# Lecture 4.
# Homogeneously disordered SC films and S-N arrays

## Plan of the Lecture

1) Suppression of $T_c$ by disorder-enhanced Coulomb
2) Mesoscopic fluctuations of $T_c$ near the QPT
3) Quantum S-M transition: SC islands on top of a poor metal film
4) Experimental realization: graphen
5) Open field: Coulomb effects at $T << Tc$

# 1) Suppression of $T_c$ in amorphous thin films by disorder-enhanced Coulomb interaction

## Theory

S. Maekawa and H. Fukuyama, J. Phys. Soc. Jpn. **51**, 1380 (1982).
H. Takagi and Y. Kuroda, Solid Stat. Comm. **41**, 643 (1982).
A. M. Finkel'stein, Pis'ma ZhETF **45**, 37 (1987), [JETP Lett **45**, 46 (1987)]; A. M. Finkel'stein in *Proc. Int. Symp. on Anderson Localization*, edited by T. Ando and H. Fukuyama (Springer-Verlag,Berlin, 1988), p. 230, Springer Proc. in Physics Vol. 28.

## Experiment

J. M. Graybeal and M. R. Beasly, Phys. Rev. B **29**, 4167 (1984), J. M. Graybeal, M. R. Beasly, and R. L. Green, Physica B+C **126** 731 (1984).

P. Xiong, A. V. Herzog, and R. C. Dynes, Phys. Rev. Lett. **78**, 927 (1997).

Review: A. M. Finkel'stein, Physica B **197**, 636 (1994).

Generalization to quasi-1D stripes: Yu. Oreg and A. M. Finkel'stein Phys. Rev. Lett. **83**, 191 (1999)

Similar approach for 3D poor conductor near Anderson transition: P. W. Anderson, K. A. Muttalib, and T. V. Ramakrishnan, Phys. Rev. B **28**, 117 (1983)

# Amorphous v/s Granular films

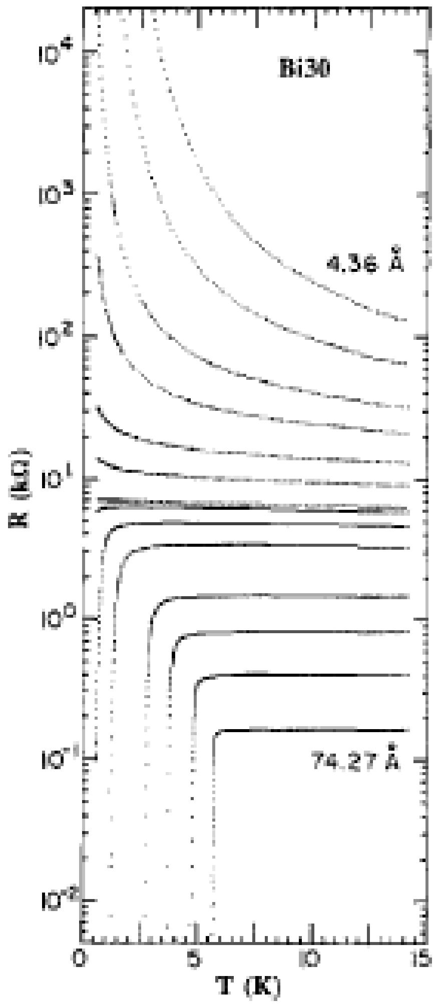



Fig. 5. Temperature dependence of the sheet resistance $R(T)$ for Bi films deposited onto Ge [15]. The films are considered to be homogeneous.

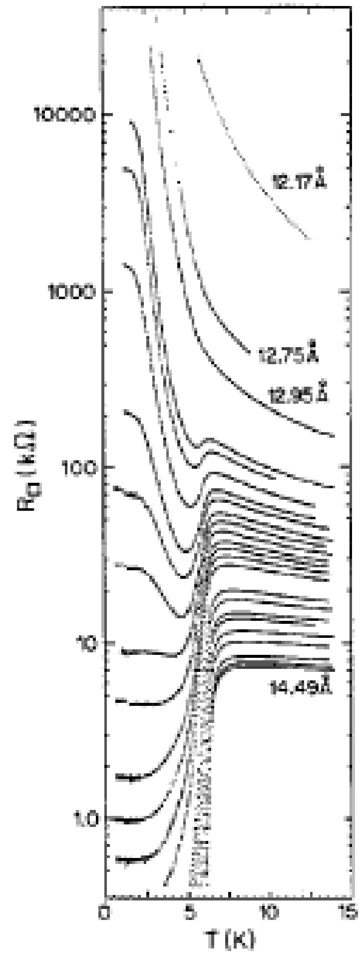



Fig. 6. Evolution of the resistive transition $R_{\square}(T)$ for granular Ga films [16].

# Suppression of $T_c$ in amorphous thin films: experimental data and fits to theory

J.Graybeal and M.Beasley

P. Xiong, A. V. Herzog, and R. C. Dynes,

Pb films

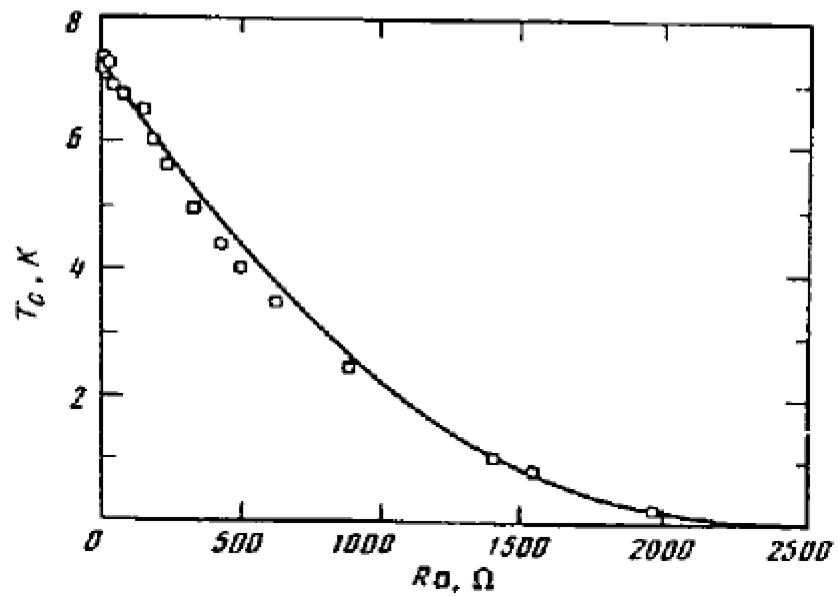



Fig. 14. Suppression of superconductivity in amorphous $Mo_{79}Ge_{21}$ [42]. The solid line is a theoretical fit with Eq. (13).

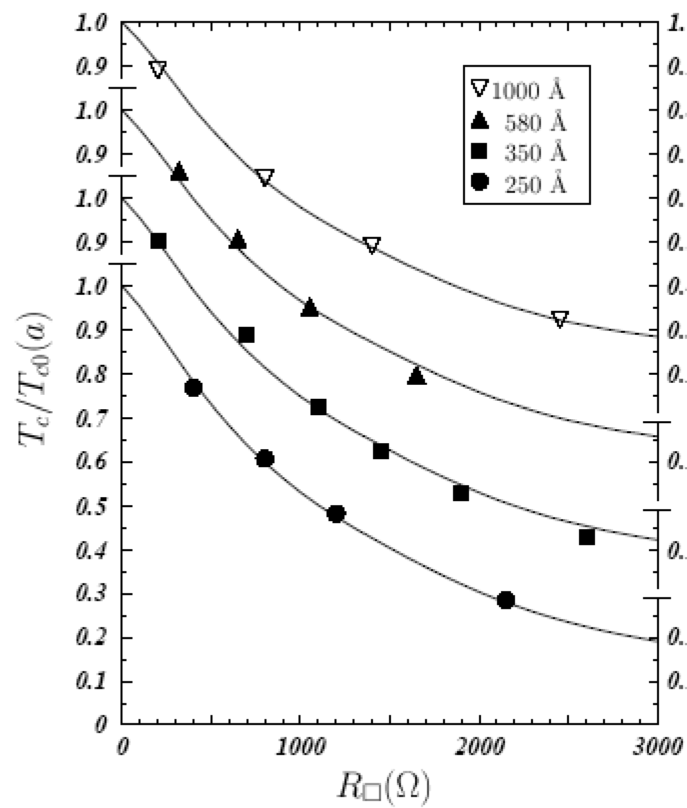



FIG. 3 Comparison between the theory (solid line) and the experimental data [13] for wires of different width. The short

# Suppression of $T_c$ in amorphous thin films: qualitative picture

- Disorder increases Coulomb interaction and thus decreases the pairing interaction (sum of Coulomb and phonon attraction). In perturbation theory:

$$\lambda(\varepsilon) = \lambda_0 - \frac{1}{24\pi g} Log\left(\frac{1}{\varepsilon\tau}\right)$$

Return probability in 2D

Roughly, $\frac{\delta T_c}{T_c} = -\frac{\delta\lambda}{\lambda^2}$

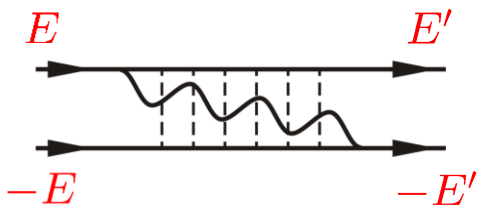
But in fact BCS-type problem with energy-dependent coupling must be solved

The origin of the correction term: ???????

