

Quantum phase transitions of insulators, superconductors and metals in two dimensions

Talk online: sachdev.physics.harvard.edu

PHYSICS



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Outline

1. Phenomenology of the cuprate superconductors
(and other compounds)
2. QPT of antiferromagnetic insulators
(and bosons at rational filling)
3. QPT of d-wave superconductors:
Fermi points of massless Dirac fermions
4. QPT of Fermi surfaces:
 - A. Finite wavevector ordering (SDW/CDW):
“Hot spots” on Fermi surfaces
 - B. Zero wavevector ordering (Nematic):
“Hot Fermi surfaces”

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Max Metlitski

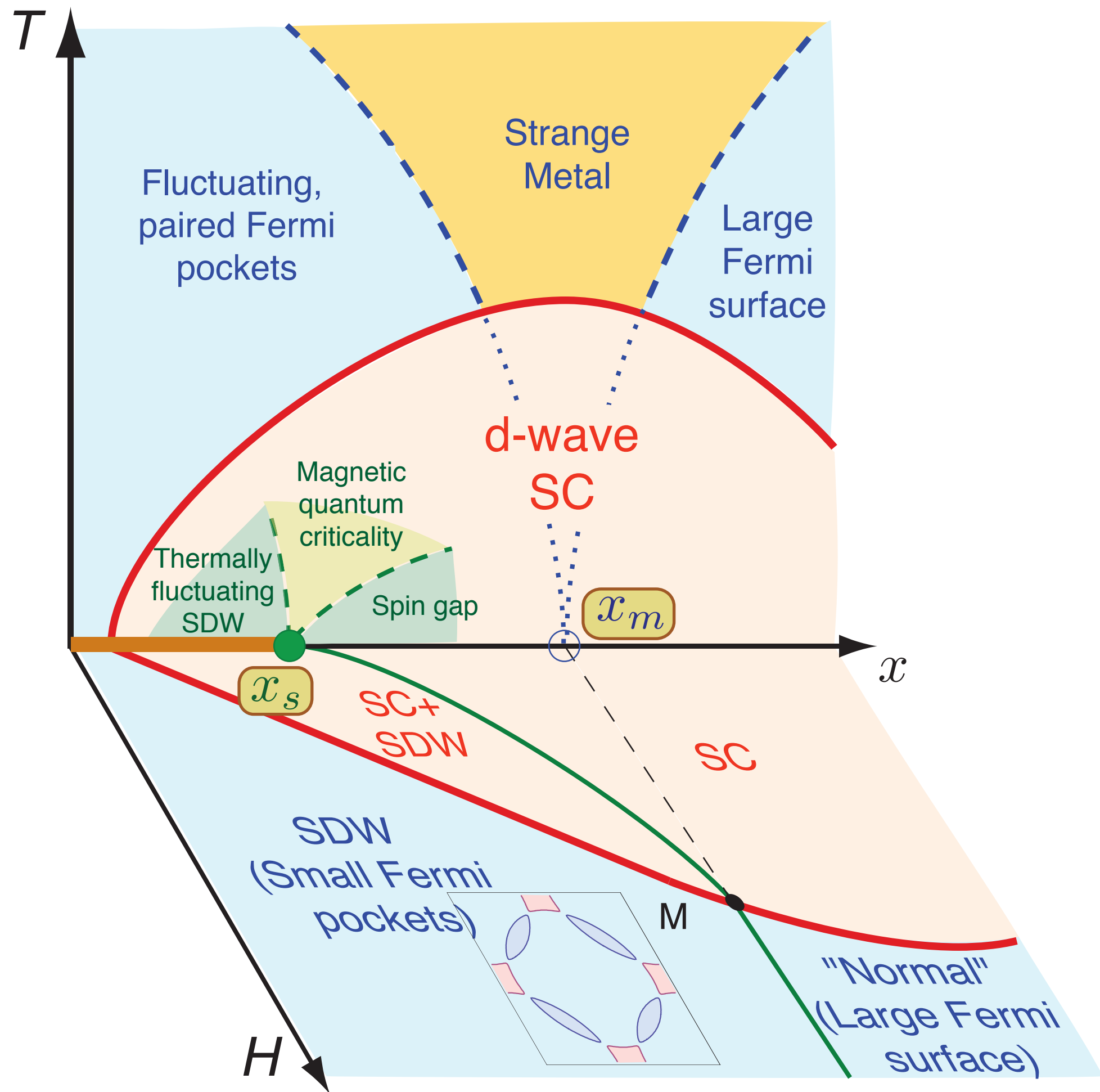
Strategy

1. Write down local field theory for order parameter and fermions
2. Apply renormalization group to field theory

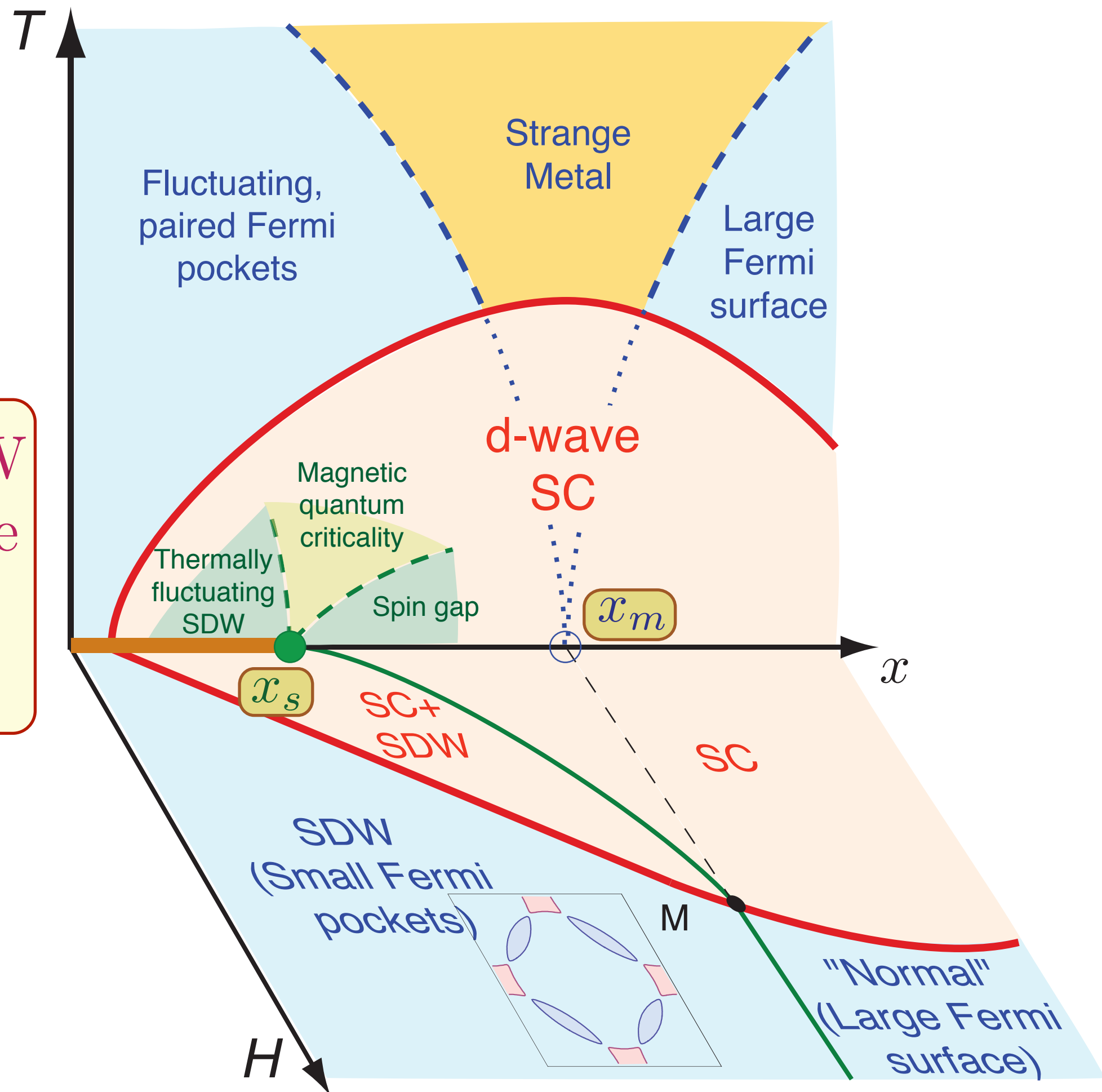
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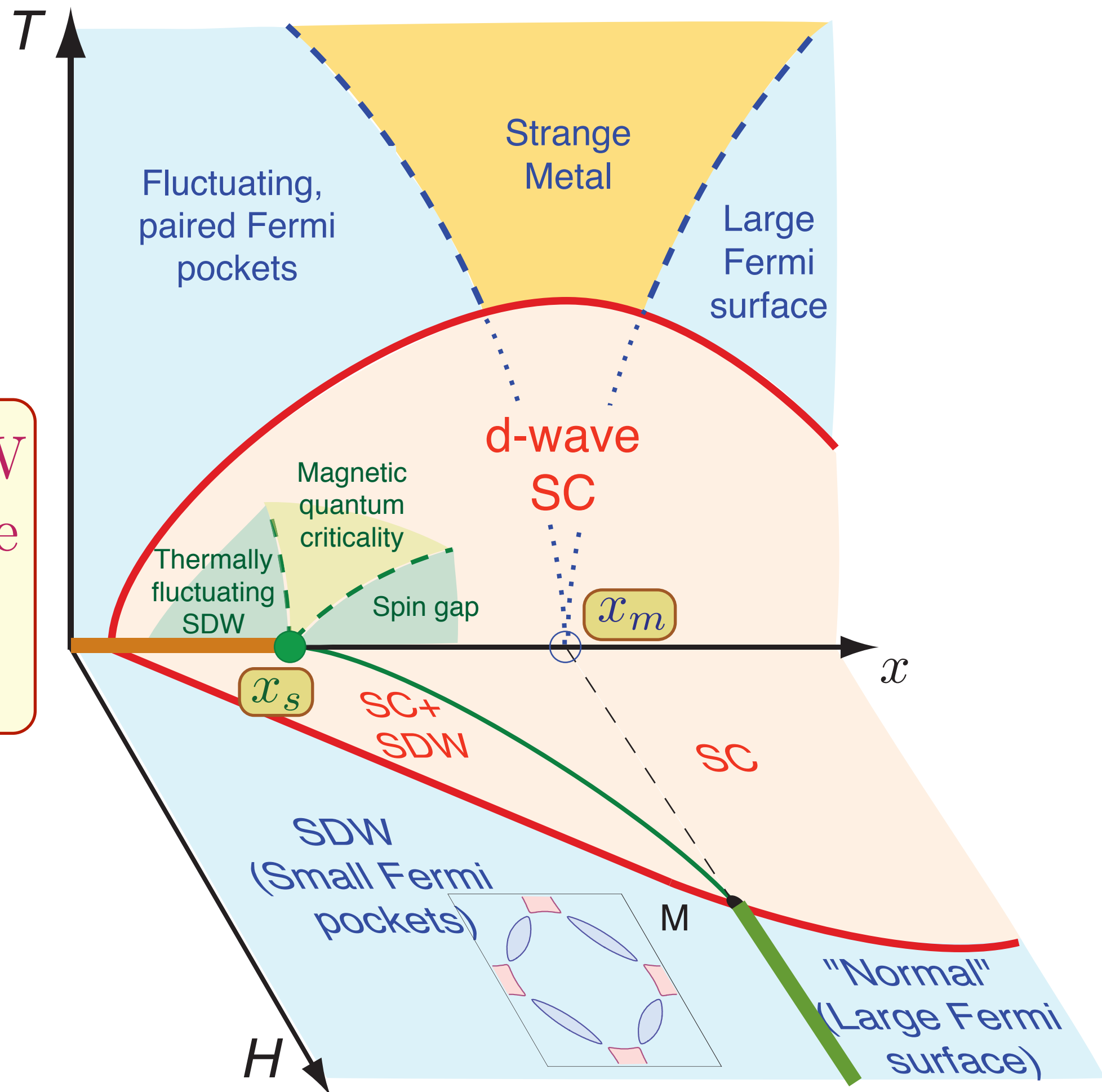
Order parameter at a non-zero wavevector: "Hot spots" on the Fermi surface.



Theory of SDW quantum phase transition in metal

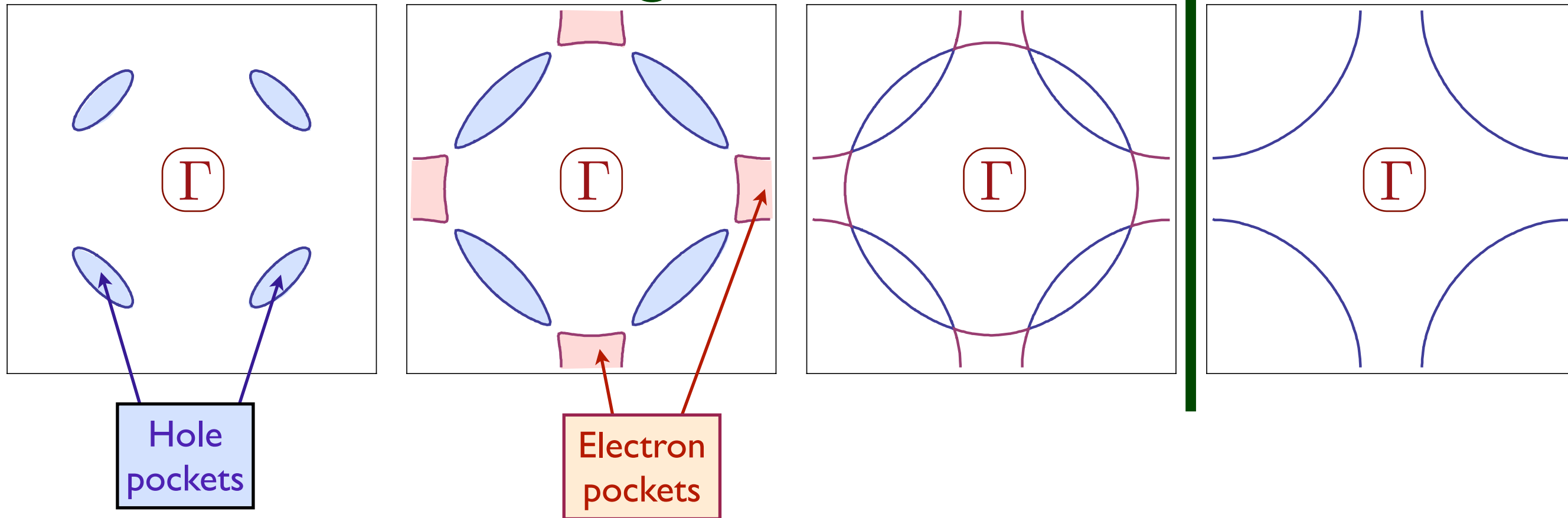


Theory of SDW quantum phase transition in metal



Hole-doped cuprates

← Increasing SDW order →

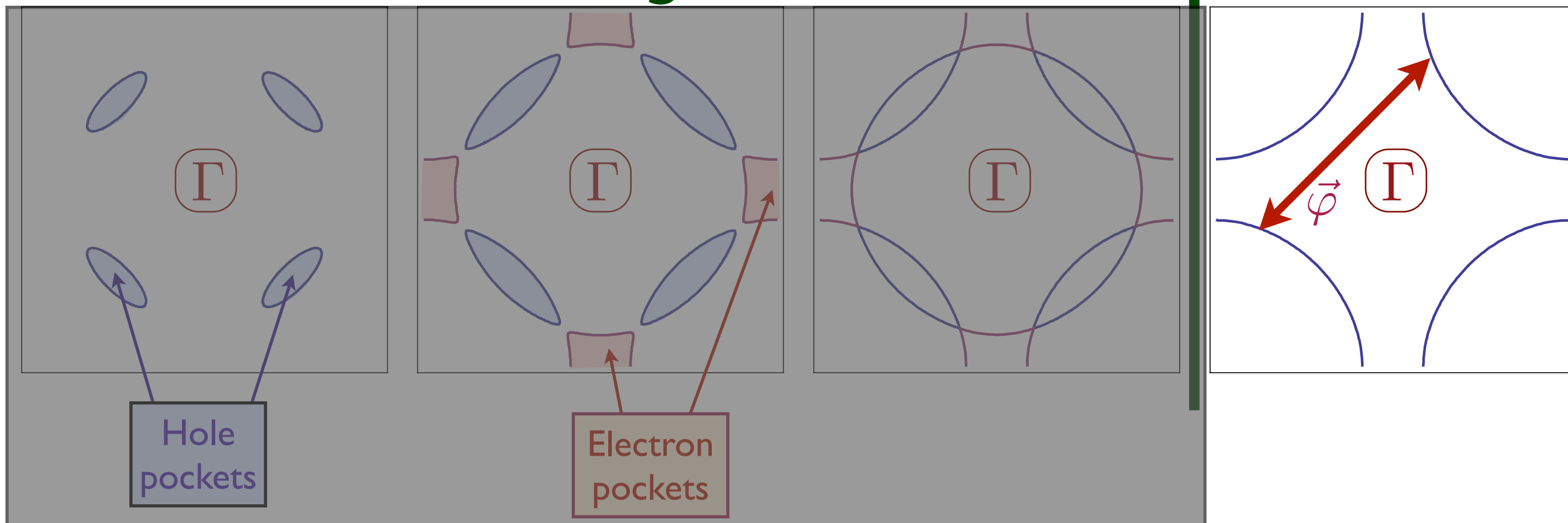


Large Fermi surface breaks up into
electron and hole pockets

S. Sachdev, A. V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A. V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Hole-doped cuprates

