$
	$-2\alpha^6$		$-\alpha^8$		$-\frac{3\alpha^{10}}{2}$
	$-2\alpha^6$		0		$-\frac{3\alpha^{10}}{2}$
	$-2\alpha^6$		0		$-\frac{3\alpha^{10}}{2}$
	0		0		0
	0				

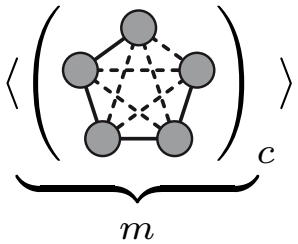
Kagome antiferromagnet



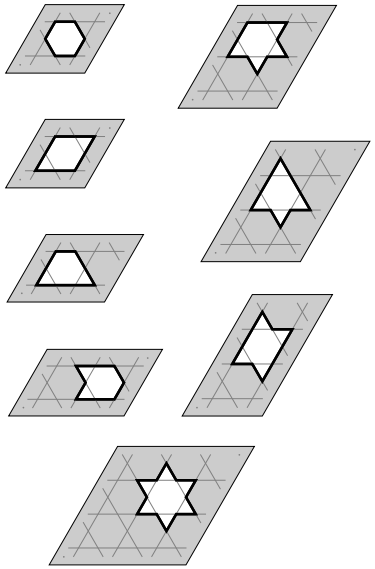
Kagome antiferromagnet



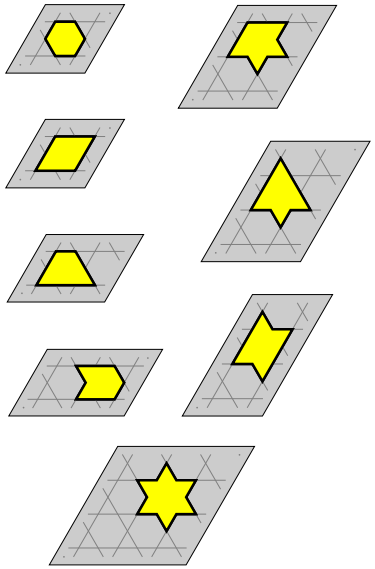
Kagome antiferromagnet



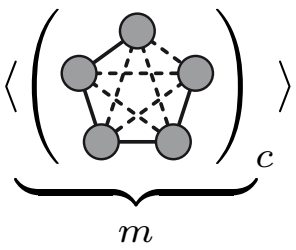
for m odd



for m even



Kagome antiferromagnet

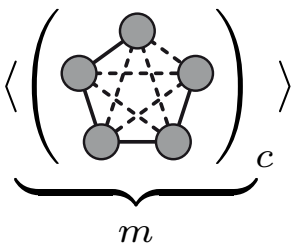


for m odd $\hat{Z} = \ldots + \beta\alpha^q e^{\mu h}$ $+ \ldots$

for m even

$$\ln \hat{Z} = \ldots + \sum_{m \geq 1} \beta^m \frac{(-1)^m}{m} \alpha^{qm} e^{\mu h m}$$ $^m + \ldots$

Kagome antiferromagnet



for m odd

$$\hat{Z} = \ldots + \beta\alpha^q e^{\mu h}$$ $$+ \ldots$$

for m even

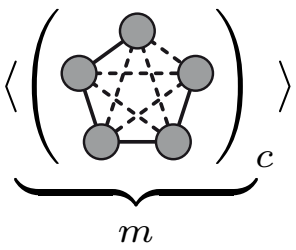
$$\ln \hat{Z} = \ldots + \sum_{m \geq 1} \beta^m \frac{(-1)^m}{m} \alpha^{qm} e^{\mu h m}$$ ^{m} $$+ \ldots$$

$$\ln \hat{Z}$$

$$\frac{1}{2} \ln \frac{1 + \beta\alpha^q e^{\mu h}}{1 - \beta\alpha^q e^{\mu h}}$$

$$\frac{1}{2} \ln \left(1 - \beta^2 \alpha^{2q} e^{2\mu h}\right)$$

Kagome antiferromagnet



for m odd

$$\hat{Z} = \ldots + \beta\alpha^q e^{\mu h}$$ $$+ \ldots$$

for m even

$$\ln \hat{Z} = \ldots + \sum_{m\geq 1} \beta^m \frac{(-1)^m}{m} \alpha^{qm} e^{\mu h m}$$ ^{m} $$+ \ldots$$