# Coulomb + disorder in 2D superconducting films

Finkelstein (1987)

Coulomb correction to the Cooper channel at $E < \tau^{-1}$:




$$\delta\lambda \sim \frac{1}{\nu}\int \frac{d\mathbf{q}}{Dq^2 + |E+E'|} \sim \frac{1}{g}\ln\frac{1}{|E+E'|\tau}$$

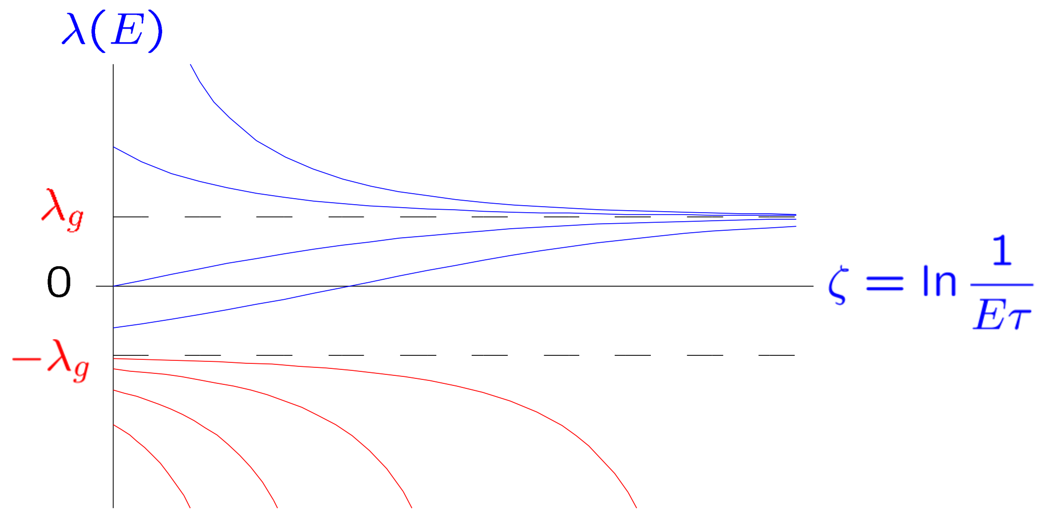
RG equation for $\lambda(E)$:

$$\frac{\partial\lambda}{\partial\zeta} = -\lambda^2 + \frac{1}{2\pi g}$$

$$\zeta = \ln\frac{1}{E\tau} \qquad g = \frac{h}{e^2 R_\square} \gg 1$$

fixed points: $\pm\lambda_g$

$$\lambda_g = \frac{1}{\sqrt{2\pi g}}$$




$$\lambda(\zeta) = \frac{\lambda_0 + \lambda_g \tanh\lambda_g\zeta}{1 + \frac{\lambda_0}{\lambda_g}\tanh\lambda_g\zeta}$$

# Coulomb suppression of $T_c$

Critical temperature

$$\frac{1}{|\lambda_0|} = \frac{1}{\lambda_g}\tanh(\lambda_g \zeta_c)$$

$$T_c\tau = \left(\frac{1 - \frac{1}{\sqrt{2\pi g}}\ln\frac{1}{T_{c0}\tau}}{1 + \frac{1}{\sqrt{2\pi g}}\ln\frac{1}{T_{c0}\tau}}\right)^{\sqrt{\pi g/2}}$$

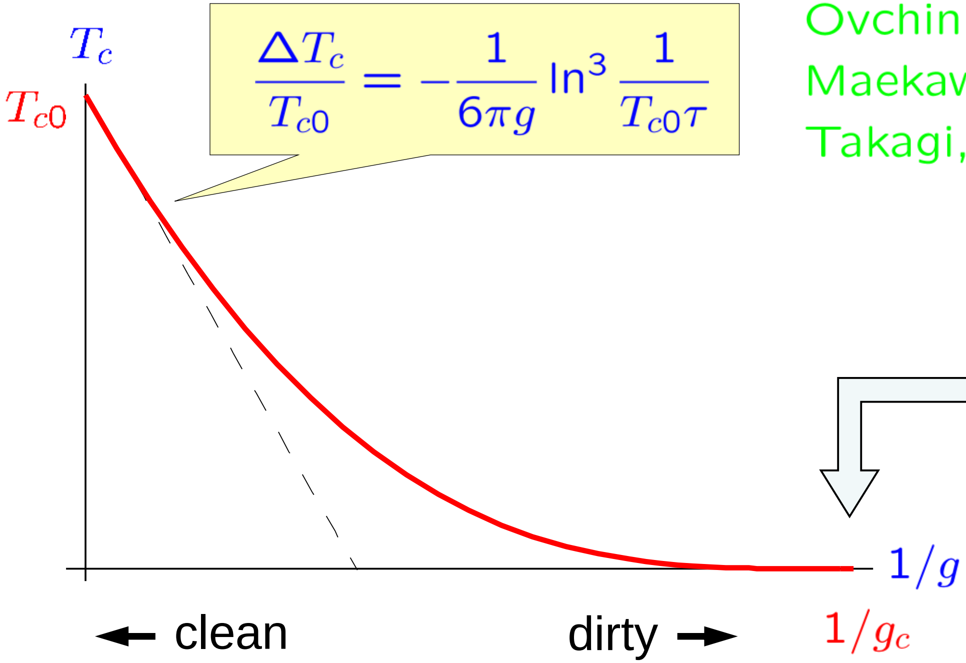
Finkelstein (1987)

$$\frac{\Delta T_c}{T_{c0}} = -\frac{1}{6\pi g}\ln^3\frac{1}{T_{c0}\tau}$$

Ovchinnikov (1973) **(wrong sign)**

Maekawa, Fukuyama (1982)

Takagi, Kuroda (1982)