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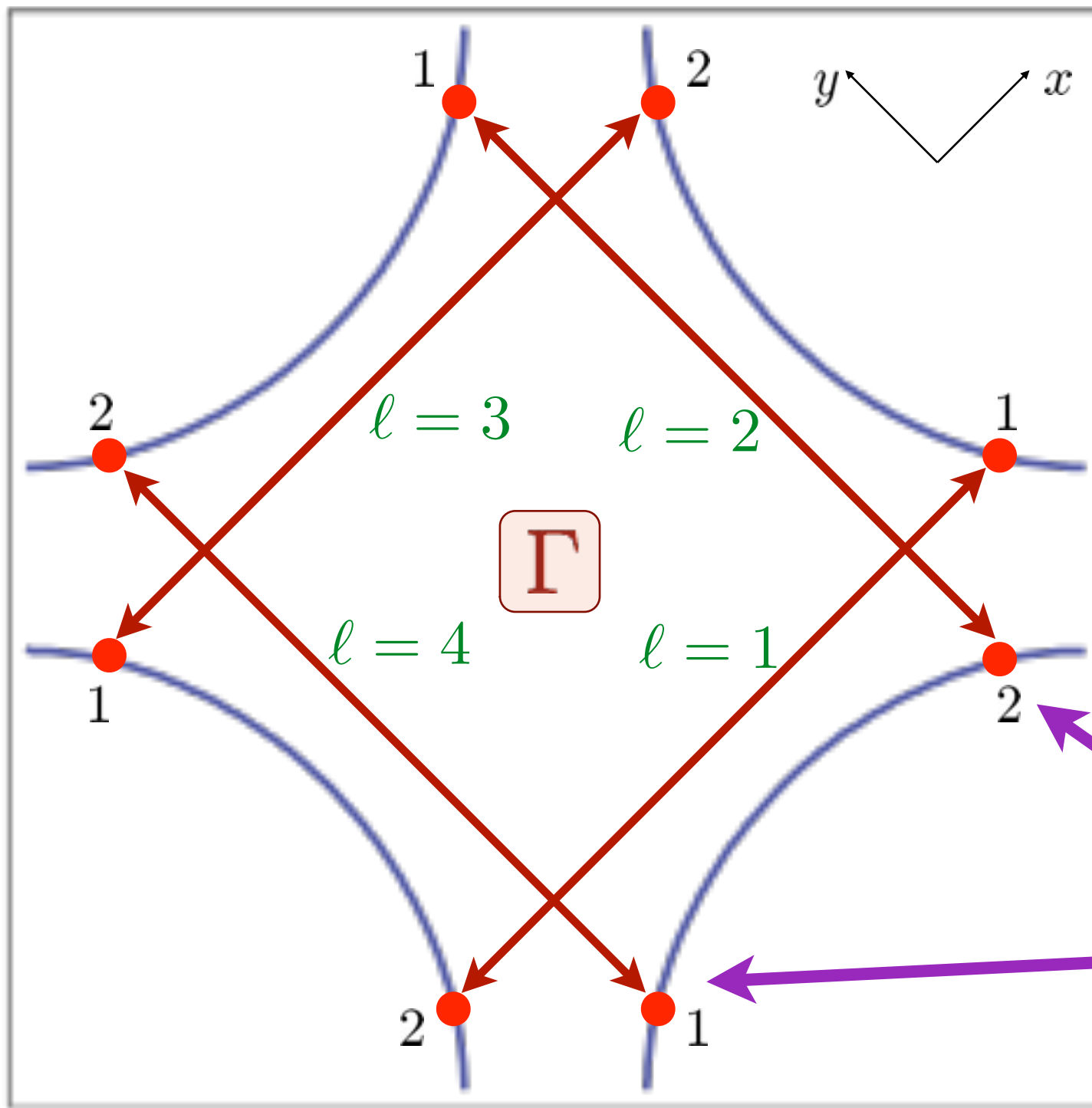


$\vec{\varphi}$ fluctuations act on the
large Fermi surface

S. Sachdev, A. V. Chubukov, and A. Sokol, *Phys. Rev. B* **51**, 14874 (1995).
A. V. Chubukov and D. K. Morr, *Physics Reports* **288**, 355 (1997).

Start from the “spin-fermion” model

$$\begin{aligned}
 \mathcal{Z} &= \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp(-\mathcal{S}) \\
 \mathcal{S} &= \int d\tau \sum_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger \left(\frac{\partial}{\partial \tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha} \\
 &\quad - \lambda \int d\tau \sum_i c_{i\alpha}^\dagger \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K} \cdot \mathbf{r}_i} \\
 &\quad + \int d\tau d^2r \left[\frac{1}{2} (\nabla_r \vec{\varphi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]
 \end{aligned}$$



Low energy fermions

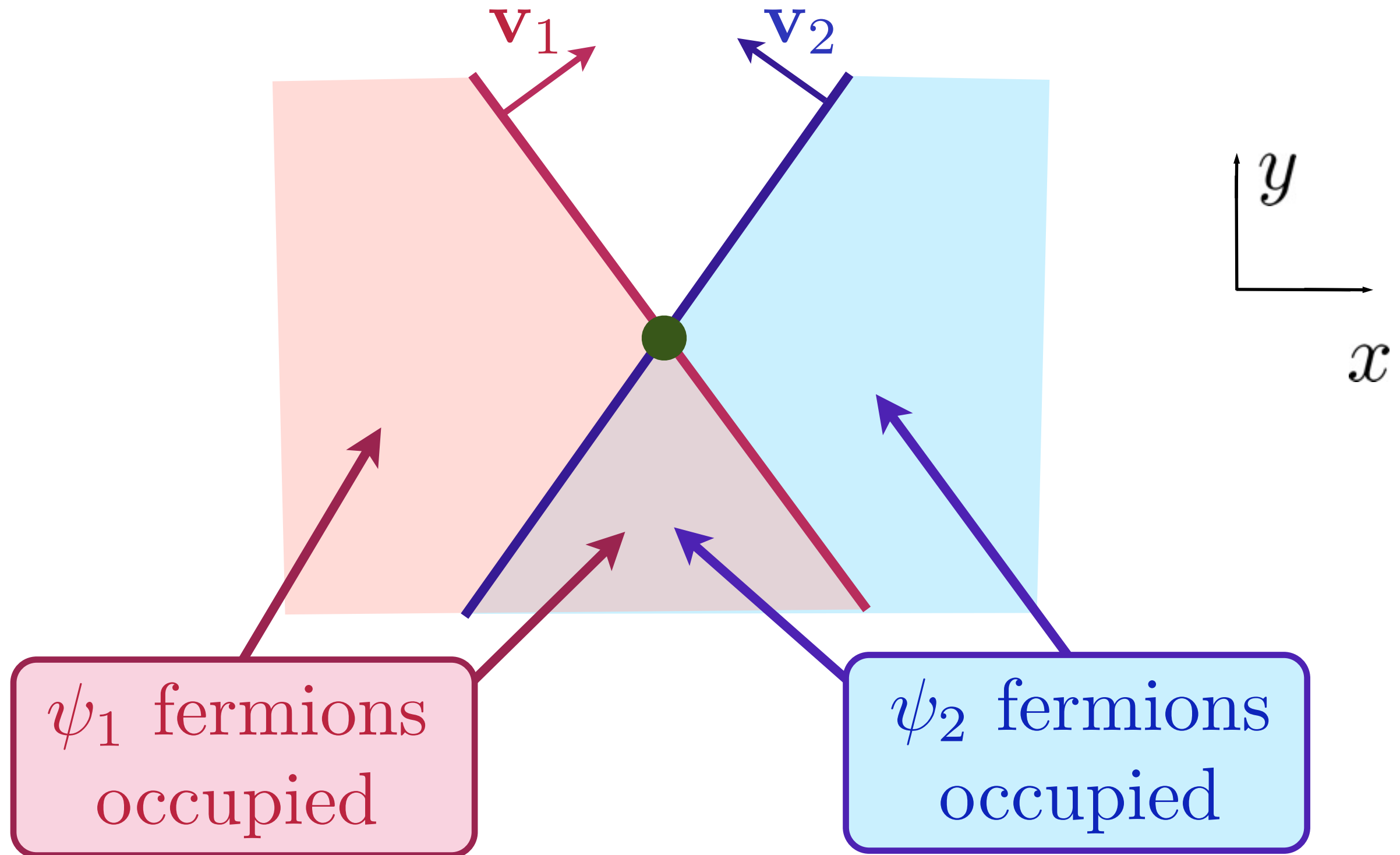
$$\psi_{1\alpha}^\ell, \psi_{2\alpha}^\ell$$

$$\ell = 1, \dots, 4$$

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$

$$\mathbf{v}_1^{\ell=1} = (v_x, v_y), \quad \mathbf{v}_2^{\ell=1} = (-v_x, v_y)$$

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$



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Order parameter: $\mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \vec{\varphi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4$

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“Yukawa” coupling: $\mathcal{L}_c = -\lambda \vec{\varphi} \cdot \left(\psi_{1\alpha}^{\ell\dagger} \vec{\sigma}_{\alpha\beta} \psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger} \vec{\sigma}_{\alpha\beta} \psi_{1\beta}^\ell \right)$

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Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain non-local corrections to $\mathcal{L}_\varphi$

$$\mathcal{L}_\varphi = \frac{1}{2} \vec{\varphi}^2 [\mathbf{q}^2 + \gamma |\omega|] / 2 \quad ; \quad \gamma = \frac{2}{\pi v_x v_y}$$

Exponent $z = 2$ and mean-field criticality (upto logarithms)

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$

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Exponent $z = 2$ and mean-field criticality (upto logarithms)

But, higher order terms contain an infinite number of marginal couplings

Ar.Abanov and A.V. Chubukov, *Phys. Rev. Lett.* **93**, 255702 (2004).