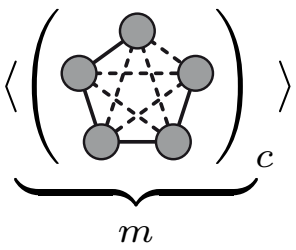
$$\ln \hat{Z}$$

$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0\\ \beta=1}}{=} \sum_{p=0}^{\infty} \frac{1}{2^{2p}} \frac{\left(\left[\ln \hat{Z}, \right)^{2p}}{(2p+1)!} \partial_\mu \ln \hat{Z}$$

$$\frac{1}{2} \ln \frac{1 + \beta\alpha^q e^{\mu h}}{1 - \beta\alpha^q e^{\mu h}}$$

$$\frac{1}{2} \ln \left(1 - \beta^2 \alpha^{2q} e^{2\mu h}\right)$$

Kagome antiferromagnet



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$$\hat{Z} = \ldots + \beta\alpha^q e^{\mu h}$$ $$+ \ldots$$

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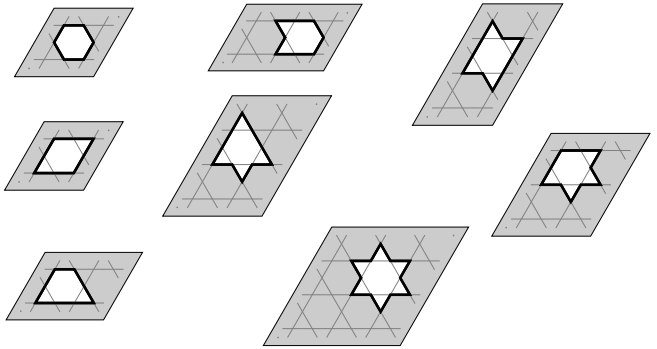
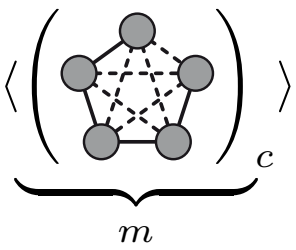
$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0 \\ \beta=1}}{=} \sum_{p=0}^{\infty} \frac{1}{2^{2p}} \frac{\left(\left[\ln \hat{Z}, \right.\right)^{2p}}{(2p+1)!} \partial_\mu \ln \hat{Z}$$



Only order p=0 contributions !

Kagome antiferromagnet



for m odd

$$\hat{Z} = \ldots + \beta\alpha^q e^{\mu h}$$ $$+ \ldots$$

for m even

$$\ln \hat{Z} = \ldots + \sum_{m\geq 1} \beta^m \frac{(-1)^m}{m} \alpha^{qm} e^{\mu h m}$$ ^{m} $$+ \ldots$$

$\ln \hat{Z}$

$$\frac{1}{2} \ln \frac{1+\beta\alpha^q e^{\mu h}}{1-\beta\alpha^q e^{\mu h}}$$

$$\frac{1}{2} \ln \left(1 - \beta^2 \alpha^{2q} e^{2\mu h}\right)$$

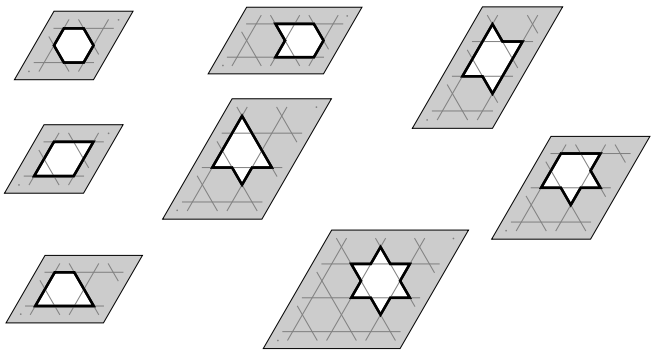
$\partial_\mu \ln \hat{Z} \big|_{\substack{\mu=0 \\ \beta=1}}$

$$h \frac{\alpha^q}{1-\alpha^{2q}}$$

$$-h \frac{\alpha^{2q}}{1-\alpha^{2q}}$$

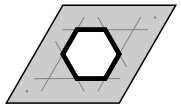
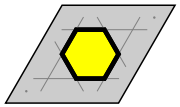
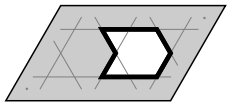
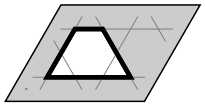
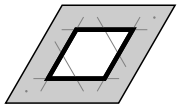
$$\hat{\mathcal{H}}_{\text{eff}}$$