$$g_c = \frac{1}{2\pi}\ln^2\frac{1}{T_{c0}\tau}$$

At $\ln(1/T_{c0}\tau) > 5$

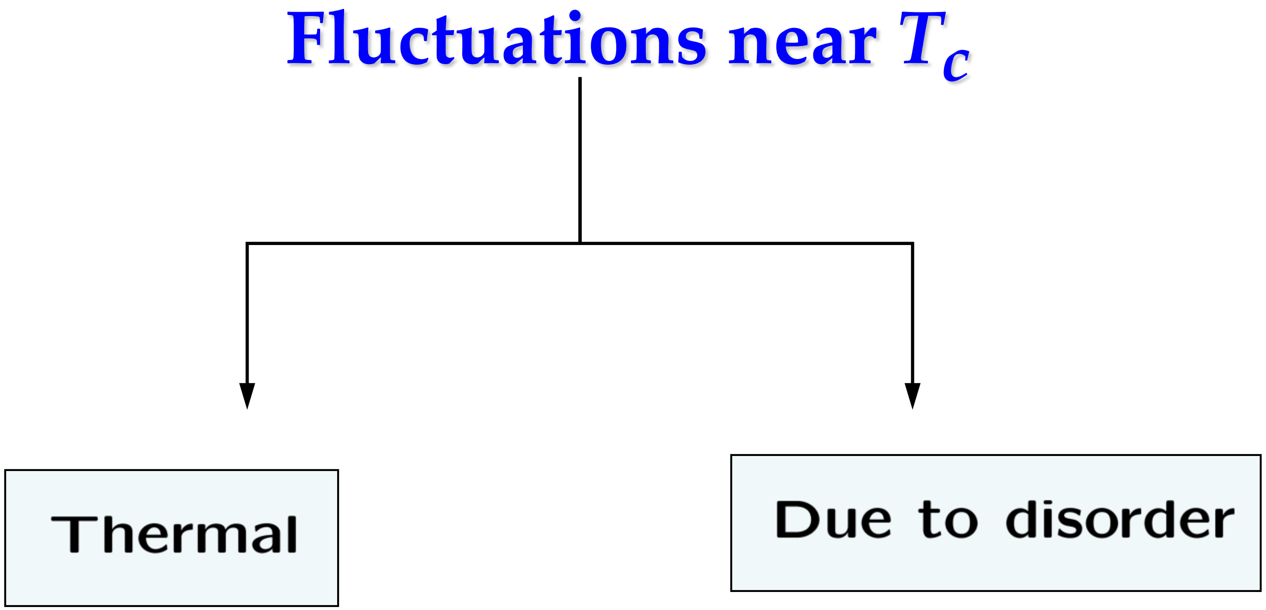
$g_c > 4$ i.e. $R_c < R_Q$

## Conclusion:

Superconductor – Metal

Quantum phase transition

# 2) Mesoscopic fluctuations of $T_c$ near the Finkelstein's Quantum Phase Transition



# Thermal fluctuations: disorder and dimensionality

Ginzburg number

| | clean ($l > \xi_0$) | dirty ($l < \xi_0$) |
|---|---|---|
| 3D | $\frac{1}{(k_F\xi_0)^4}$ | $\frac{1}{k_F^4 \xi_0 l^3}$ |
| 2D | $\frac{1}{k_F^2 \xi_0 d}$ | $\frac{1}{k_F^2 l d}$ |
| 1D | $\frac{1}{(k_F^2 S)^{2/3}}$ | $\frac{(\xi_0/l)^{1/3}}{(k_F^2 S)^{2/3}}$ |

film of thickness $d$: (2D)

wire of area $S$: (1D)

2D dirty → $\frac{1}{g}$

$$g = \frac{h}{e^2 R_\square} \gg 1$$

dimensionless
sheet conductance

$$R_Q = 25\,\mathrm{k\Omega}$$

# Fluctuations due to disorder

Linearized gap equation: $\Delta(\mathbf{r}) = \lambda \int K(\mathbf{r}, \mathbf{r}')\, \Delta(\mathbf{r}')\, d\mathbf{r}'$

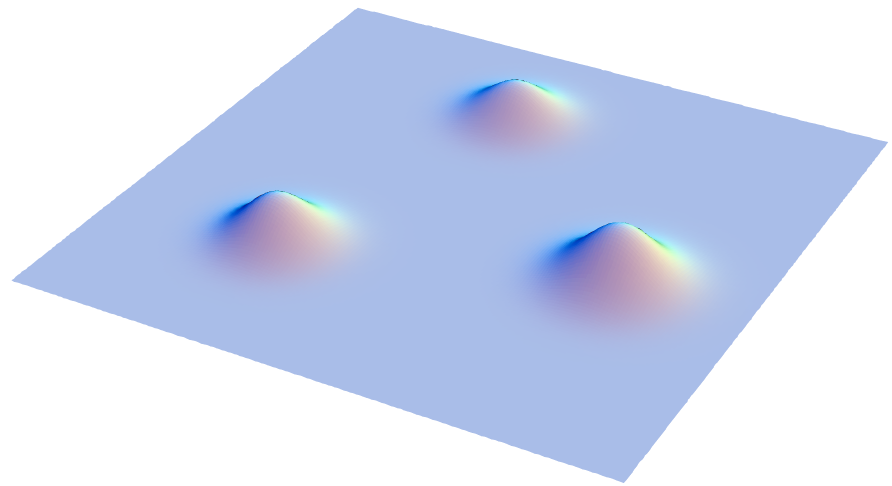
disorder dependent!

- **Mean field**

$$\langle \Delta(\mathbf{r}) \rangle = \lambda \int \langle K(\mathbf{r} - \mathbf{r}') \rangle \, \langle \Delta(\mathbf{r}') \rangle \, d\mathbf{r}'$$

Uniform solutions: $\Delta(\mathbf{r}) = \text{const}$

- **Exact solution:** localized superconducting droplets at $T > T_c^{\mathrm{MF}}$.



mesoscopic disorder at $T = 0$, $H \to H_{c2}(0)$:

Spivak, Zhou (1995)

Galitski, Larkin (2001)

$$\frac{\delta H_{c2}}{\overline{H}_{c2}} \sim \frac{1}{g}$$

## The goal:

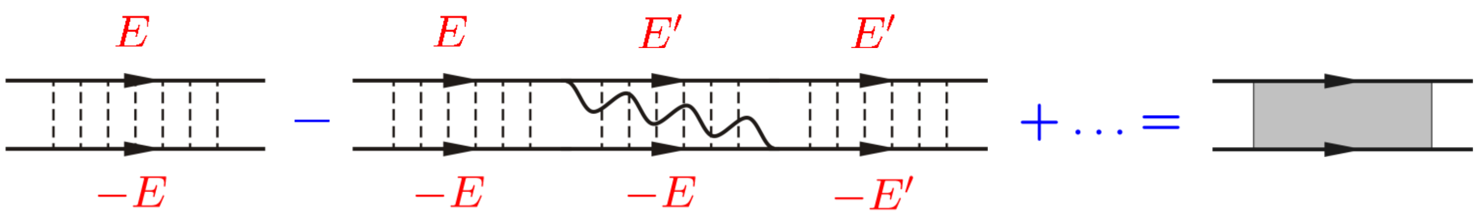
To study mesoscopic fluctuation
near the Coulomb-suppressed $T_c$

## Way to go:

1) Derive the Ginzburg-Landau expansion
2) Include mesoscopic fluctuations of K(r,r′)
3) Droplets of superconductive phase above $T_c$

# Ginzburg-Landau expansion: step 1

$$F = \frac{\nu}{|\lambda_0|}\int d\mathbf{r}\,\Delta^*(\mathbf{r})\Delta(\mathbf{r}) - \int d\mathbf{r}\,d\mathbf{r}'\,\Delta^*(\mathbf{r})\Delta(\mathbf{r}')K(\mathbf{r},\mathbf{r}')$$
$$+\frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4\,\Delta^*(\mathbf{r}_1)\Delta(\mathbf{r}_2)\Delta^*(\mathbf{r}_3)\Delta(\mathbf{r}_4)K^{(4)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)+\dots$$

$\langle K(\mathbf{q})\rangle =$ 


$$T\sum_E \frac{1}{|E|} \qquad T^2\sum_{E,E'}\frac{1}{|E|}\frac{1}{|E'|}\frac{1}{g}\ln\frac{1}{|E+E'|\tau}$$

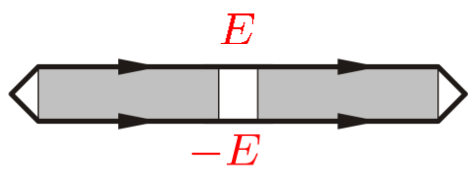
$$\int_0^{\zeta_T} d\zeta = \zeta_T \qquad \frac{1}{g}\int_0^{\zeta_T} d\zeta\,d\zeta'\,\min(\zeta,\zeta') = \frac{\zeta_T^3}{3g}$$

Uniform free energy:

$$F = \nu\int d\mathbf{r}\left(\frac{1}{|\lambda_0|} - \frac{1}{\lambda_g}\tanh(\lambda_g\zeta_T)\right)|\Delta(\mathbf{r})|^2 \Longrightarrow \nu(T/T_c-1)\int d\mathbf{r}\,\frac{|\Delta(\mathbf{r})|^2}{\cosh^2(\lambda_g\zeta_c)}$$

# Ginzburg-Landau expansion: step 2

- Gradient term:

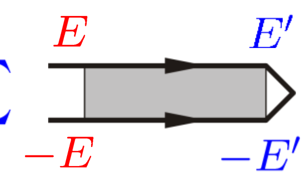



$$\frac{\partial}{\partial D q^2}\frac{1}{Dq^2+2|E|}\bigg|_{q=0} \sim \frac{1}{E^2}$$

$$\gamma \propto T\sum_E \frac{w^2(\zeta)}{E^2} \approx w^2(\zeta_T)\, T\sum_E \frac{1}{E^2}$$