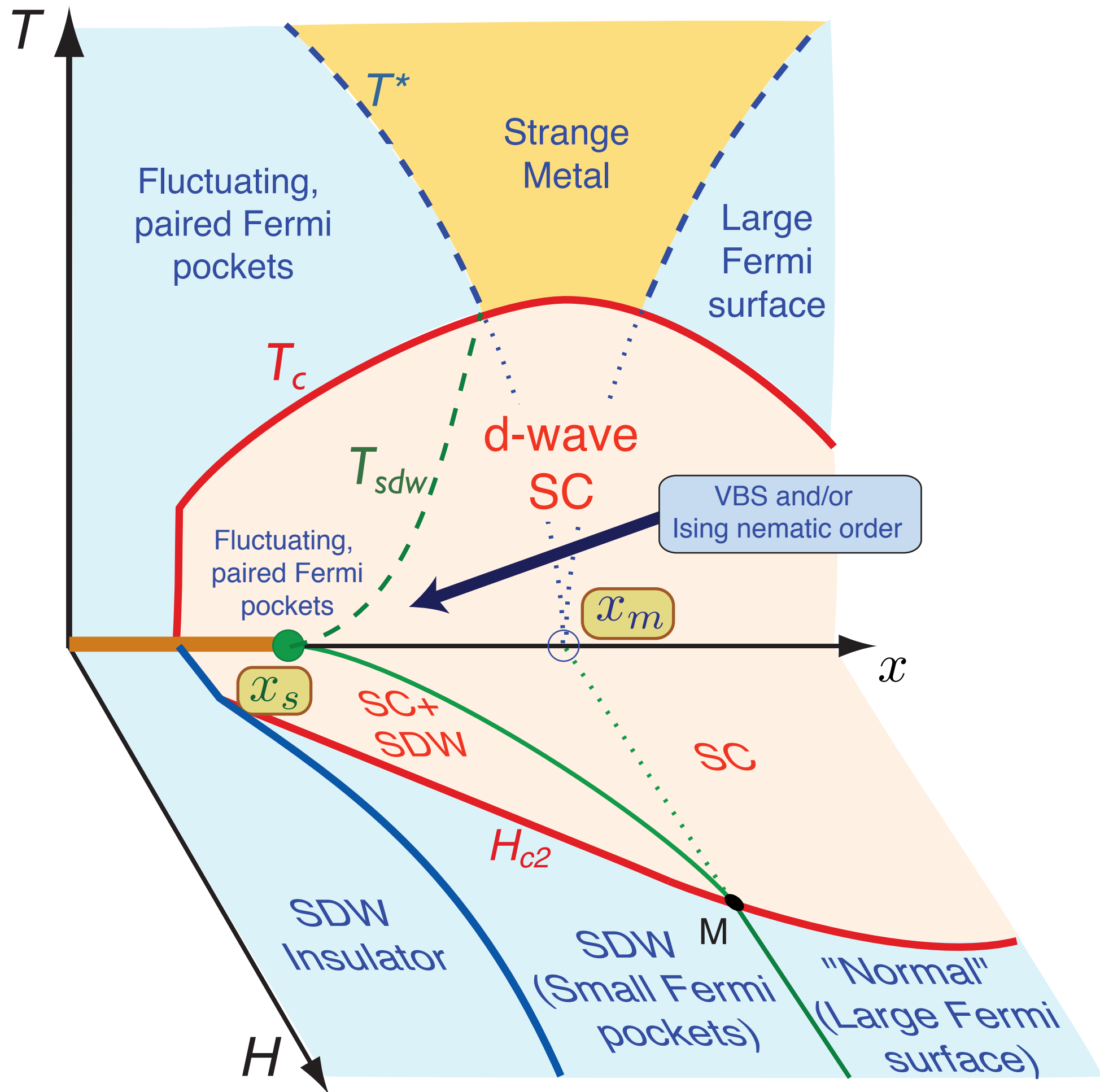
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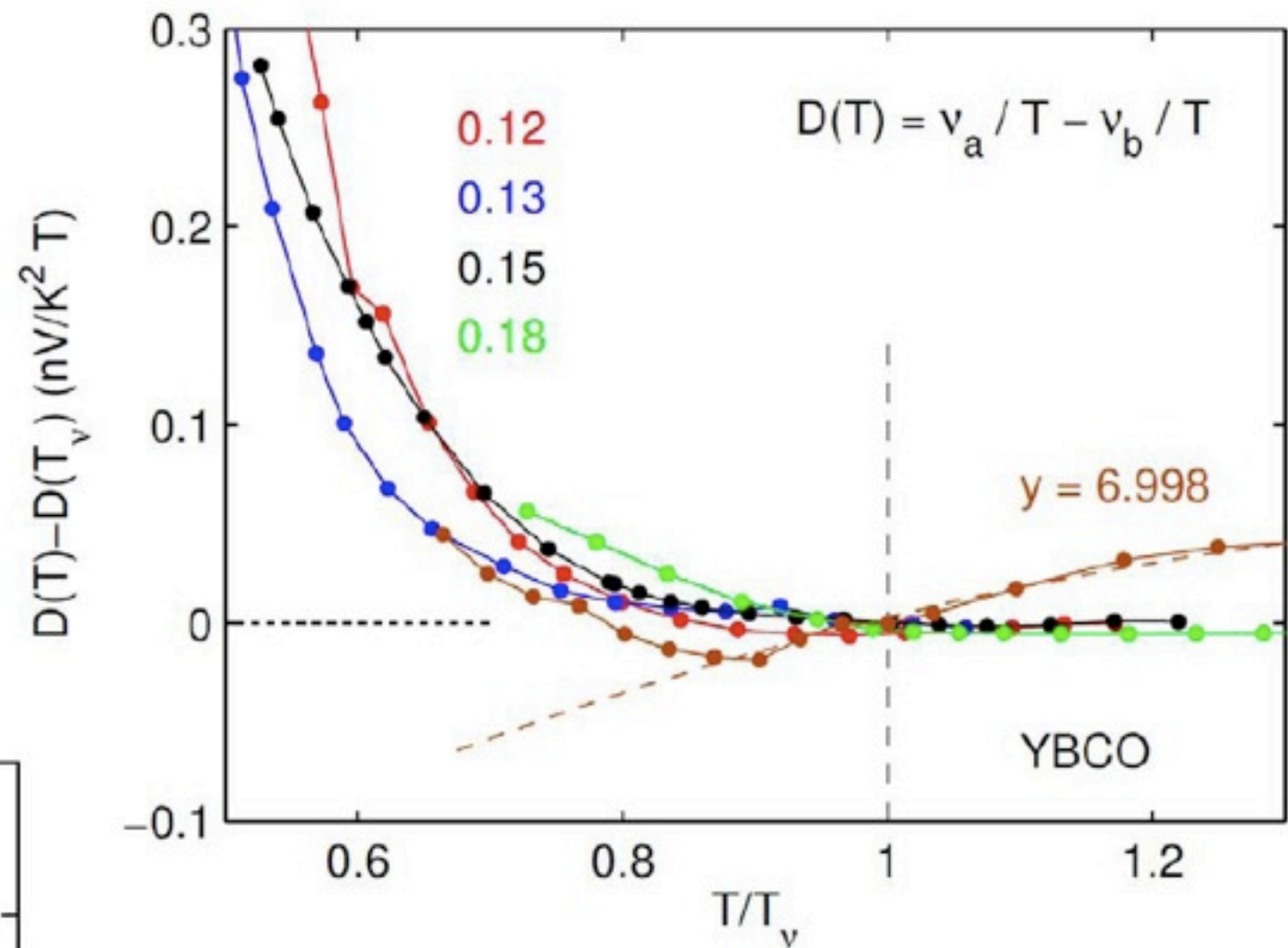
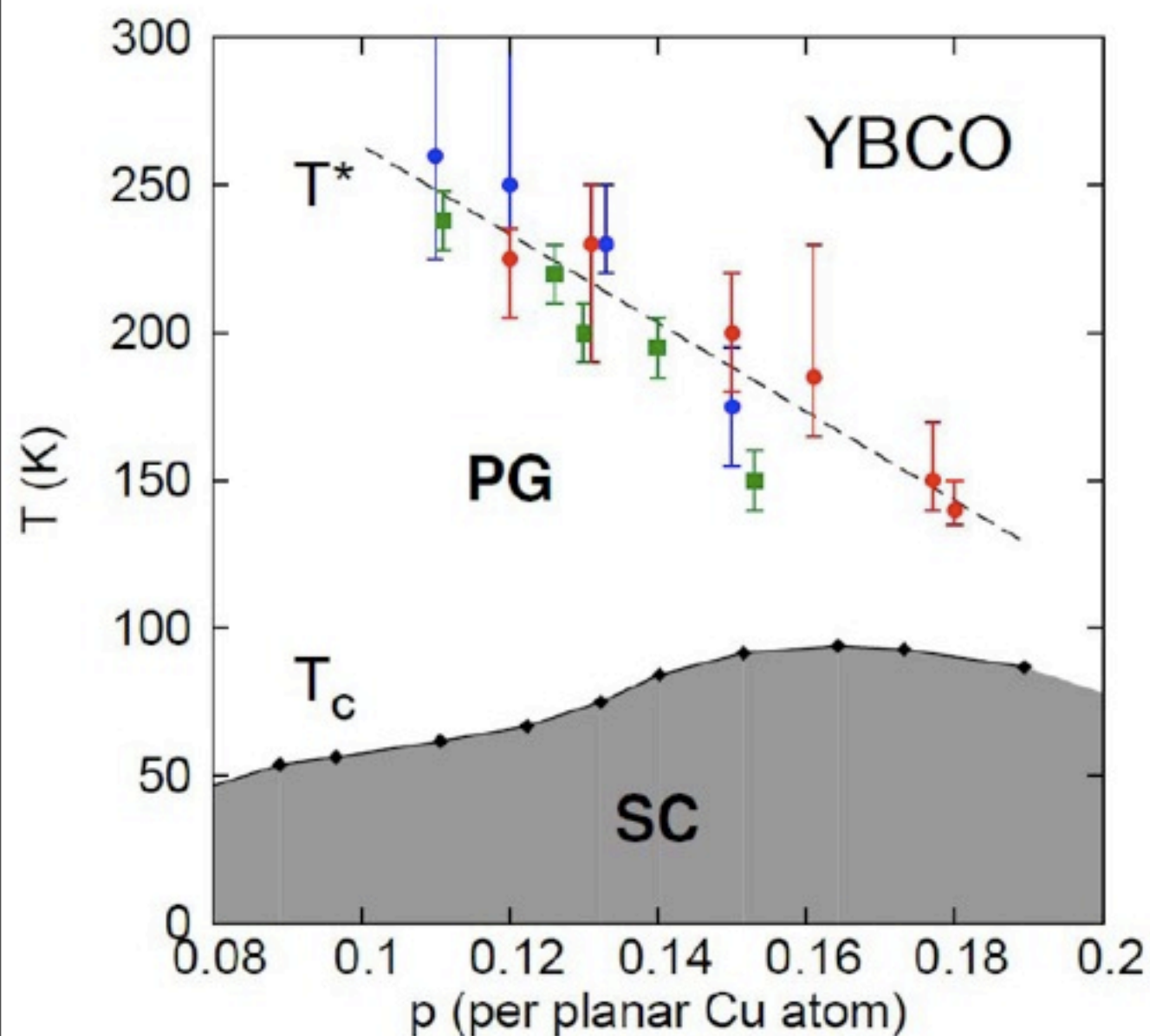
Perform RG on both fermions and $\vec{\varphi}$,
using a *local* field theory.

Order parameter at zero
wavevector:
"Hot Fermi surface".

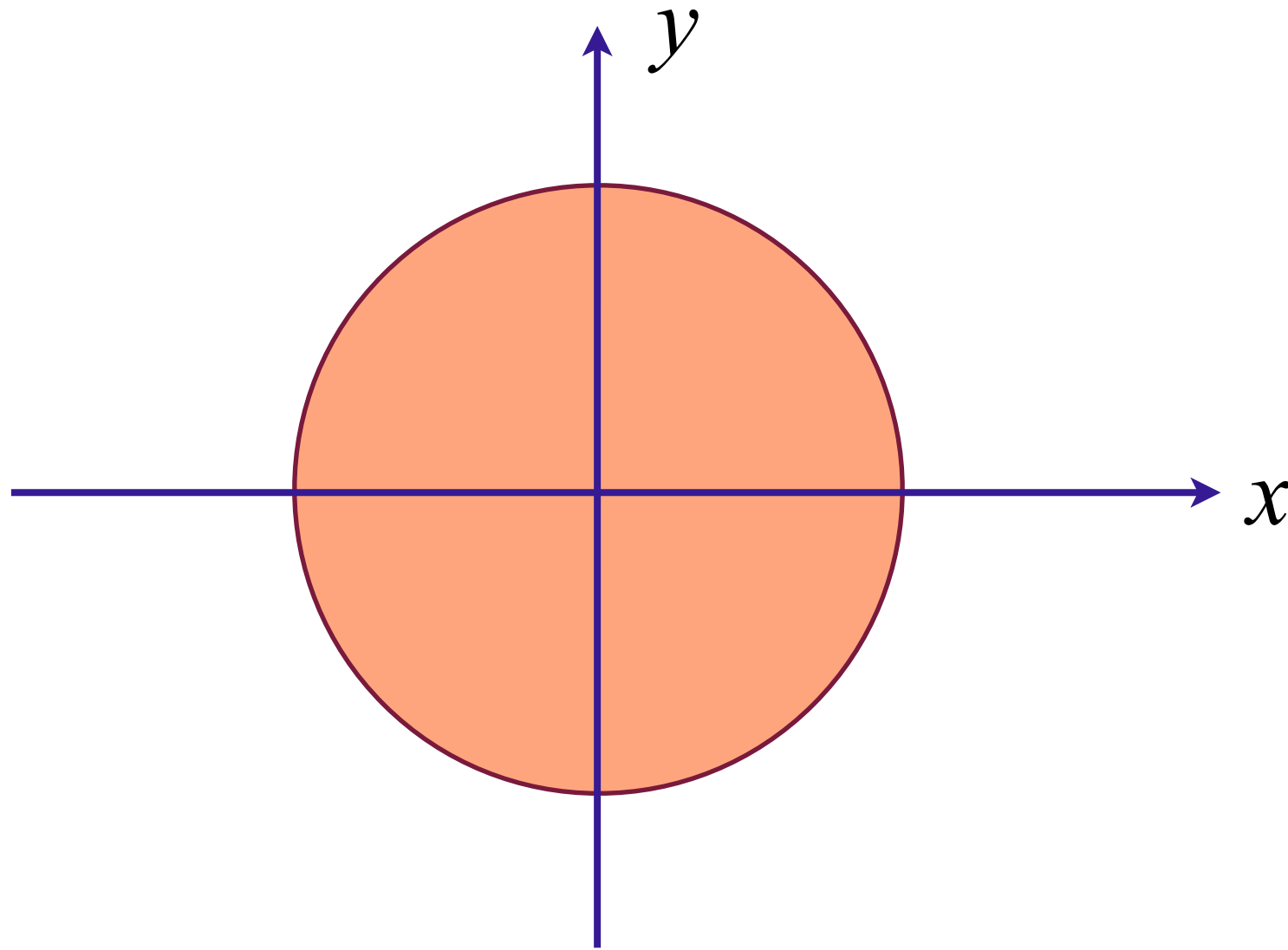


Broken rotational symmetry in the pseudogap phase of a high- T_c superconductor

R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choiniere, Francis Laliberte, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D.A. Bonn, W. N. Hardy, and Louis Taillefer
arXiv: 0909.4430, Nature, in press