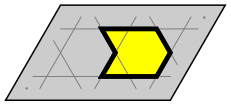
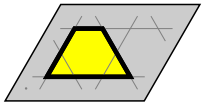
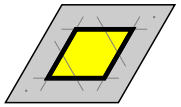
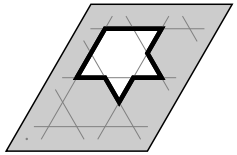
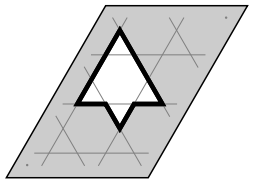
$$\hat{\mathcal{H}}_{\text{eff}} \underset{\substack{\mu=0 \\ \beta=1}}{=} \sum_{p=0}^{\infty} \frac{1}{2^{2p}} \frac{\left(\left[\ln \hat{Z}, \right)^{2p}}{(2p+1)!} \partial_\mu \ln \hat{Z}$$

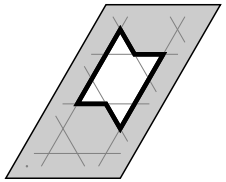
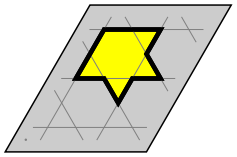
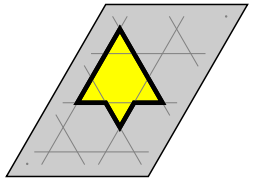
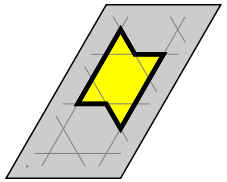


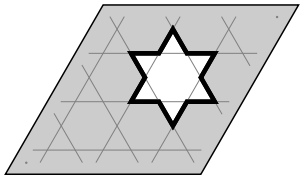
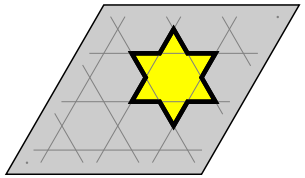
Only order p=0 contributions !

Kagome antiferromagnet

$\hat{\mathcal{H}}_{\text{eff}} = -\frac{4}{5}$  $+\frac{1}{5}$  $+\frac{16}{63}$  $+\frac{16}{63}$  $+\frac{16}{63}$ 

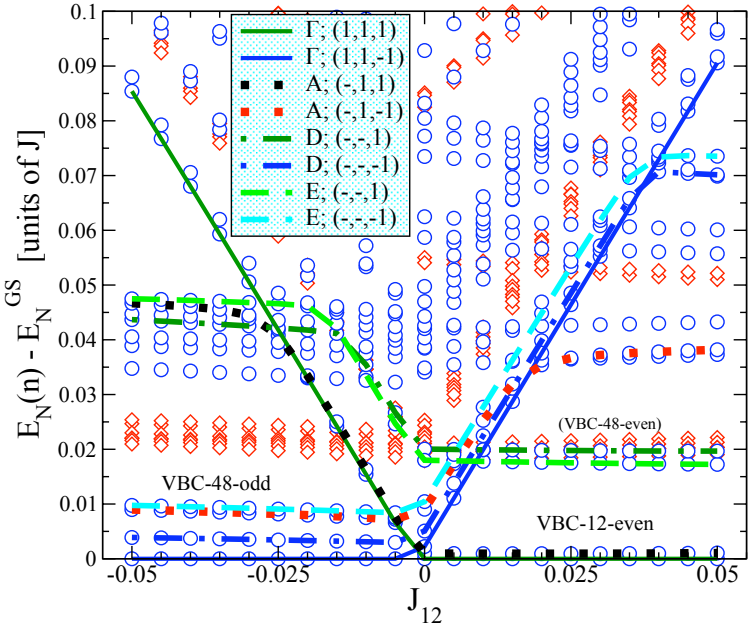
$+\frac{2}{63}$  $+\frac{2}{63}$  $+\frac{2}{63}$  $-\frac{16}{255}$  $-\frac{16}{255}$ 