**Screening factor**

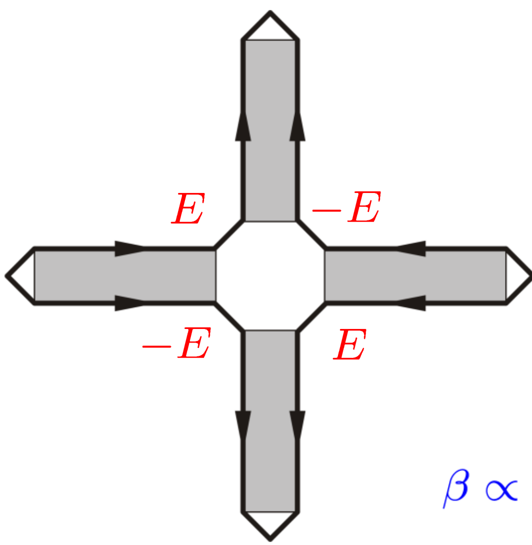
$$\frac{w(E)}{|E|} = T\sum_{E'}$$




$$w(\zeta) = \cosh\lambda_g\zeta - \tanh\lambda_g\zeta_T \sinh\lambda_g\zeta$$

$$w(\zeta_T) = \frac{1}{\cosh\lambda_g\zeta_T}$$

- Quartic term:




$$\beta \propto T\sum_E |E|\frac{w^4(\zeta)}{E^4} \approx w^4(\zeta_T)\, T\sum_E \frac{1}{|E|^3}$$

# Ginzburg-Landau expansion: result

$$F[\Delta] = \int \left[ \alpha(T/T_c - 1)|\tilde{\Delta}|^2 + \gamma|\nabla\tilde{\Delta}|^2 + \frac{\beta}{2}|\tilde{\Delta}|^4 \right] d\mathbf{r}$$

$\alpha$, $\beta$ and $\gamma$ are the usual GL parameters for dirty superconductors, and

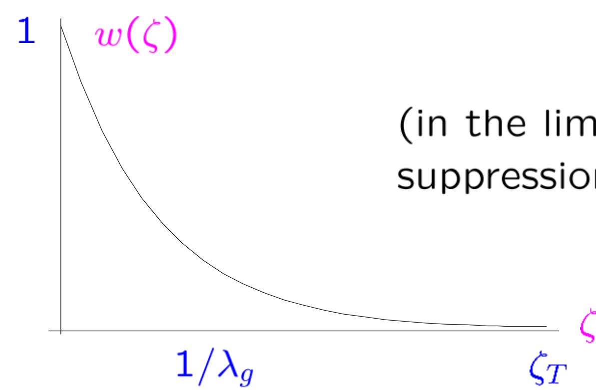
$$\tilde{\Delta} = \Delta w(\zeta_{T_c}) = \frac{\Delta}{\cosh \lambda_g \zeta_{T_c}} \qquad \left( \lambda_g = \frac{1}{\sqrt{2\pi g}}, \ \zeta = \ln \frac{1}{E\tau} \right)$$

Gi is the same as in the absense of the Coulomb repulsion:

$$\mathrm{Gi} = \frac{\pi}{8\,g}$$

**Physical meaning of $\Delta$**

A quasiparticle propagating at energy $E$ feels the pairing potential $\Delta w(\zeta)$



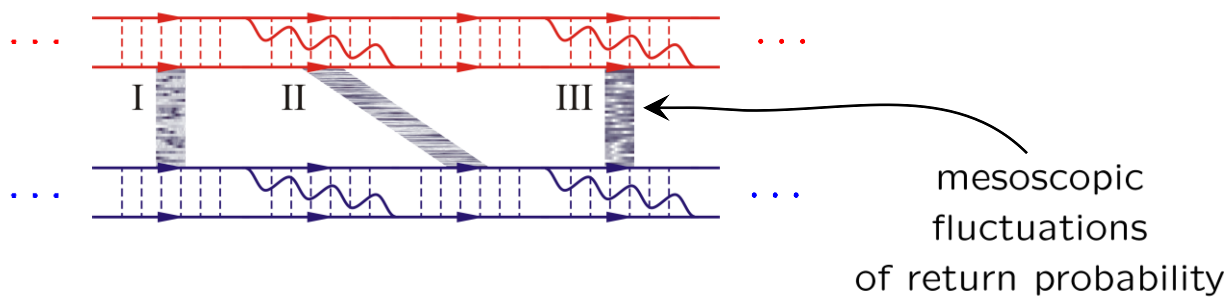

(in the limit of large suppression of $T_c$)

# Mesoscopic fluctuations of the kernel $K(\mathbf{r},\mathbf{r}')$: step 1

$$F^{(2)} = \int d\mathbf{r}\, d\mathbf{r}'\, \Delta^*(\mathbf{r}) \left( \frac{\nu\delta(\mathbf{r}-\mathbf{r}')}{|\lambda_0|} - K(\mathbf{r},\mathbf{r}') \right) \Delta(\mathbf{r}')$$

$$\langle K(\mathbf{r}-\mathbf{r}')\rangle + \delta K(\mathbf{r},\mathbf{r}')$$

$$\langle\langle \delta K(\mathbf{r}_1,\mathbf{r}_2)\delta K(\mathbf{r}_3,\mathbf{r}_4)\rangle\rangle =$$



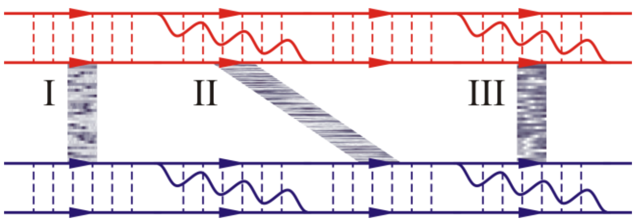

Length scales:

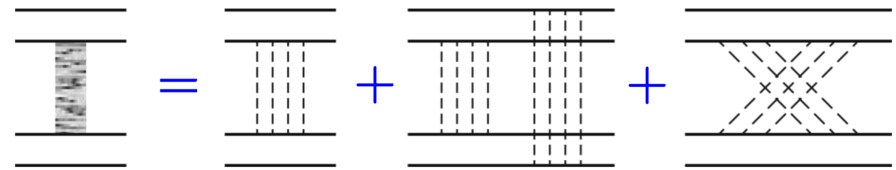
- $\langle\langle \delta K(\mathbf{r}_1,\mathbf{r}_2)\delta K(\mathbf{r}_3,\mathbf{r}_4)\rangle\rangle$: decays at $\mathbf{r}_i - \mathbf{r}_j \sim L_T = \sqrt{D/T_c}$
- superconductive coherence length $\xi(T) = L_T\sqrt{\frac{T_c}{T-T_c}} \gg L_T$

From the viewpoint of the superconductive system, fluctuations of $K(\mathbf{r},\mathbf{r}')$ are short-ranged and characterized by a single number:

$$C = \int \langle\langle \delta K(\mathbf{r}_1,\mathbf{r}_2)\delta K(\mathbf{r}_3,\mathbf{r}_4)\rangle\rangle\, d\mathbf{r}_2\, d\mathbf{r}_3\, d\mathbf{r}_4$$

# Mesoscopic fluctuations of the kernel $K(\mathbf{r},\mathbf{r}')$: step 2




*** Calculation was here ***

$$C = \frac{w_0^2(T)}{\pi D}\left(T^2 \sum_{\varepsilon,\varepsilon'>0} \frac{1}{\varepsilon\varepsilon'(\varepsilon+\varepsilon')}\right)\left[w_0(T) + \underbrace{\frac{1}{g} T \sum_{\varepsilon>0} \frac{w_0(\varepsilon)}{\varepsilon} \ln\frac{1}{\varepsilon\tau}}_{1-w_0(T)}\right]^2$$

mesoscopic fluctuations of return probability

$$C = \frac{7\zeta(3)}{8\pi^4 DT} \frac{1}{\cosh^2(\lambda_g \zeta_T)}$$

2 instead of 4!