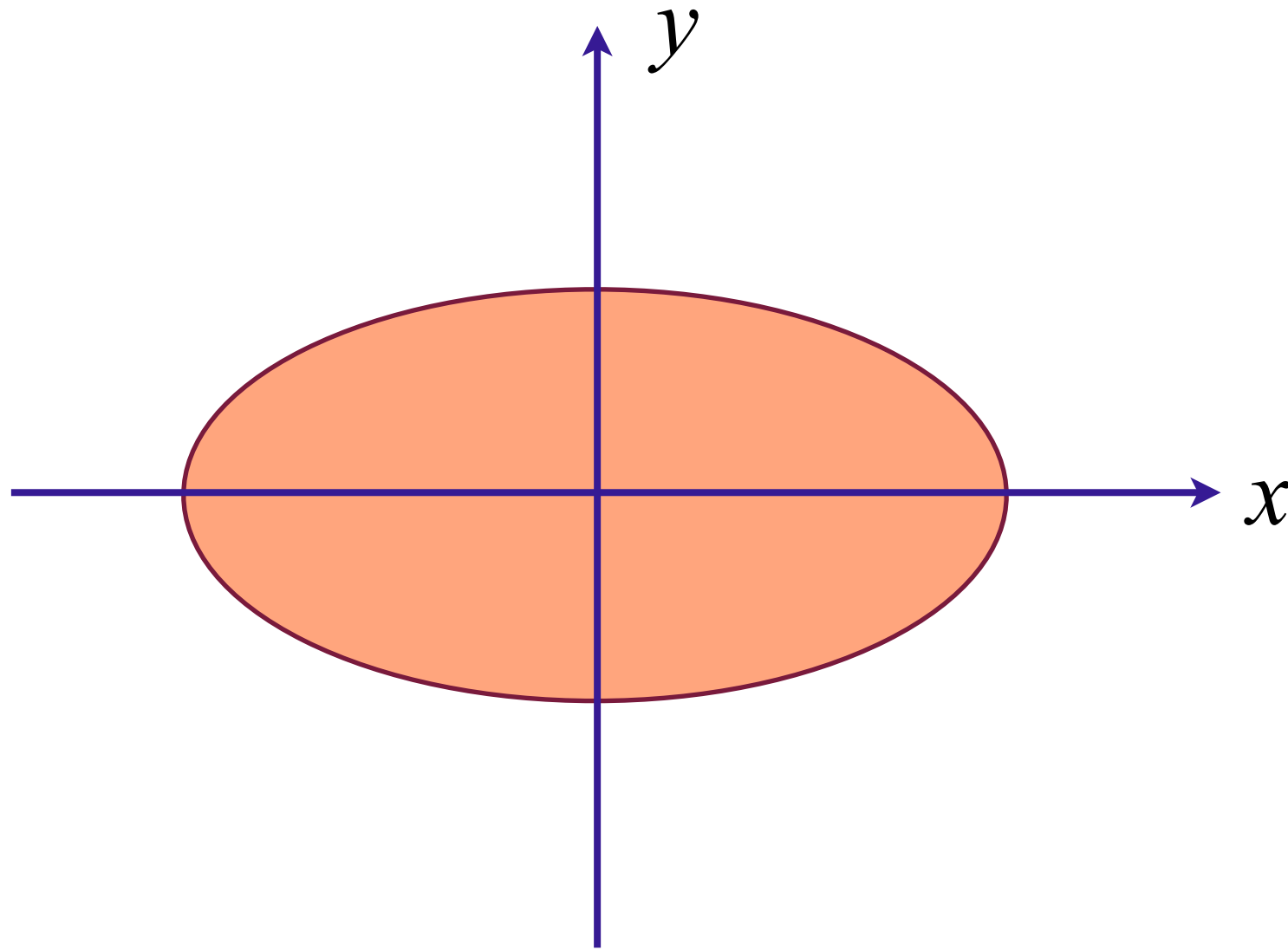


Quantum criticality of Pomeranchuk instability



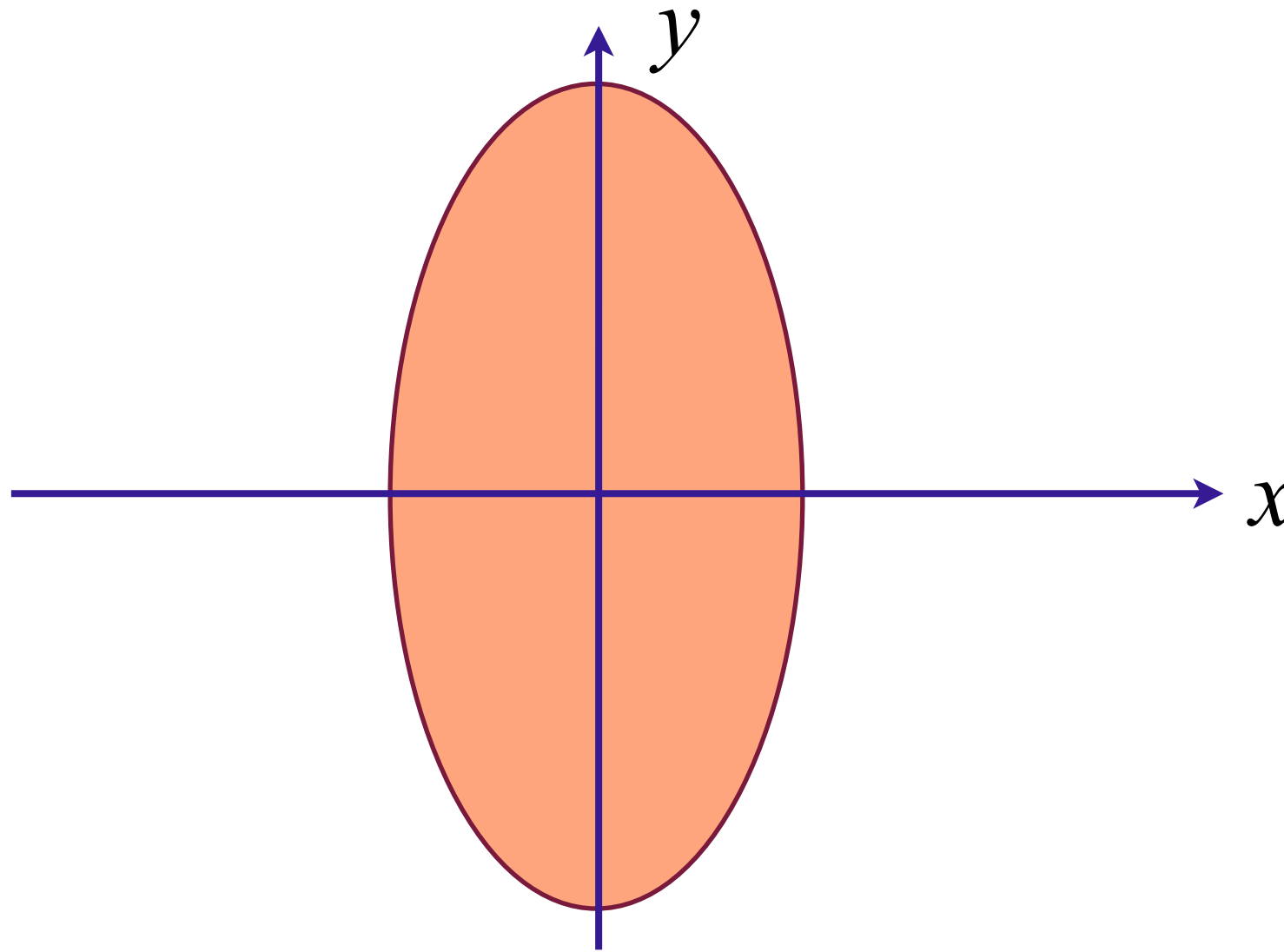
Fermi surface with full square lattice symmetry

Quantum criticality of Pomeranchuk instability



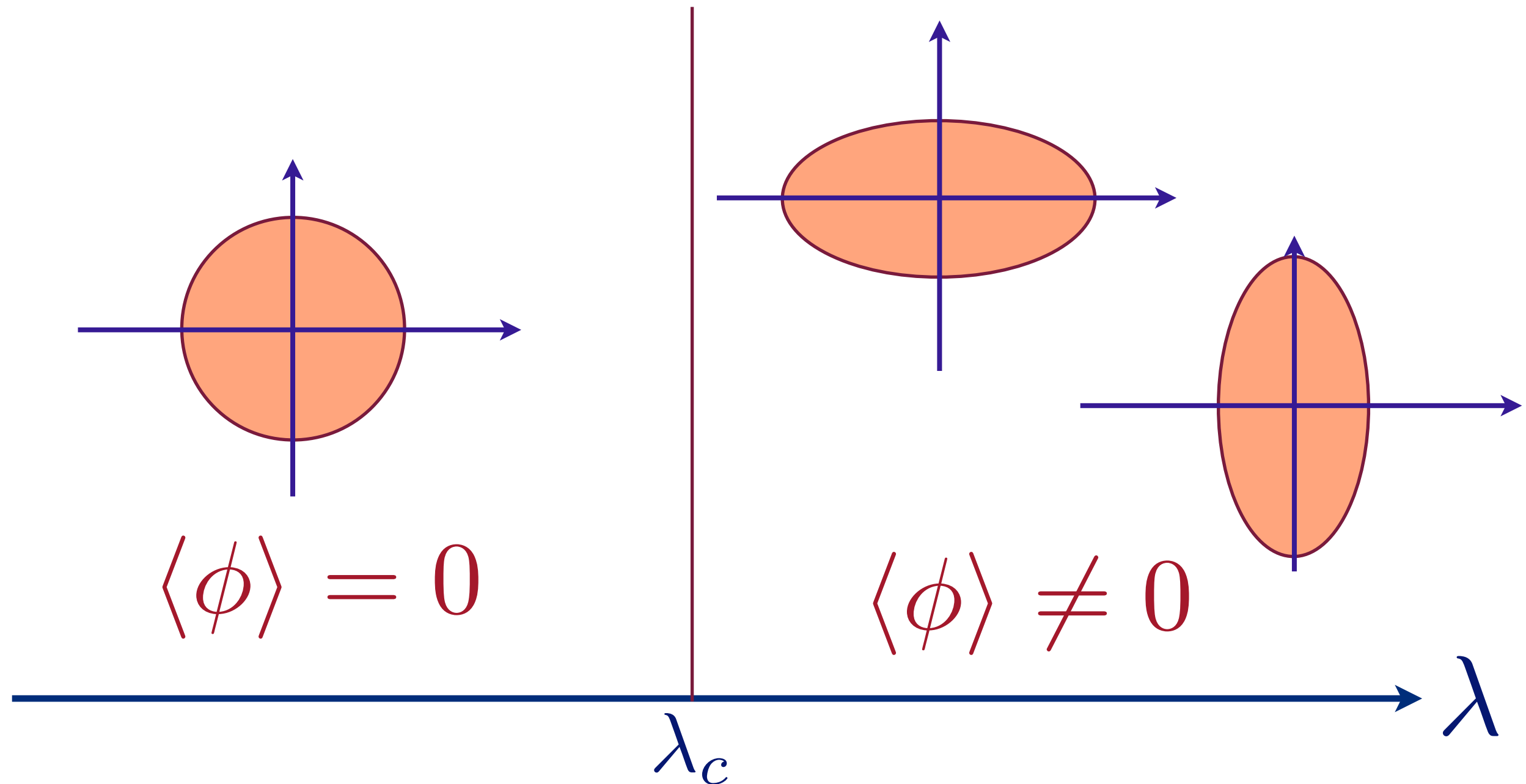
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Pomeranchuk instability



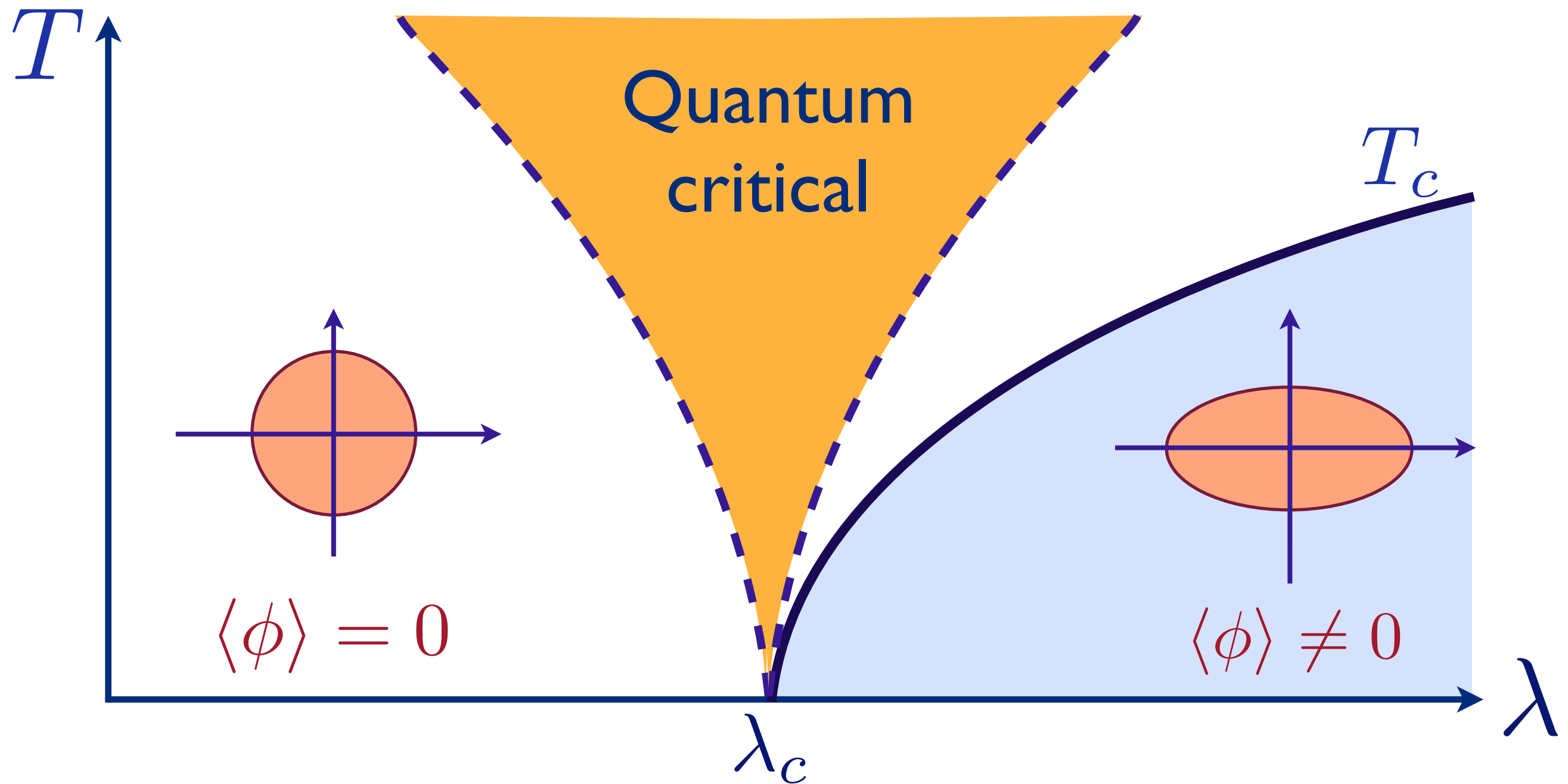
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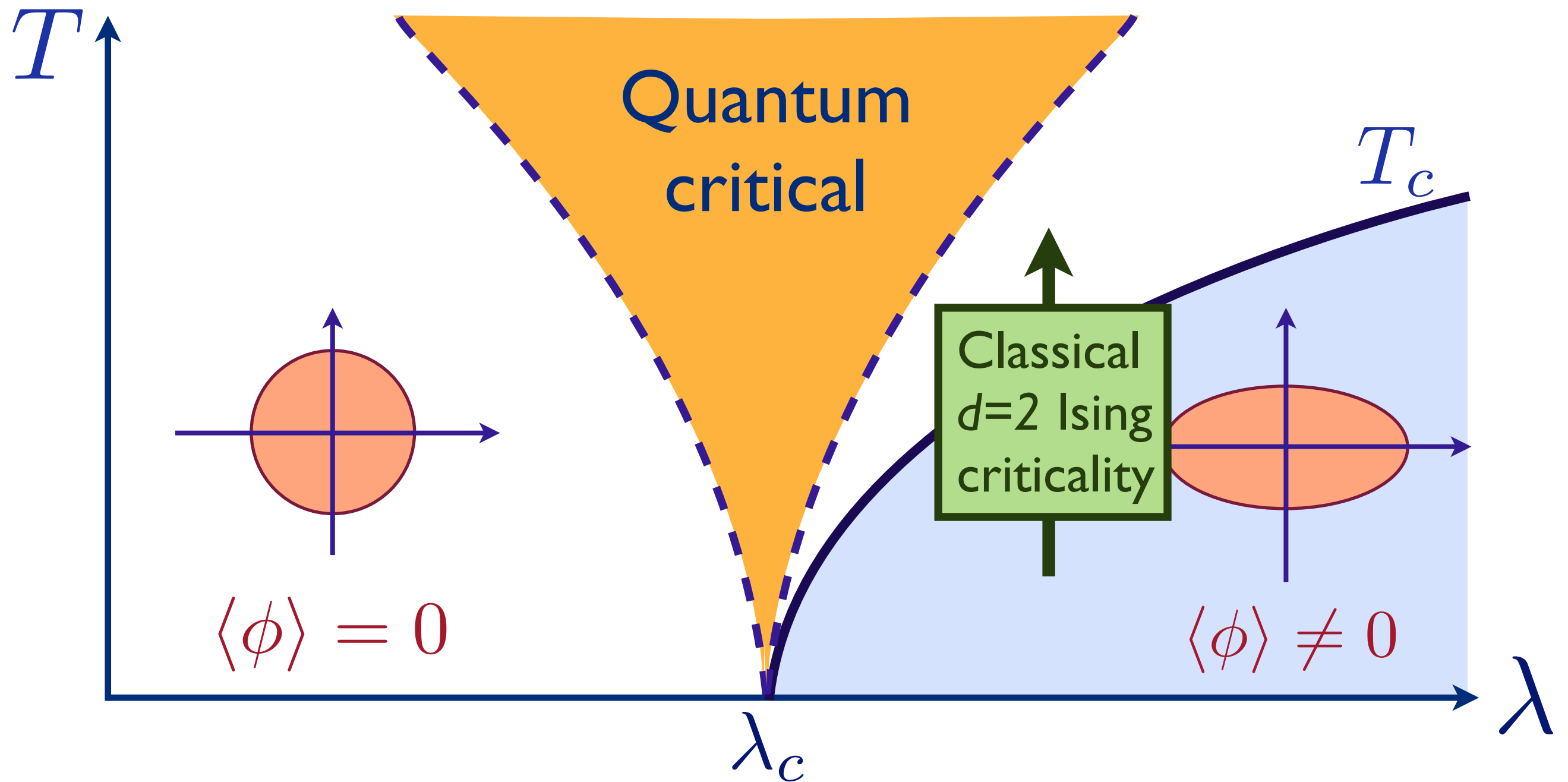
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Pomeranchuk instability



