Introduction to the Finite Element Method

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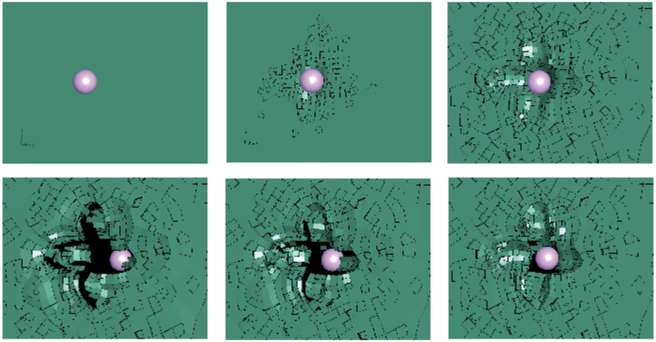
- Widely used by researchers – structural and solid mechanics, fluid flow, heat transfer, electricity and magnetism and various coupled problems
- Routinely used in the industry – design of buildings, airframes, electric motors, automobiles, materials

Most problems of interest to engineers and scientists cannot be solved analytically

From 'Projectile Impact on a Carbon Fiber Reinforced Plate', Abaqus Technology Brief, Simulia Corporation, 2007



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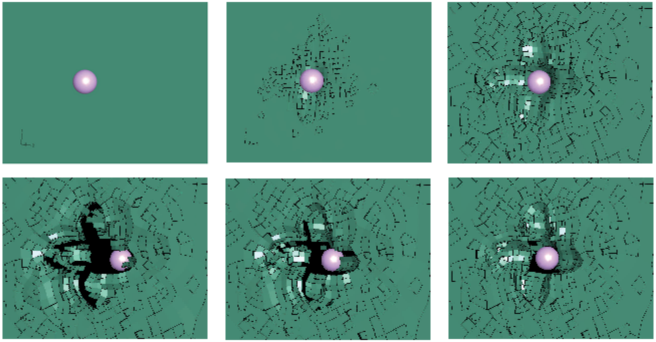

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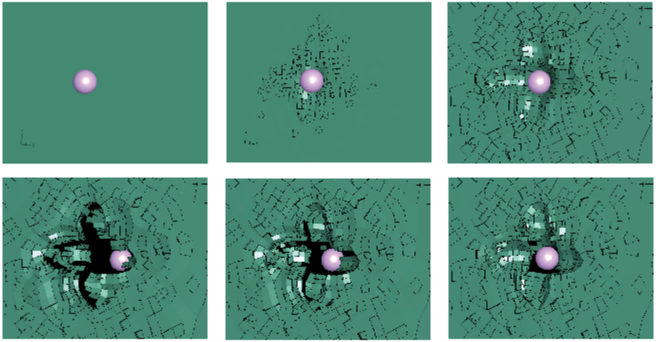

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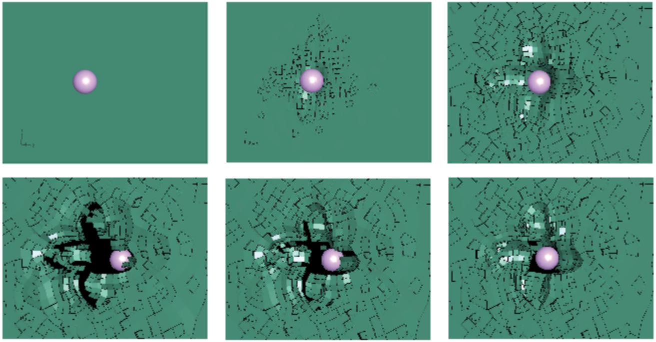

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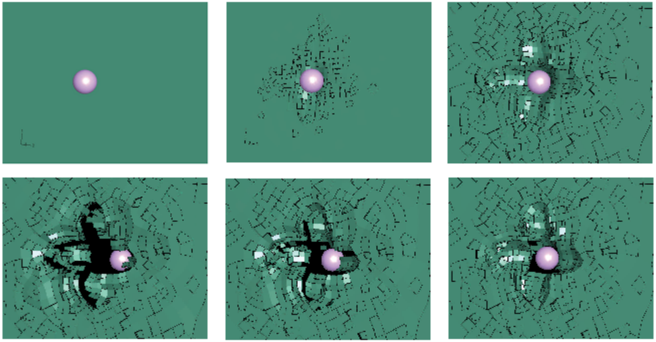

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- Analytical solution is out of the question!

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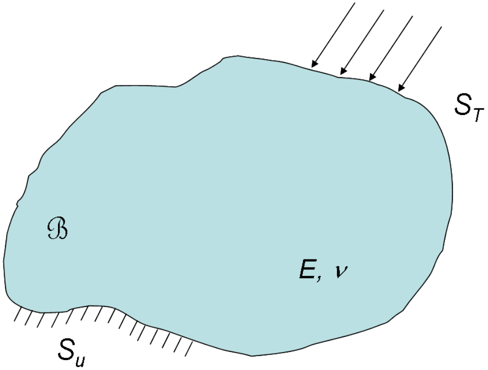
- 100 unknowns large in 1960s, now 10^6 unknowns routine!