# Superconductor with fluctuating $T_c$

Superconductor with fluctuating $T_c$ (Ioffe, Larkin (1981)):

Sov.Phys.JETP **54**, 378 (1981)

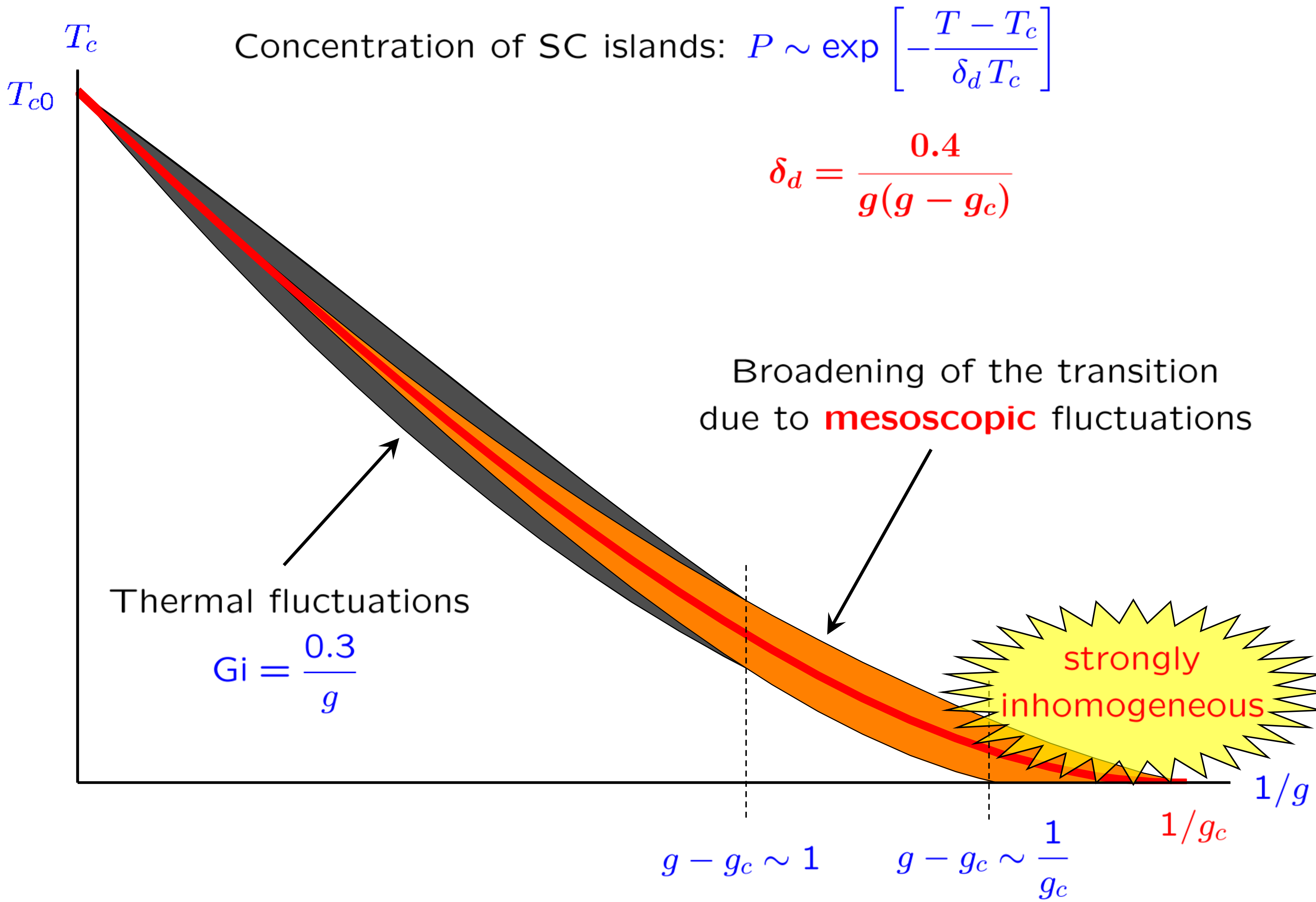
$$F = \int \left\{ \left[ \alpha(T/T_c - 1) + \delta\alpha(\mathbf{r}) \right] |\tilde{\Delta}|^2 + \gamma |\nabla \tilde{\Delta}|^2 + \frac{\beta}{2} |\tilde{\Delta}|^4 \right\} d\mathbf{r}$$

$$\langle \delta\alpha(\mathbf{r}) \delta\alpha(\mathbf{r}') \rangle = \frac{C}{w^4(T)} \delta(\mathbf{r} - \mathbf{r}') = \frac{7\zeta(3)}{8\pi^4 DT} \cosh^2(\lambda_g \zeta_T)\, \delta(\mathbf{r} - \mathbf{r}')$$

GL equation looks like a Schrödinger equation with random potential
Zittartz, Langer; Halperin, Lax (1966)

Instantons

# Mesoscopic vs. thermal fluctuations



# Conclusion of the part 2):

**Close to the quantum critical point, at $g - g_c \leq 1$,**
**the superconducting phase transition**
**is due to overlapping superconducting droplets**
**created by mesoscopic fluctuations**

# 3) Quantum S-M transition: SC islands on the top of a poor metal film

- M. V. Feigel’man and A. I. Larkin

  Chem. Phys. Lett. 235, 107 (1998)

- B. Spivak, A. Zyuzin, and M. Hruska

  Phys. Rev. B **64**, 132502 (2001)

- M. V. Feigel’man, A. I. Larkin and M.A.Skvortsov

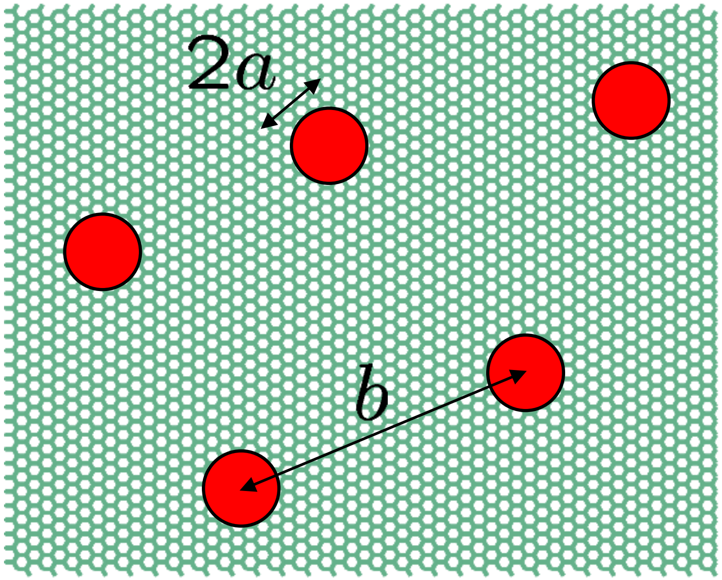
  Phys. Rev. Lett. **86**, 1869 (2001)

# 4) Experimental realization: graphen

Suggested in M. Feigel'man, M. Skvortsov and K. Tikhonov
JETP Letters **88**(11), 747-751 (2008); arXiv:0810.0109]

First experiment: B.M. Kessler, C.O. Girit, A. Zettl, V. Bouchiat
arxiv: 0907.3661

# Graphene + superconducting islands




Graphene can be made superconductive
if small superconductive islands
are placed on top of it