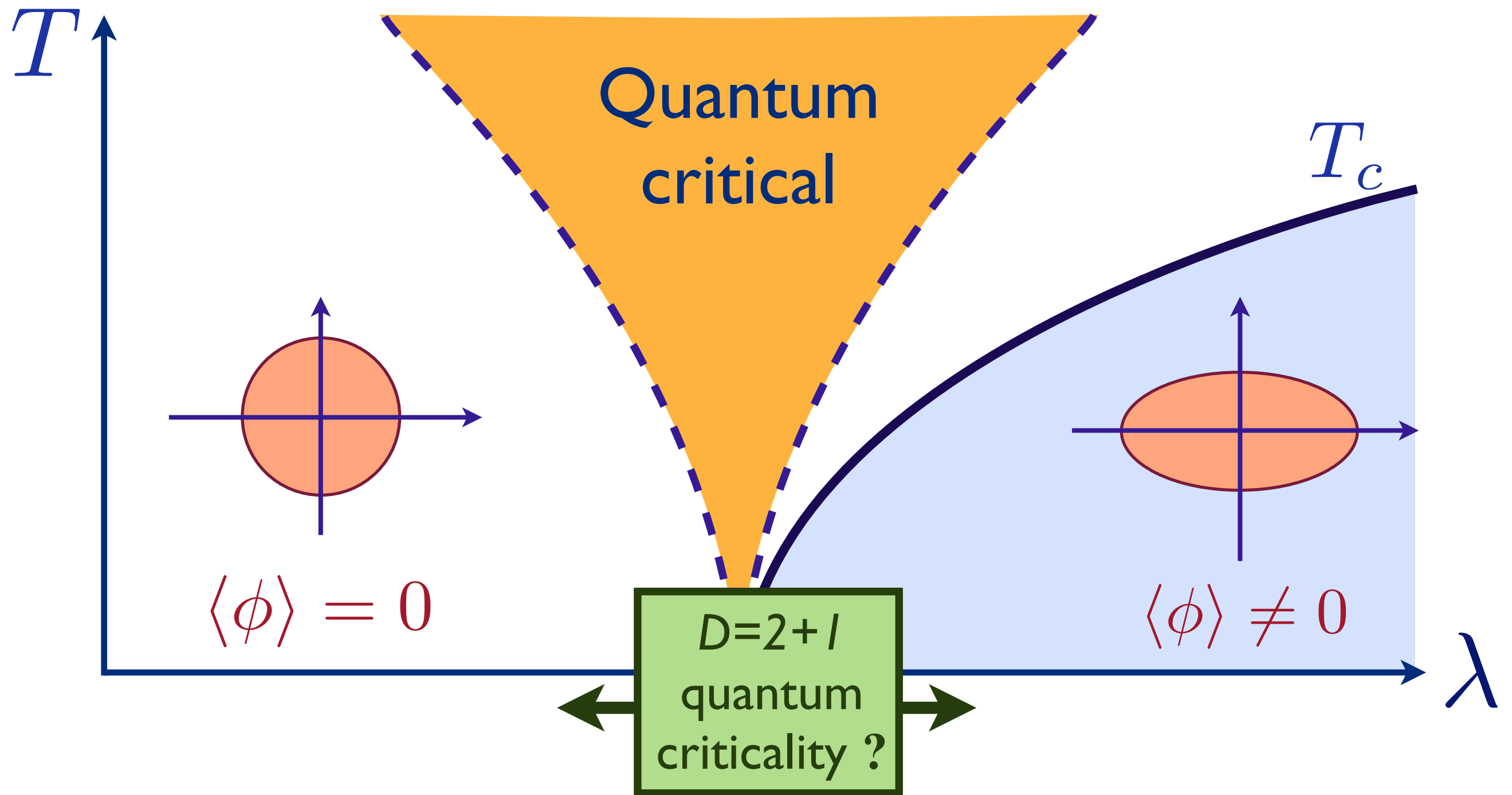
Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability



Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability



Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

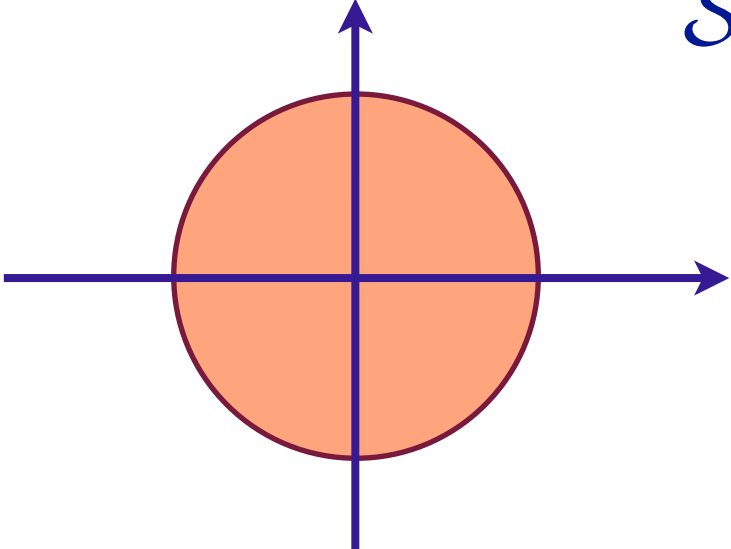
$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

Effective action for electrons:

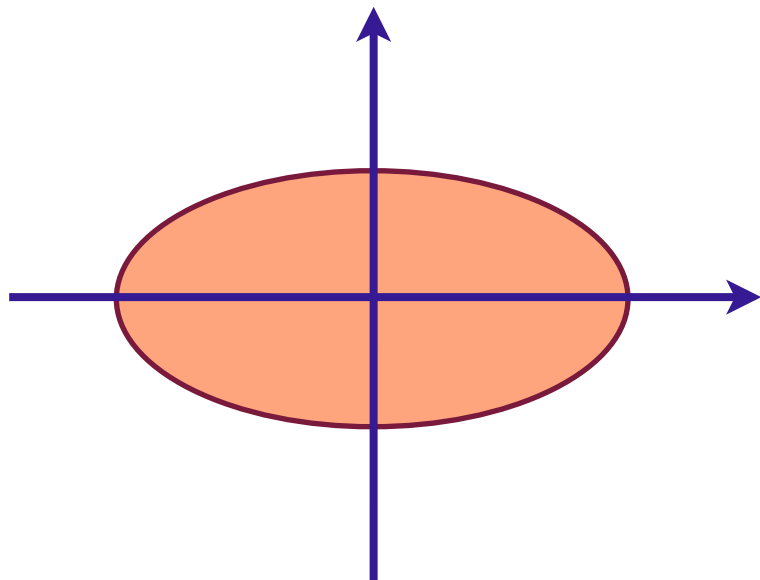

$$\begin{aligned} \mathcal{S}_c &= \int d\tau \sum_{\alpha=1}^{N_f} \left[\sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} \right] \\ &\equiv \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger (\partial_\tau + \varepsilon_{\mathbf{k}}) c_{\mathbf{k}\alpha} \end{aligned}$$

Quantum criticality of Pomeranchuk instability

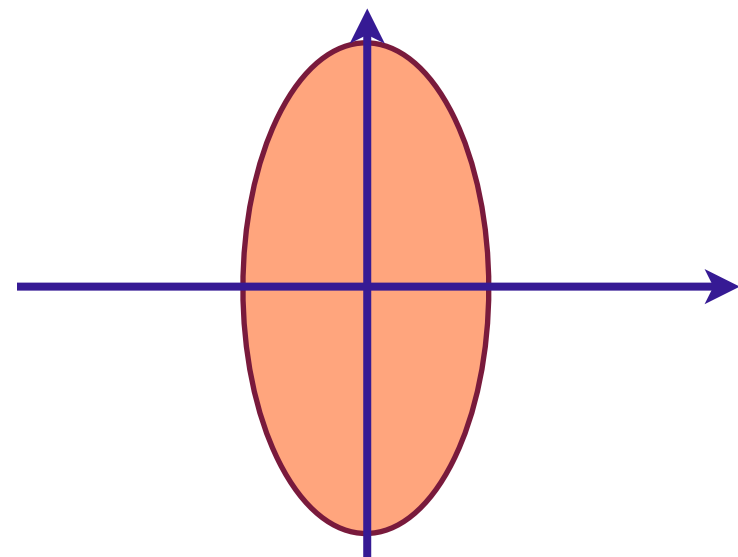
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \phi \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} (\cos k_x - \cos k_y) c_{\mathbf{k}\alpha}^{\dagger} c_{\mathbf{k}\alpha}$$

for spatially independent ϕ



$$\langle \phi \rangle > 0$$



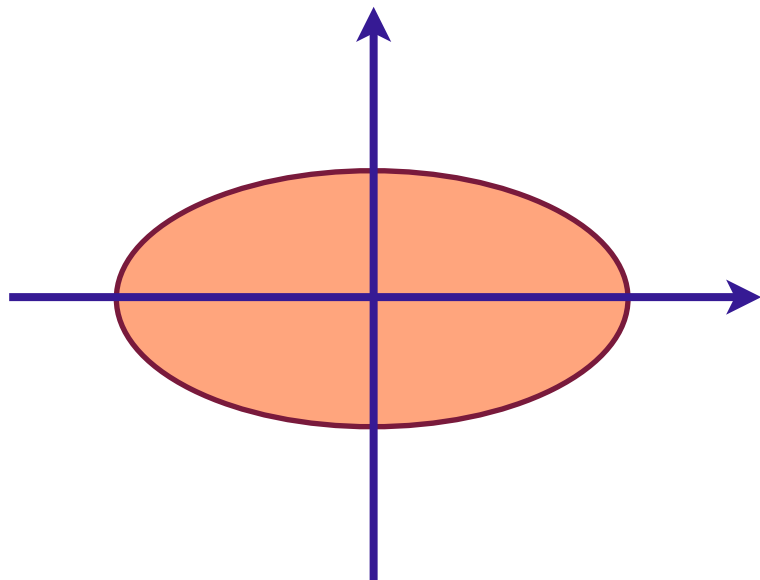
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Quantum criticality of Pomeranchuk instability

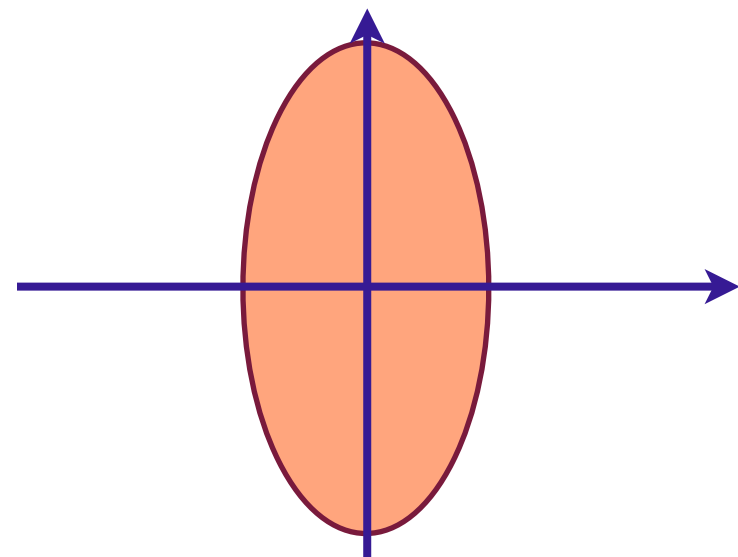
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}, \mathbf{q}} \phi_{\mathbf{q}} (\cos k_x - \cos k_y) c_{\mathbf{k}+\mathbf{q}/2, \alpha}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2, \alpha}$$

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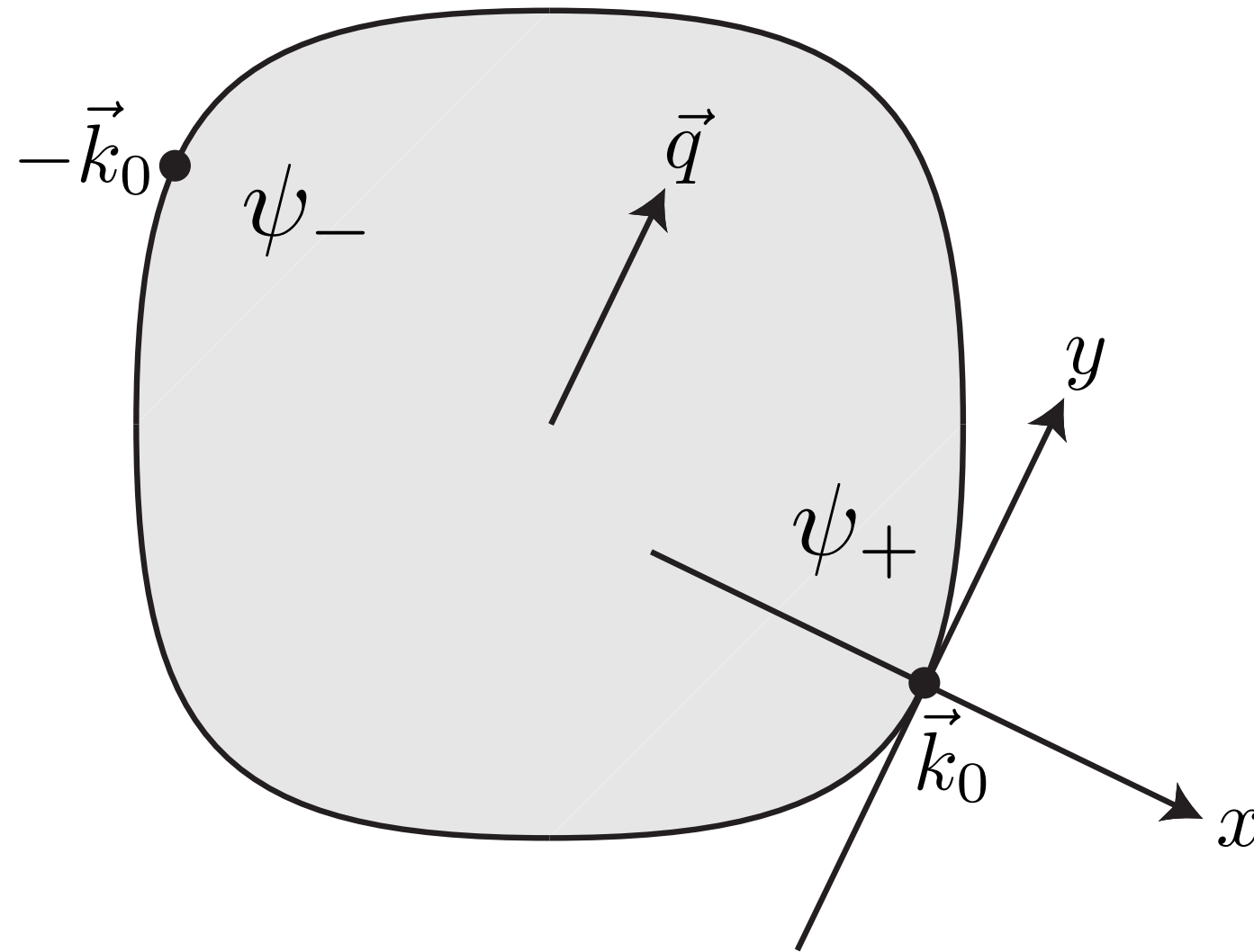
$$\langle \phi \rangle < 0$$

Quantum criticality of Pomeranchuk instability

$$\mathcal{S}_\phi = \int d^2r d\tau [(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4]$$