$-\frac{16}{255}$  $+\frac{1}{255}$  $+\frac{1}{255}$  $+\frac{1}{255}$ 

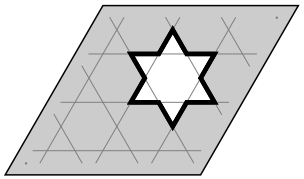
$+0$  $+0$ 

D. Schwandt, M. Mambrini & D. Poilblanc, ArXiv early 2010

Kagome antiferromagnet



D. Poilblanc, M. Mambrini & D. Schwandt, ArXiv 0912.0724

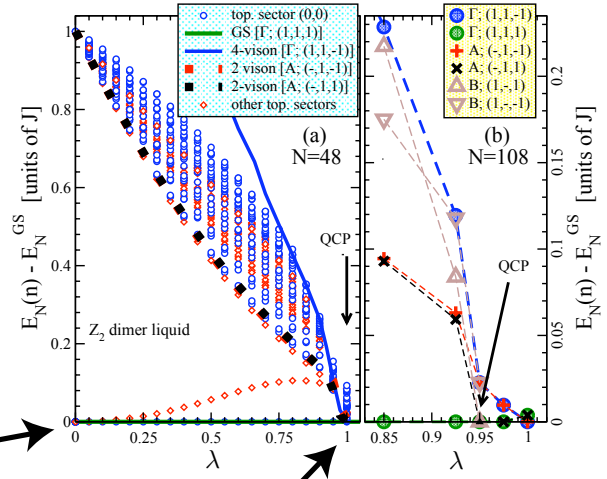


D. Poilblanc's talk (next week)

Exact diagonalizations up to 108 sites

First order phase transition between two VBC phases

Close to a QCP characterized by the melting of the even VBC with a Z_2 spin liquid

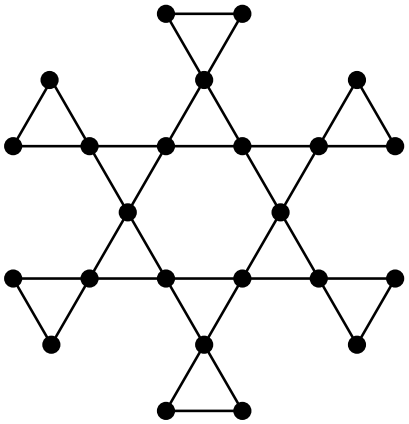


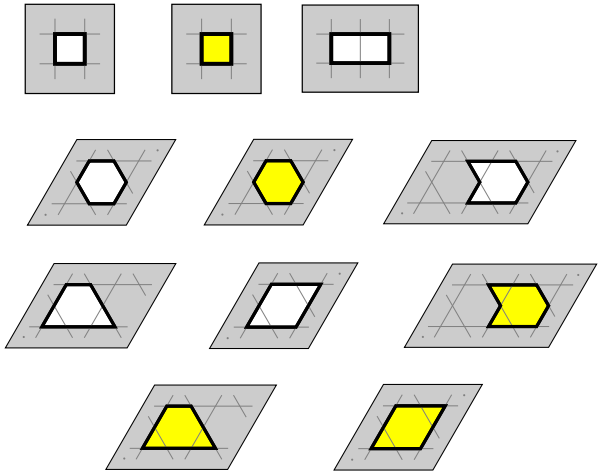
λ=0

G.Misguich, D. Serban & V. Pasquier, Phys. Rev. Lett. 89, 137202 (2002)

Our model

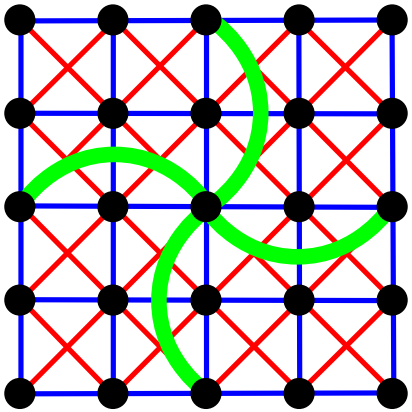
$$\mathcal{H} = \sum_{(i,j)} J_{ij} \vec{S}_i . \vec{S}_j$$





Low energy

Generalized Quantum Dimer Model



Extensions

treat other SU(2) interactions (multiple exchange,...)

include spinons, triplets