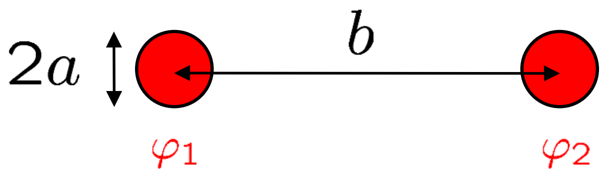
$$T_c \sim \frac{\hbar D}{b^2}$$

can be Kelvins

disordered graphene

scales: $l \leq a \ll b$

# Josephson coupling via disordered graphene




$$\mathcal{H} = -E_J(b,T)\cos(\varphi_1 - \varphi_2)$$

- Usadel equation (at $l < r$)
- linearization ($a \ll b$): $F_E(\mathbf{r}) = F_E^{(\varphi_1)}(\mathbf{r}-\mathbf{r}_1) + F_E^{(\varphi_2)}(\mathbf{r}-\mathbf{r}_2)$

$$E_J(b,T) = 16\pi^3 gT \sum_{\omega_n>0} \frac{P(\sqrt{\omega_n r^2/2D})}{\ln^2(\omega_n a^2/D)}, \qquad P(z) = z\int_0^\infty K_0(z\cosh t)K_1(z\cosh t)\,dt$$

good contact

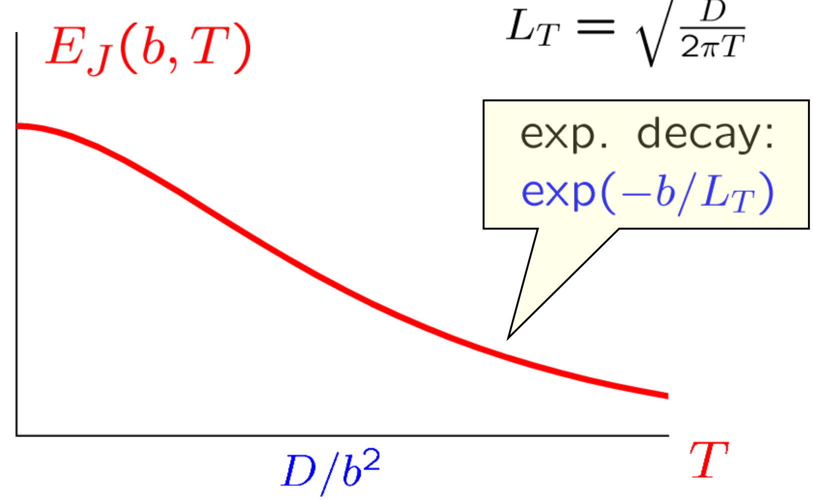
$$E_J(b,0) = \frac{\pi^3}{4}\frac{g}{\ln^2(b/a)}\frac{D}{b^2}$$

tunnel barrier

$G_T$

$$E_J(b,0) = \frac{G_T^2}{8\pi g}\frac{D}{b^2}$$

$$L_T = \sqrt{\frac{D}{2\pi T}}$$



# Thermal phase transition

JJ array:

$$\mathcal{H} = -\sum_{ij} E_J(b,T)\cos(\varphi_i - \varphi_j)$$

XY model:

$$\mathcal{H} = \frac{\Upsilon(T)}{2}\int d\mathbf{r}\,(\nabla\varphi)^2$$

Superfluid stiffness: $\Upsilon(T) = \frac{c}{2b^2}\sum_j |\mathbf{r}_i - \mathbf{r}_j|^2 E_J(\mathbf{r}_i - \mathbf{r}_j, T)$

The systems exhibits the **BKT transition** at $\Upsilon(T_c) = (2/\pi)T_c$

Josephson coupling is short-ranged at $T \sim T_c$

$$T_c = \gamma\, E_J(T_c)$$

$$E_{\mathrm{Th}} = \frac{D}{b^2}$$

for triangular lattice $\gamma = 1.5$ (Shih and Stroud 1985)

- at $T > T_c$, phases of different islands are uncorrelated
- at $T < T_c$, all phases phases look in the same direction

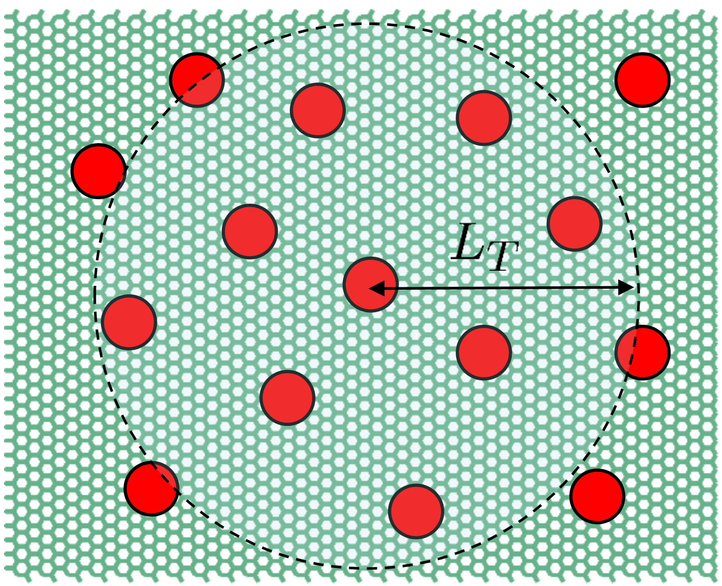
# Superfluid density below $T_c$

At $T \ll T_c$, the Josephson coupling is long-ranged.

Many islands contribute to the superfluid stiffness $\Upsilon(T)$:

$$\Upsilon(T) = \frac{1}{2b^2}\int_0^\infty r^2 E_J(r,T)\, 2\pi r\, dr = \frac{\pi^5}{4}\frac{g}{\ln^2(b/a)}\frac{E_{\mathrm{Th}}^2}{T}$$

$L_T^2$ $L_T^{-2}$ $L_T^2$



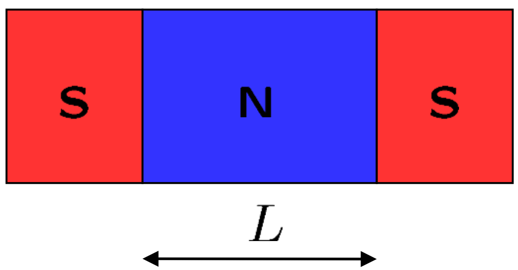

Strong divergence of $\Upsilon(T \to 0)$

Did we forget something?

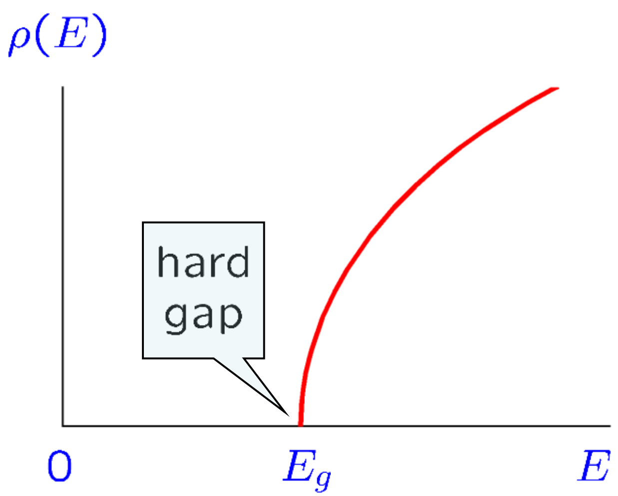
?

# Spectral gap

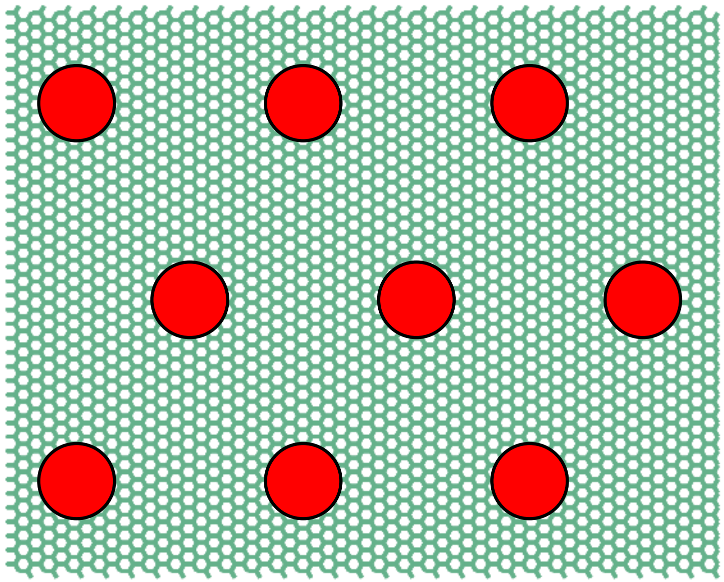
**Minigap in an SNS junction**




$$E_g \simeq 3.1 \frac{D}{L^2}$$




**Regularly placed islands on graphene**



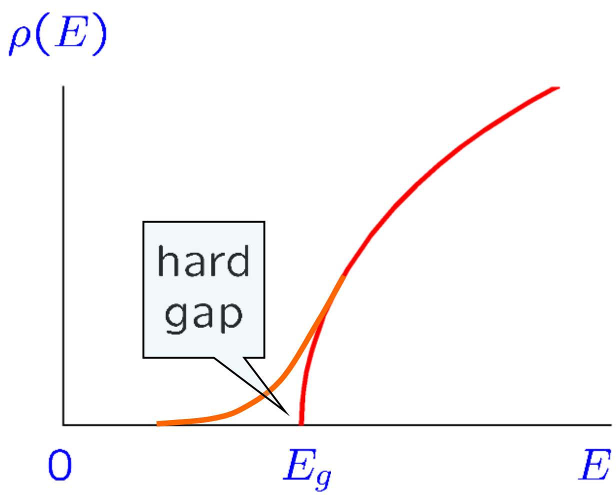
Numeric integration of the Usadel eq. gives