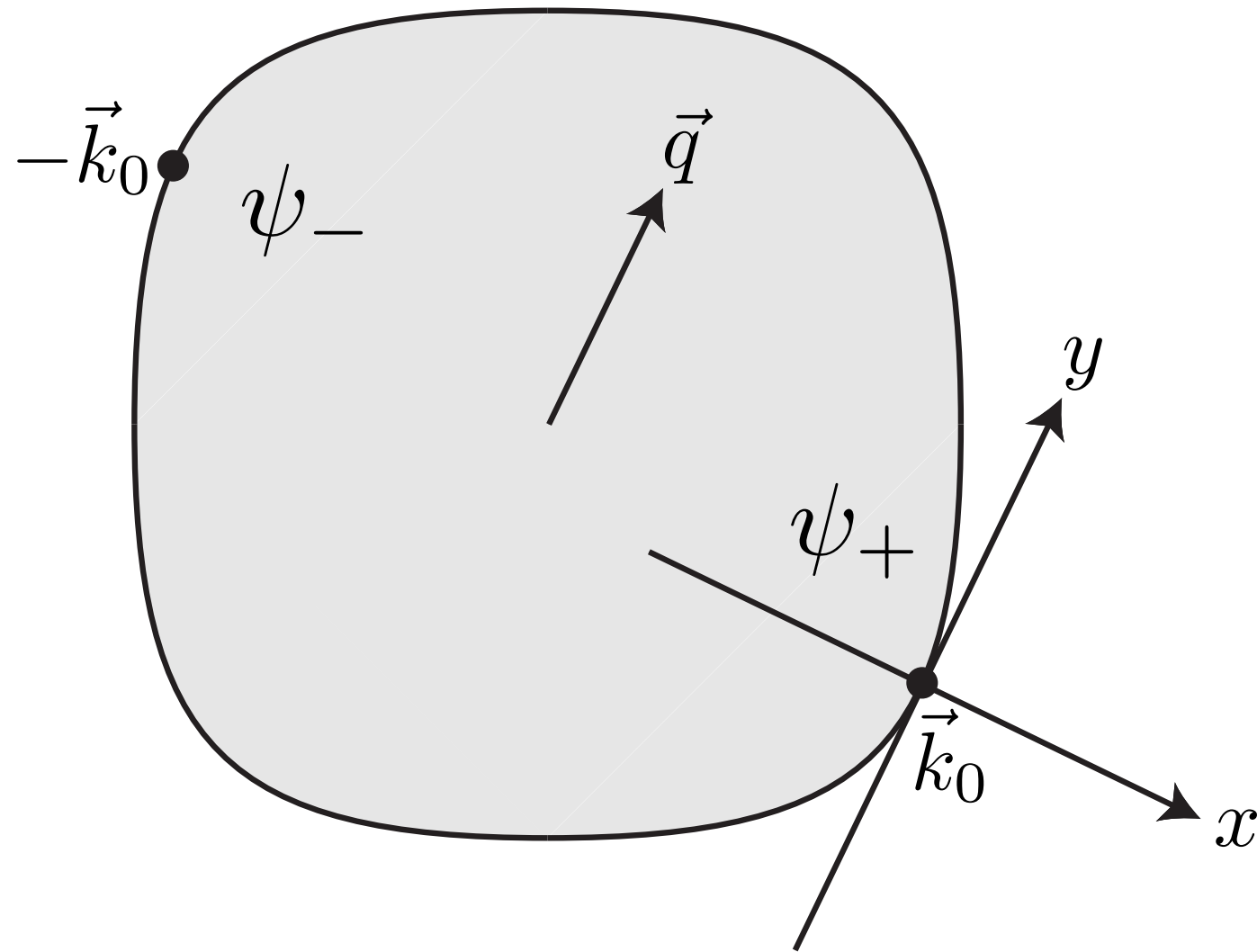
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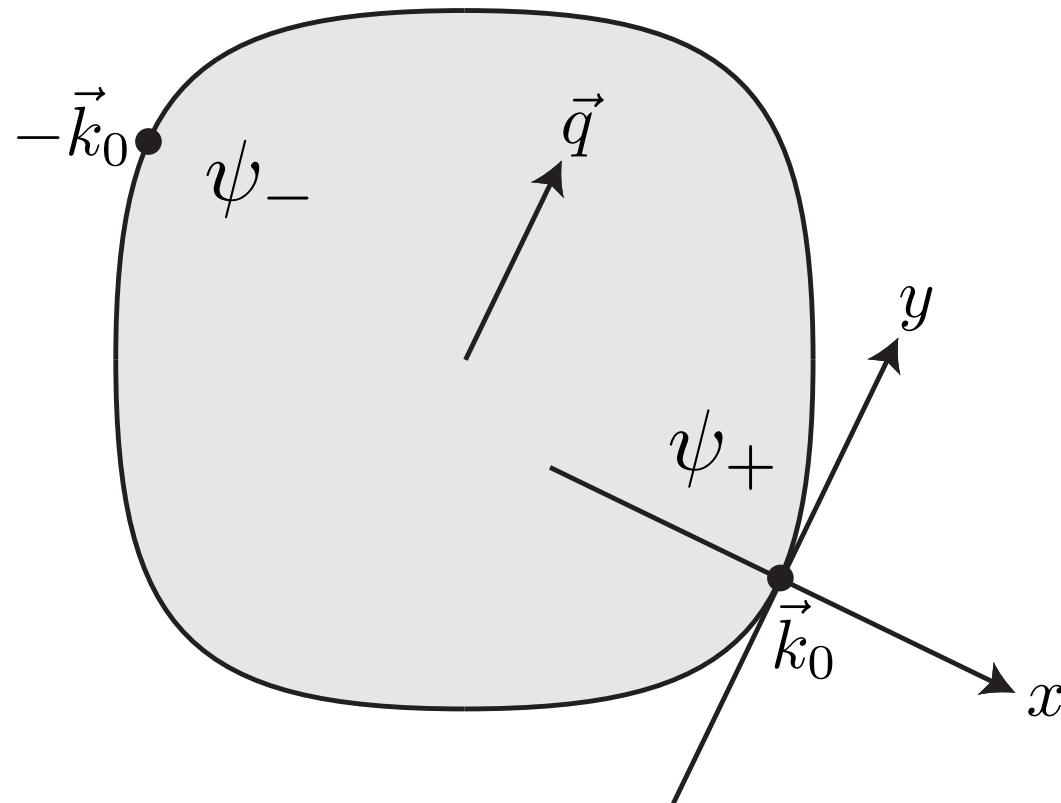


A ϕ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm\vec{k}_0$.

Expand fermion kinetic energy at wavevectors about $\vec{k}_0$



$$\begin{aligned} \mathcal{L} = & \psi_+^\dagger (\zeta \partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\zeta \partial_\tau + i \partial_x - \partial_y^2) \psi_- \\ & - \lambda \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g} (\partial_y \phi)^2 + \frac{r}{2} \phi^2 \end{aligned}$$



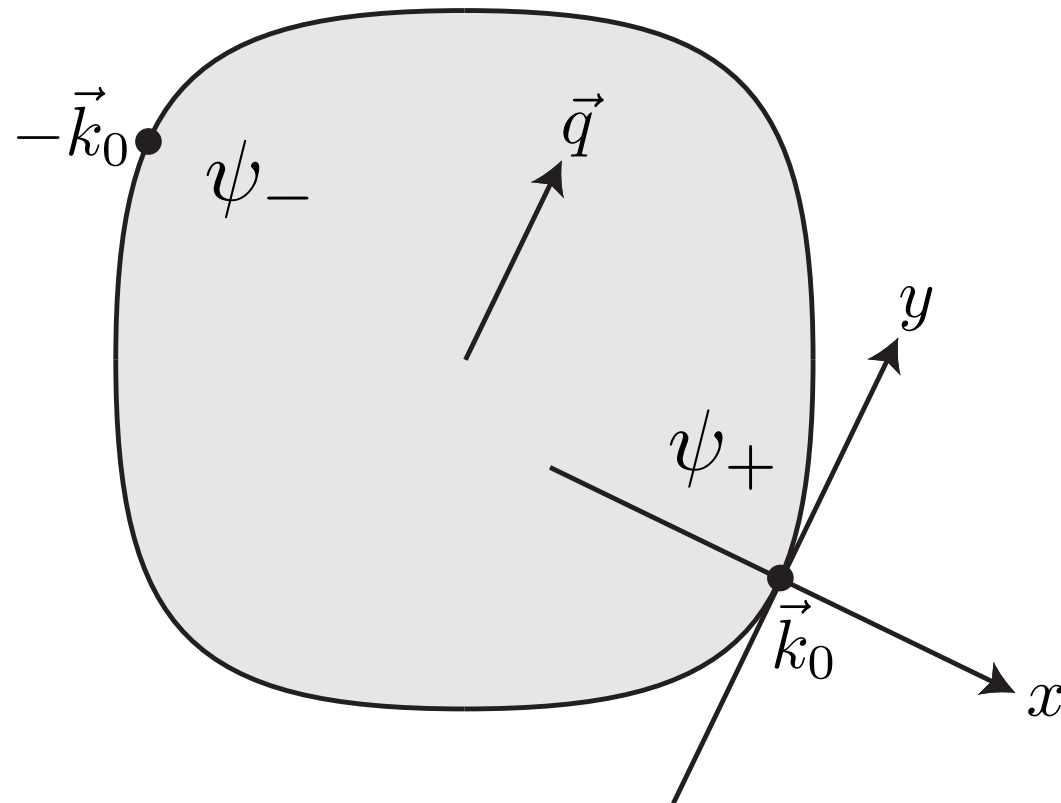
Emergent “Galilean invariance” at low energy ($s = \pm$):

$$\phi(x, y) \rightarrow \phi(x, y + \theta x), \quad \psi_s(x, y) \rightarrow e^{-is(\frac{\theta}{2}y + \frac{\theta^2}{4}x)} \psi_s(x, y + \theta x)$$

which implies for the fermion Green’s function

$$G(q_x, q_y) = G(sq_x + q_y^2).$$

Every point on the Fermi surface $sq_x + q_y^2 = 0$ has the same singularity: “Hot Fermi surface”.



Hertz-Moriya-Millis (HMM) theory

Integrate out fermions and obtain effective action for ϕ

$$\mathcal{L}_\phi = \frac{1}{2} \phi^2 \left[\frac{q_y^2}{g} + \frac{|\omega|}{4\pi |q_y|} \right]$$

Exponent $z = 3$ and mean-field criticality ?

Strategy

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Order parameter at a non-zero wavevector: "Hot spots" on the Fermi surface.

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$

Order parameter:
$$\mathcal{L}_\varphi = \frac{1}{2} (\nabla_r \vec{\varphi})^2 + \frac{\tilde{\zeta}}{2} (\partial_\tau \vec{\varphi})^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4$$

“Yukawa” coupling:
$$\mathcal{L}_c = -\lambda \vec{\varphi} \cdot \left(\psi_{1\alpha}^{\ell\dagger} \vec{\sigma}_{\alpha\beta} \psi_{2\beta}^\ell + \psi_{2\alpha}^{\ell\dagger} \vec{\sigma}_{\alpha\beta} \psi_{1\beta}^\ell \right)$$

Under the rescaling $x' = x e^{-\ell}$, $\tau' = \tau e^{-z\ell}$, the spatial gradients are fixed if the fields transform as

$$\vec{\varphi}' = e^{(d+z-2)\ell/2} \vec{\varphi} \quad ; \quad \psi' = e^{(d+z-1)\ell/2} \psi.$$

Then the Yukawa coupling transforms as

$$\lambda' = e^{(4-d-z)\ell/2} \lambda$$

For $d = 2$, with $z = 2$ the Yukawa coupling is invariant, and the bare time-derivative terms $\zeta, \tilde{\zeta}$ are irrelevant.

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$

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With $z = 2$ scaling, ζ is irrelevant.

So we take $\zeta \rightarrow 0$

( watch for dangerous irrelevancy).

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$

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Set $\vec{\varphi}$ wavefunction renormalization by keeping co-efficient of $(\nabla_r \vec{\varphi})^2$ fixed (as usual).

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Set fermion wavefunction renormalization by keeping Yukawa coupling fixed.

Y. Huh and S. Sachdev, *Phys. Rev. B* **78**, 064512 (2008).

$$\mathcal{L}_f = \psi_{1\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_1^\ell \cdot \nabla_r) \psi_{1\alpha}^\ell + \psi_{2\alpha}^{\ell\dagger} (\zeta \partial_\tau - i \mathbf{v}_2^\ell \cdot \nabla_r) \psi_{2\alpha}^\ell$$

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We find consistent two-loop RG factors, as $\zeta \rightarrow 0$, for the velocities v_x , v_y , and the wavefunction renormalizations.

Consistency check: the expression for the boson damping constant, $\gamma = \frac{2}{\pi v_x v_y}$, is preserved under RG.

RG-improved Migdal-Eliashberg theory

RG flow can be computed a $1/N$ expansion (with N fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{d\ell} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$

RG-improved Migdal-Eliashberg theory

RG flow can be computed a $1/N$ expansion (with N fermion species) in terms of a single dimensionless coupling $\alpha = v_y/v_x$ whose flow obeys

$$\frac{d\alpha}{d\ell} = -\frac{3}{\pi N} \frac{\alpha^2}{1 + \alpha^2}$$

The velocities flow as

$$\frac{1}{v_x} \frac{dv_x}{d\ell} = \frac{\mathcal{A}(\alpha) + \mathcal{B}(\alpha)}{2} ; \quad \frac{1}{v_y} \frac{dv_y}{d\ell} = \frac{-\mathcal{A}(\alpha) + \mathcal{B}(\alpha)}{2}$$

$$\mathcal{A}(\alpha) \equiv \frac{3}{\pi N} \frac{\alpha}{1 + \alpha^2}$$

$$\mathcal{B}(\alpha) \equiv \frac{1}{2\pi N} \left(\frac{1}{\alpha} - \alpha \right) \left(1 + \left(\frac{1}{\alpha} - \alpha \right) \tan^{-1} \frac{1}{\alpha} \right)$$

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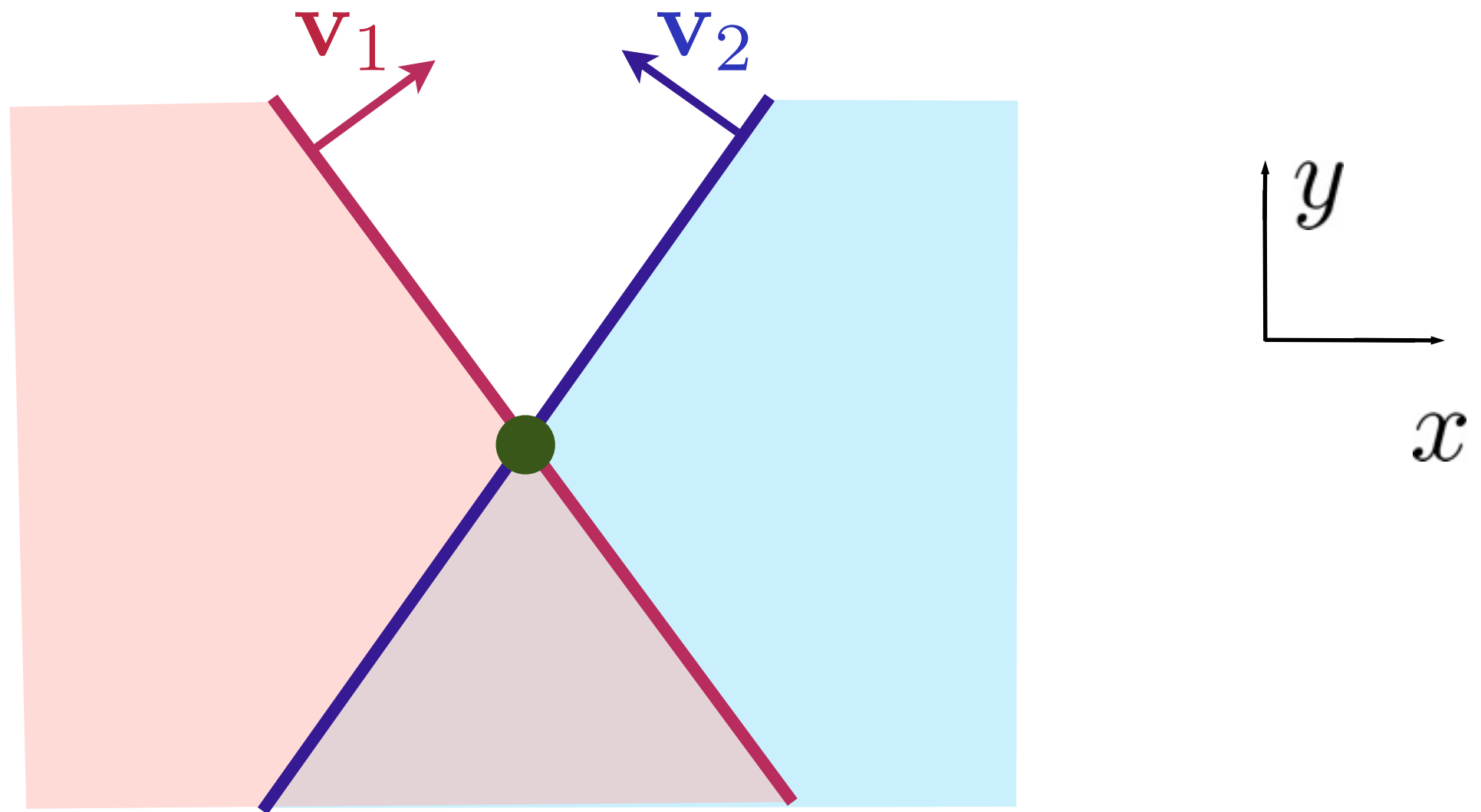
The anomalous dimensions of $\vec{\varphi}$ and ψ are

$$\eta_{\varphi} = \frac{1}{2\pi N} \left(\frac{1}{\alpha} - \alpha + \left(\frac{1}{\alpha^2} + \alpha^2 \right) \tan^{-1} \frac{1}{\alpha} \right)$$
$$\eta_{\psi} = -\frac{1}{4\pi N} \left(\frac{1}{\alpha} - \alpha \right) \left(1 + \left(\frac{1}{\alpha} - \alpha \right) \tan^{-1} \frac{1}{\alpha} \right)$$

RG-improved Migdal-Eliashberg theory

$\alpha = v_y/v_x \rightarrow 0$ logarithmically in the infrared.

Dynamical Nesting

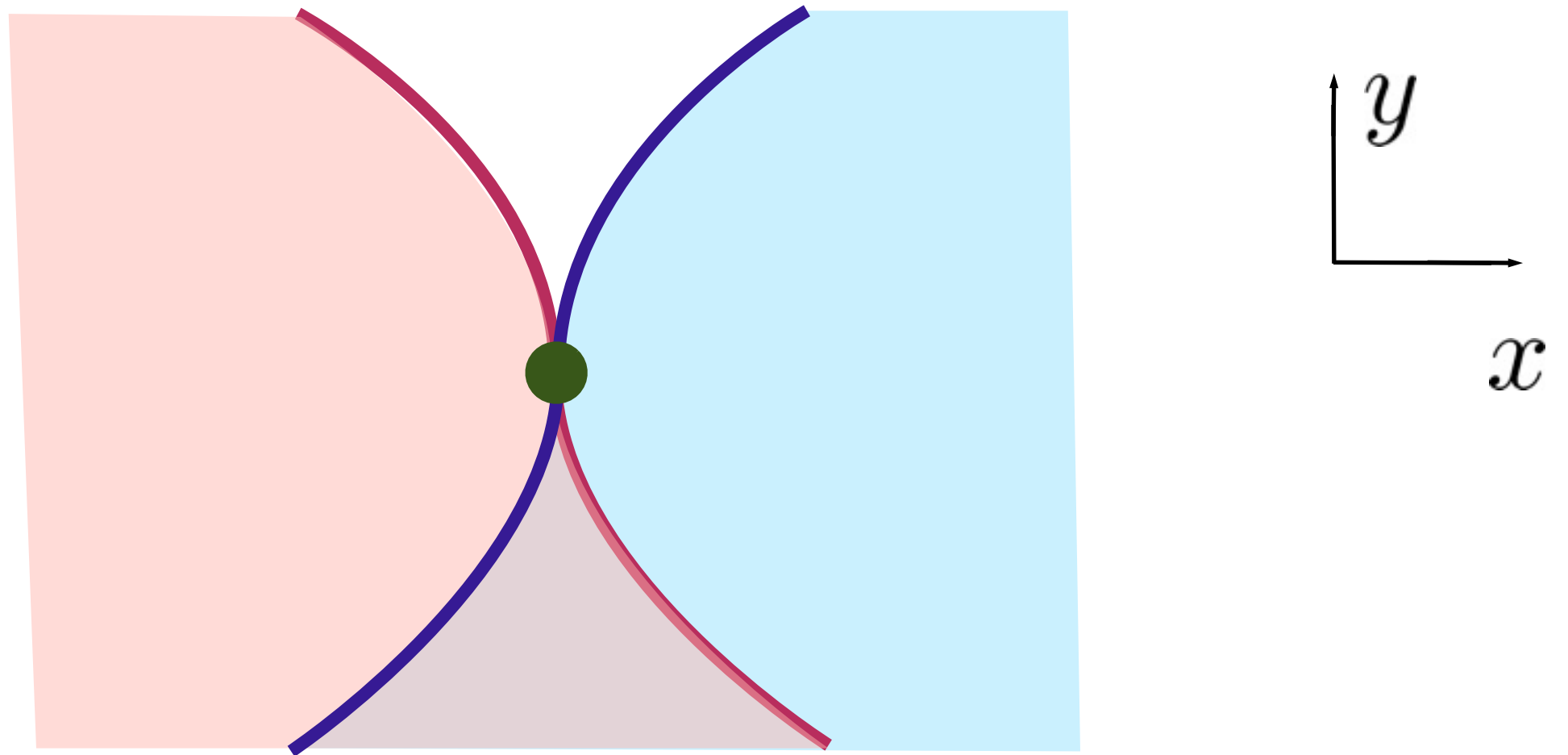


Bare Fermi surface

RG-improved Migdal-Eliashberg theory

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Dynamical Nesting

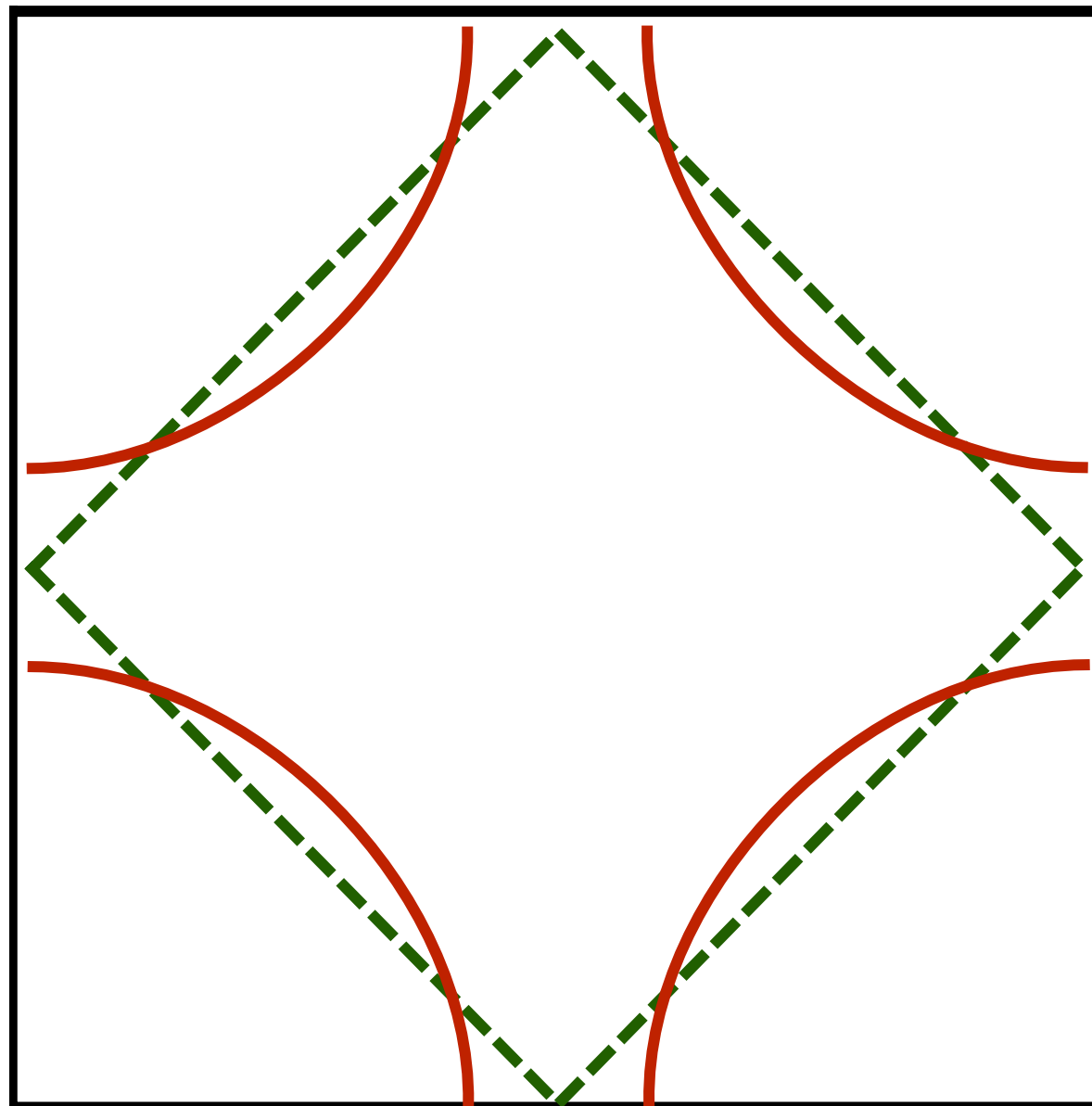


Dressed Fermi surface

RG-improved Migdal-Eliashberg theory

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Dynamical Nesting

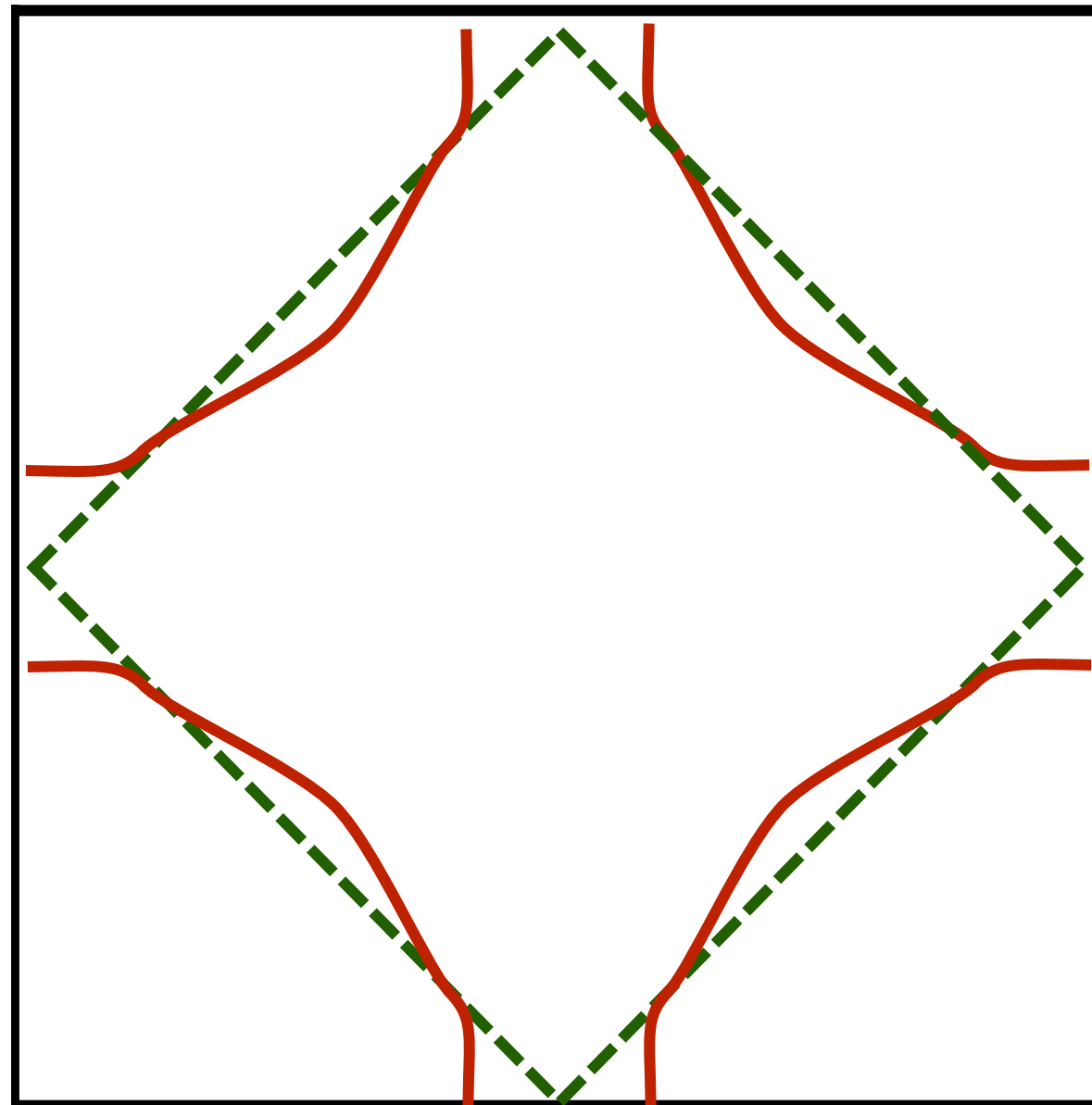


Bare Fermi surface

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Dynamical Nesting



Dressed Fermi surface

RG-improved Migdal-Eliashberg theory

$\alpha = v_y/v_x \rightarrow 0$ logarithmically in the infrared.

In $\vec{\varphi}$ SDW fluctuations, characteristic q and ω scale as

$$q \sim \omega^{1/2} \exp \left(-\frac{3}{64\pi^2} \left(\frac{\ln(1/\omega)}{N} \right)^3 \right).$$

However, $1/N$ expansion cannot be trusted in the asymptotic regime.