$$E_g \simeq \frac{2.6}{\ln(b/a)} \frac{D}{b^2}$$

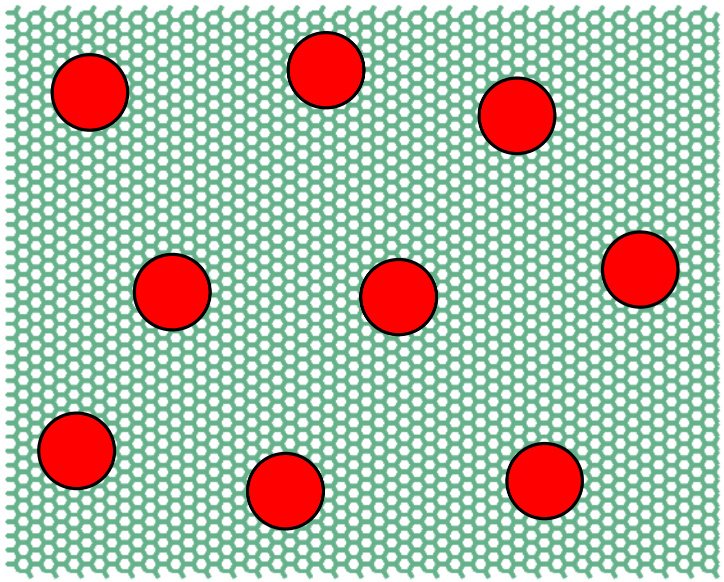
Can be seen in STM experiments

# Spectral gap in an irregular system

The hard gap will be smeared

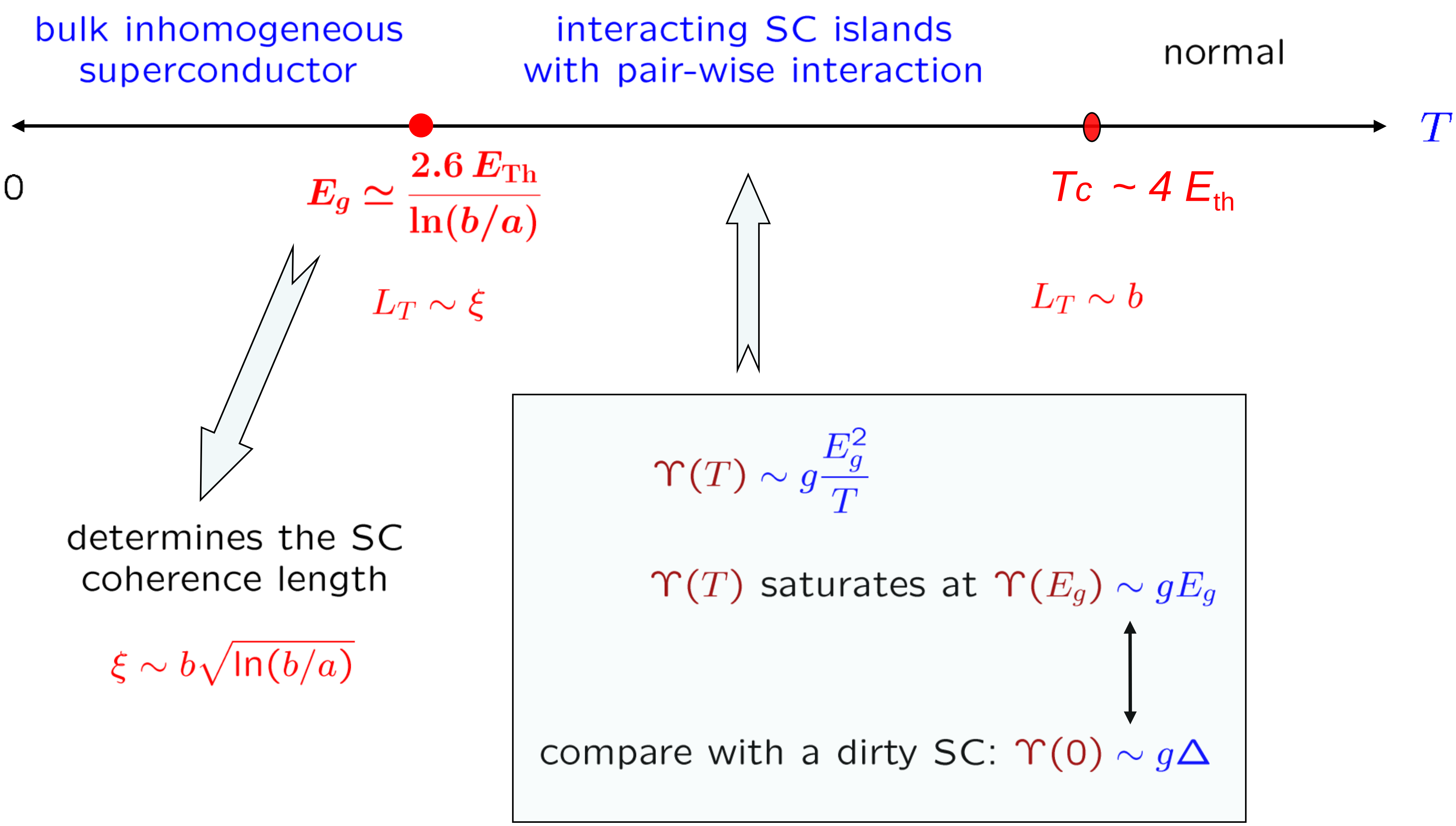



**Iregularly placed islands on graphene**



The relative smearing is expected to be small: $\dfrac{\delta E_g}{E_g} \sim \dfrac{1}{\ln(b/a)}$

# Two-energy-scale superconductivity



# Magnetic field effect (low $T$, low $H$)

Small $H$: like in a 2D superconductor with $\Delta \sim E_g$

Critical field:

$$H_g \simeq \frac{cE_g}{eD} = \frac{2.6}{\pi \ln b/a} \frac{\Phi_0}{b^2}$$

$H_g \sim 100$ G
at $b \sim 300$ nm

## $H < H_g$

- disordered vortex lattice
- core size $\xi \sim b \ln(b/a) \gg b$
- strong pinning

## $H > H_g$

- spectral gap is totally suppressed
- gapples SC $\Leftrightarrow$ SC glass
- weakly frustrated