New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)

$\vec{\varphi}$ propagator

$$\frac{1}{N} \frac{1}{(q^2 + \gamma|\omega|)}$$

fermion propagator

$$\frac{1}{\mathbf{v} \cdot \mathbf{q} + i\zeta\omega + i \frac{1}{N\sqrt{\gamma}v} \sqrt{\omega} F \left(\frac{v^2 q^2}{\omega} \right)}$$

New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)

$\vec{\varphi}$ propagator

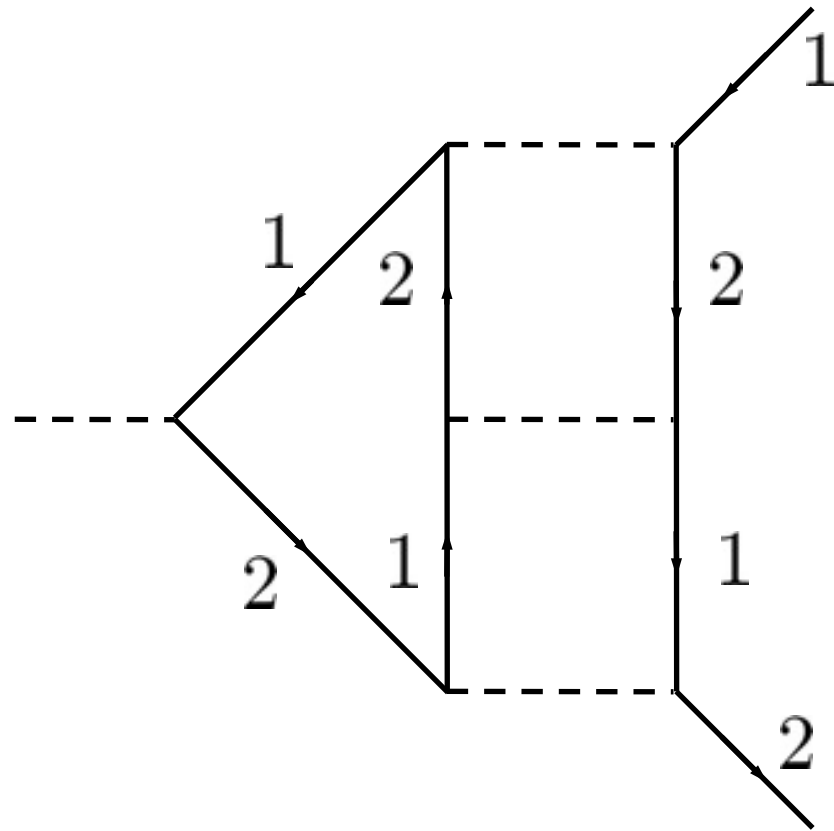
$$\frac{1}{N} \frac{1}{(q^2 + \gamma|\omega|)}$$

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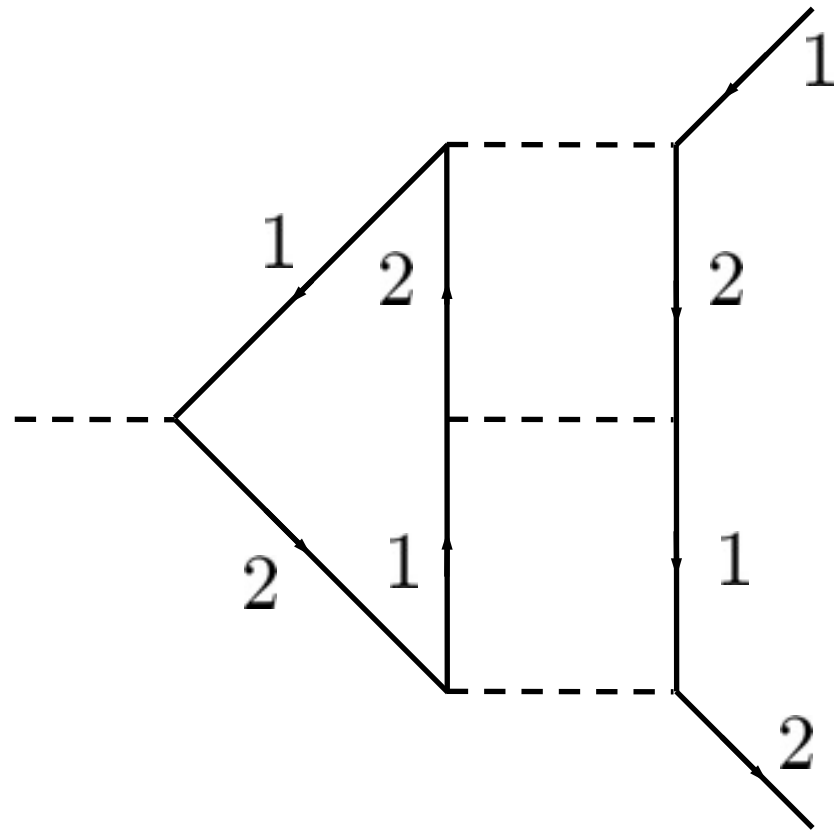
 **Dangerous**

New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)



Ignoring fermion self energy: $\sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$

New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)



Ignoring fermion self energy: $\sim \frac{1}{N^2} \times \frac{1}{\zeta^2} \times \frac{1}{\omega}$

Actual order $\sim \frac{1}{N^0}$

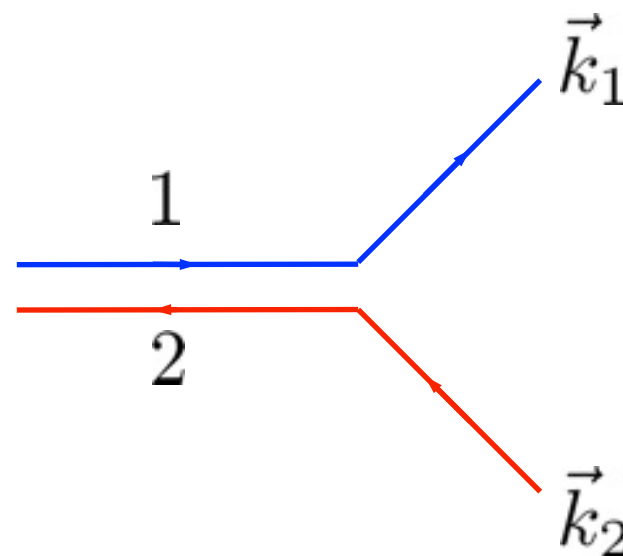
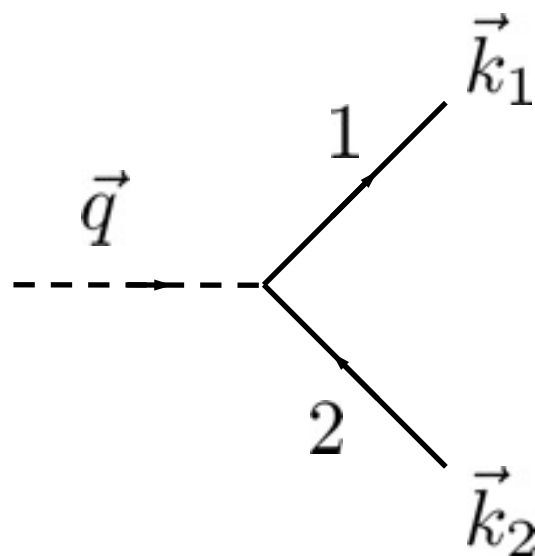
Double line representation

- A way to compute the order of a diagram.
- Extra powers of N come from the Fermi-surface

$$G(\omega, \vec{k}) = \frac{1}{-\Sigma_1(\omega, \vec{k}) - \vec{v} \cdot \vec{k}} \quad \Sigma_1 \sim \frac{1}{N}$$

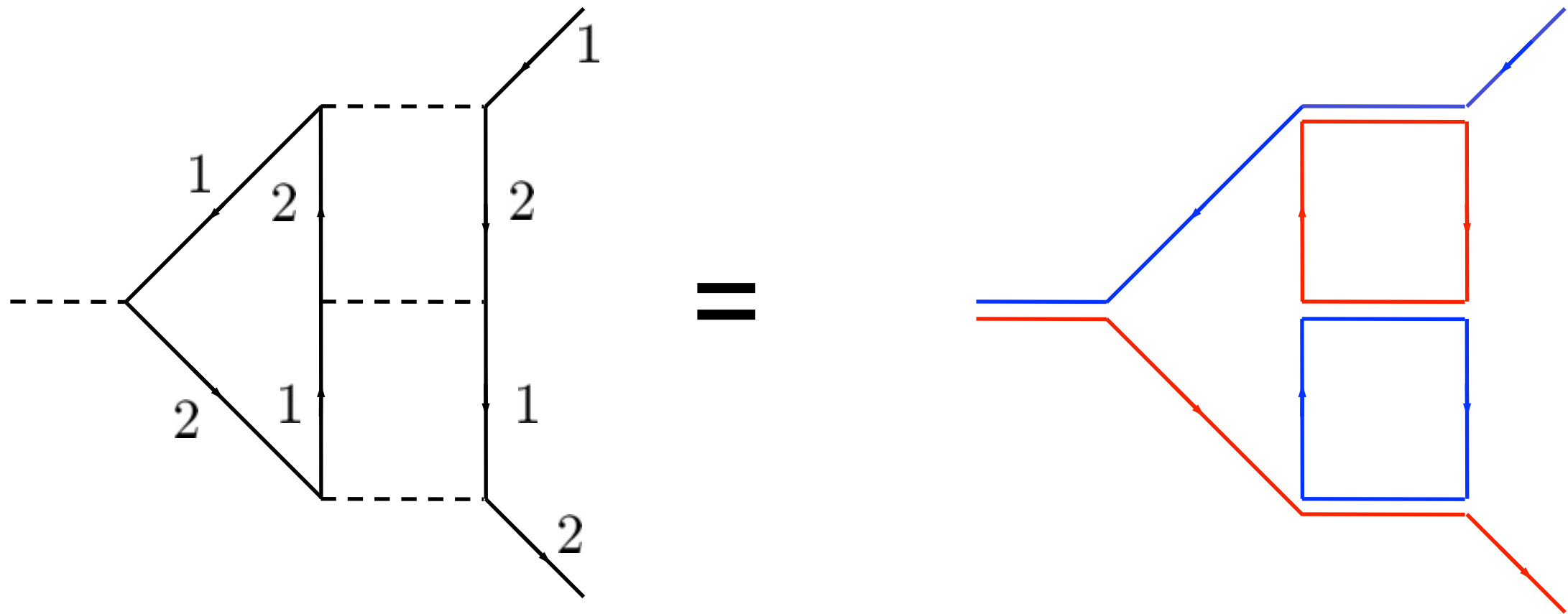
- What are the conditions for all propagators to be on the Fermi surface?
- Concentrate on diagrams involving a single pair of hot-spots
- Any bosonic momentum may be (uniquely) written as

$$\vec{q} = \vec{k}_1 - \vec{k}_2 \quad \vec{k}_1 \in \text{FS of } \psi_1 \quad \vec{k}_2 \in \text{FS of } \psi_2$$



R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).
 S.W.Tsai, A. H. Castro Neto, R. Shankar, and D. K. Campbell, Phys. Rev. B **72**, 054531 (2005).

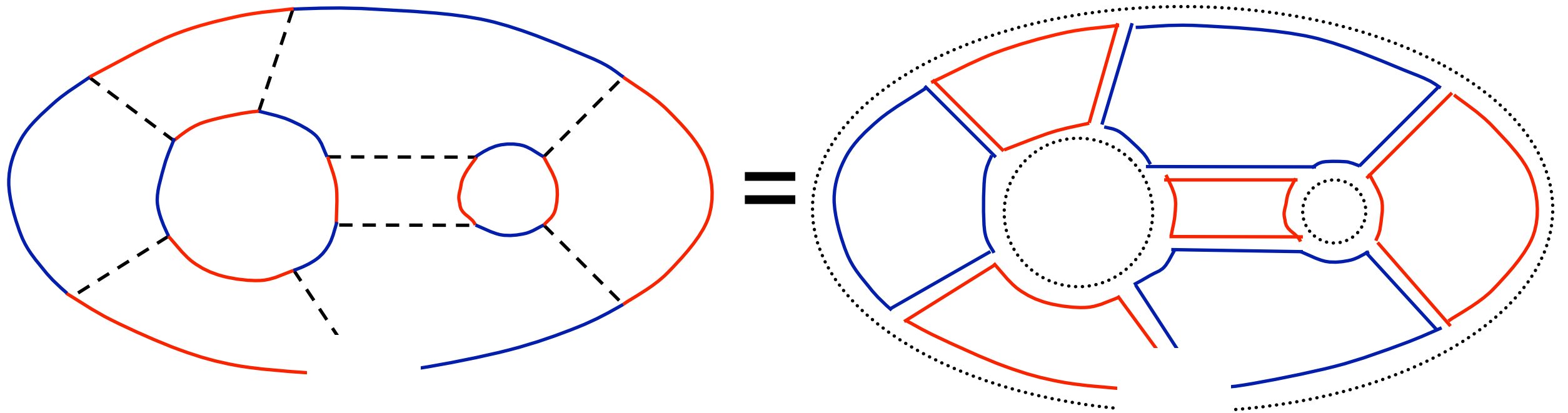
New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)



Singularities as $\zeta \rightarrow 0$ appear when fermions in closed blue and red line loops are exactly on the Fermi surface

$$\text{Actual order} \sim \frac{1}{N^0}$$

New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)

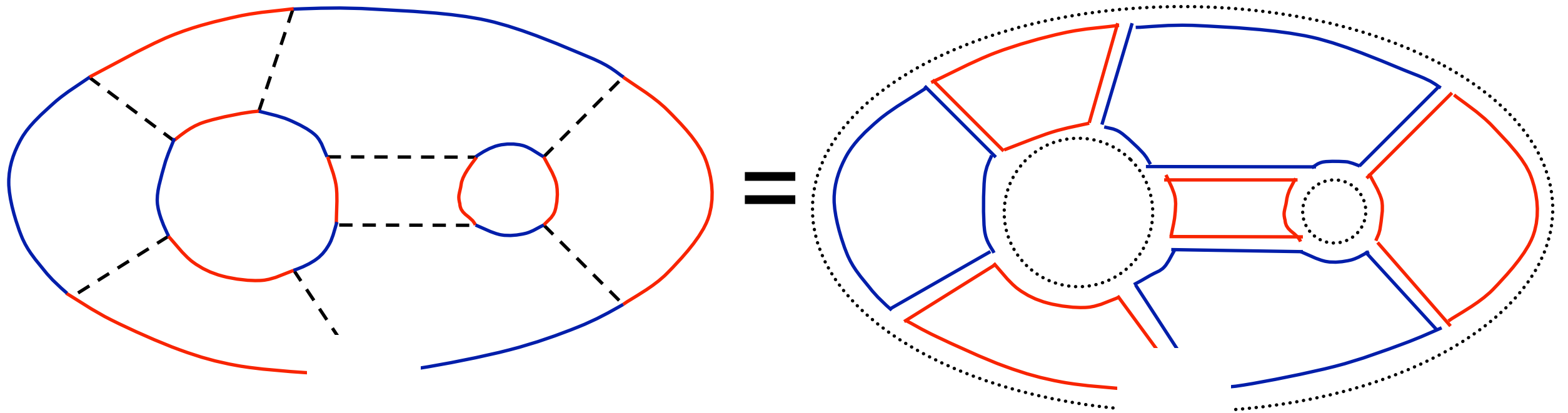


$$\text{Actual order} \sim \frac{1}{N^0}$$

Graph is **planar** after turning fermion propagators also into double lines by drawing additional dotted single line loops for each fermion loop

Sung-Sik Lee, arXiv:0905.4532

New infra-red singularities as $\zeta \rightarrow 0$ at higher loops (Breakdown of Migdal-Eliashberg)



$$\text{Actual order} \sim \frac{1}{N^0}$$



A consistent analysis requires
resummation of all planar graphs



Theory for the onset
of spin density wave
order in metals is
strongly coupled in
two dimensions