# Magnetic field effect (low $T$, high $H$)

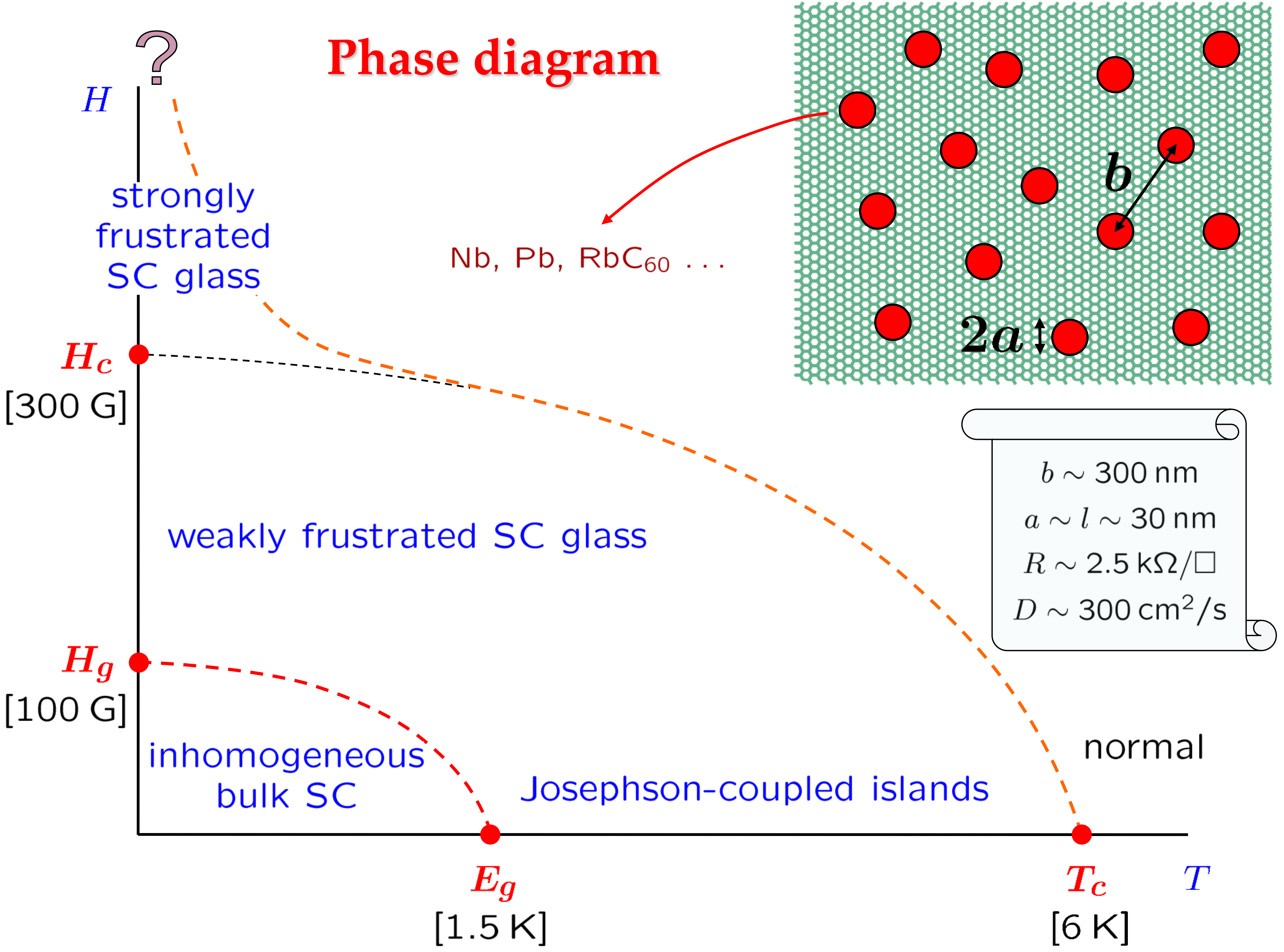
$$H_c \sim \frac{\Phi_0}{b^2} \gg H_g$$

At $H \sim H_c$ the Josephson coupling bewteen the nearest islands becomes totally frustated

average coupling: $\langle E_J(r_{ij})\rangle \sim g\,\dfrac{D}{r_{ij}^2}\,\mathrm{exp}(-r_{ij}/L_H)$

mesoscopic fluctuations: $\sqrt{\langle E_J^2(r_{ij})\rangle} \sim \dfrac{D}{r_{ij}^2}$

# Phase diagram



# Conclusions for the theory in part 4)

1. Graphene can be made superconductive at Kelvins with very small part of area covered by superconductive islands

2. Spectral gap expected due to proximity effect should be measurable by low-temperature STM (and destroyed easily by magnetic field)

3. Transformation from continous disordered superconductor to weakly coupled junction's array is predicted with growth of either T or B

4. Approaching the neutrality point will lead to island's decoupling and T=0 quantum phase transition to metal

# 1st Experiment

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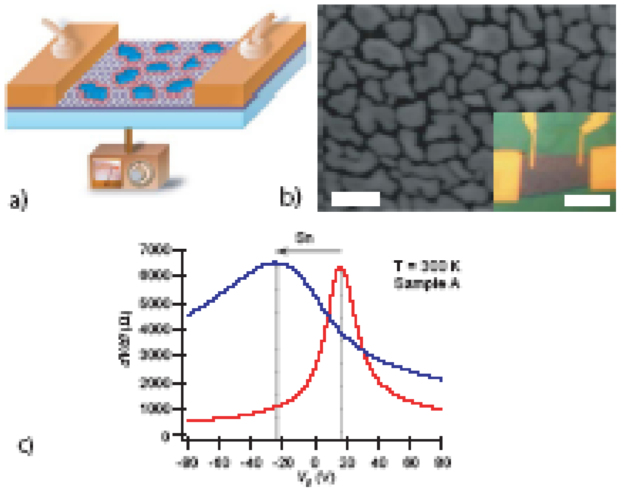


FIG. 1: (Color online) a) Schematic of device configuration and measurement setup. Blue islands correspond to Sn clusters. b) Scanning electron micrograph of Sn island morphology on the graphene sheet (Scale bar = 100 nm. Inset: optical image of a typical device showing the four probe configuration (Scale bar 10 microns) c) Four-terminal sheet resistance as a function of gate voltage for Sample A before (red online) and after (blue online) Sn deposition. The dotted lines indicate the charge neutral point and the arrow indicates the shift after Sn deposition.

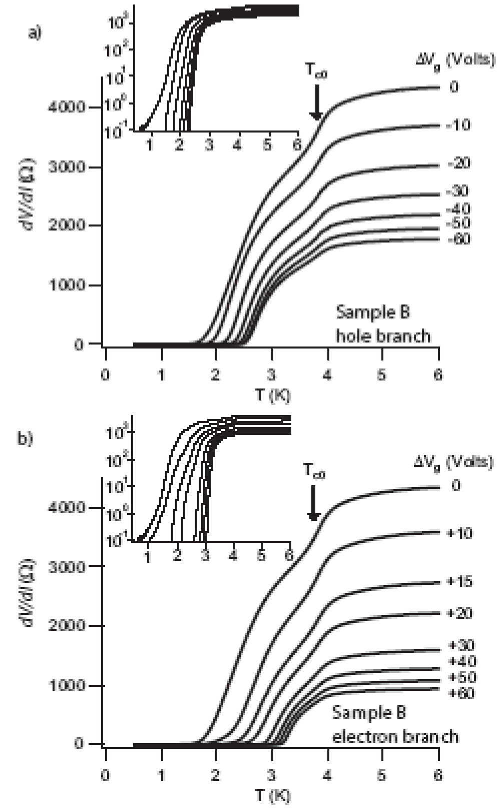


FIG. 2: Sheet resistance versus temperature for various gate voltages, $V_g$, referenced to the charge neutrality point $V_D = +40$ V for this device. In a) $\Delta V_g = V_g - V_D < 0$ corresponds to hole transport, whereas, $\Delta V_g > 0$ in b) corresponds to electron transport through the graphene sheet. The arrow labeled $T_{c0}$ indicates the first partial resistance drop corresponding to the mean-field pairing transition of the Sn islands. Inset: Same data on a log scale.

New experiments are on the way:
Bilayer graphen, STM probe,
Low-fraction covering

# 5) Open problems

DoS smearing at low temperatures

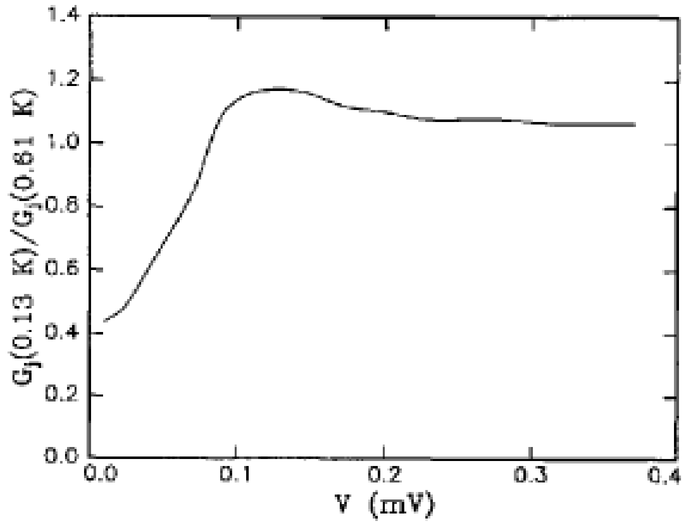


Fig. 16. Tunnel junction conductance of a 5.8 kΩ/□ Bi film at $T = 0.13$ K normalized to its conductance at $T = 0.61$ K [40].

Mesoscopic fluctuations of Hc2 at g near gc

[ $\delta H_{c2} / H_{c2} \sim 1/g$ at $g >> gc$ ]