We are interested in calculating the displacements and stresses everywhere in a deforming elastic body



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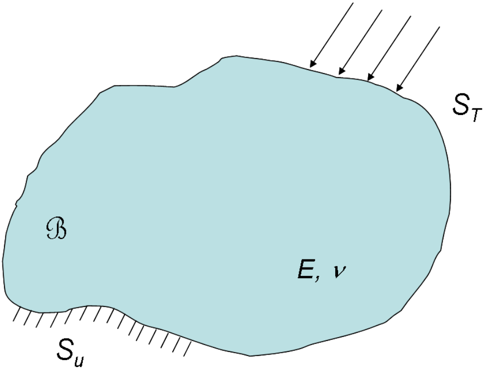

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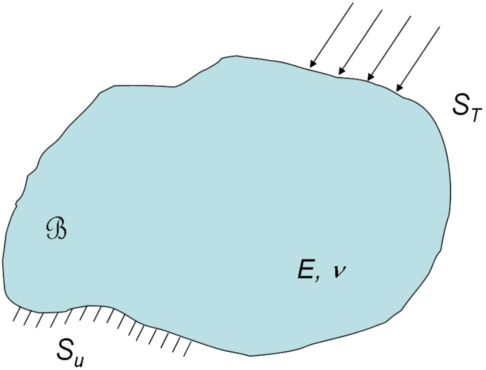

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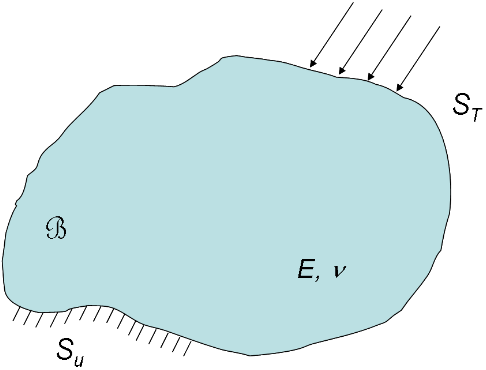

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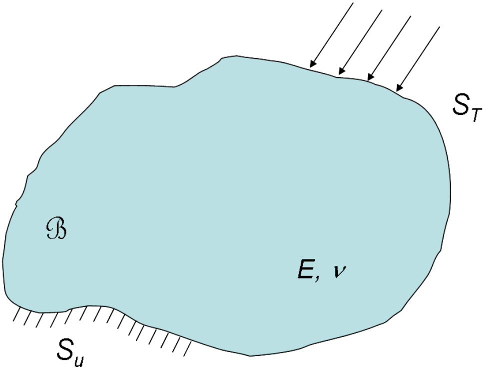

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- We know the forces and displacements imposed on the body
- Force F applied to a bar of cross-section A, stress $\sigma = F/A$
- **For arbitrary 3-D solids, how do we compute the displacements and stresses everywhere in the body?**

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The governing equation (no body forces, quasi-static case) is the **linear momentum balance** equation:

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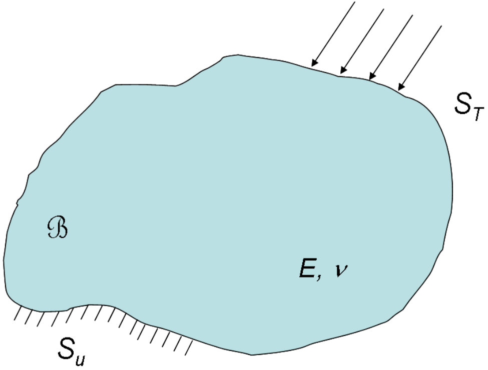
$$\boxed{\sigma_{ij,j} = 0, \qquad i = 1, 3}$$

... boundary conditions [2], ...



$$\sigma_{ij} n_j = T_i = T_i^{sp} \text{on } S_T$$
$$u_i = u_i^{sp} \text{on } S_u \qquad \boxed{2}$$

S_T and S_u are the portions of external surface S on which forces and displacements are specified respectively.

... constitutive equations $\boxed{3}$ ...

Constitutive Equation - Hooke's Law for isotropic, linear elasticity:

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right] \qquad \boxed{3}$$

E - Young's modulus, ν - Poisson's ratio, ϵ_{ij} - strain tensor components, δ_{ij} - Kronecker delta

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$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \boxed{4}$$

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Strain-displacement relations:

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$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

reduces to the familiar definition of $\epsilon = \Delta l / l$ for the 1-D case.

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$$\boxed{\sigma_{ij,j} = 0, \qquad i = 1,3},$$

- we solve

$$\boxed{\int_V u_i^* \sigma_{ij,j} dV = 0}$$

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- Using the divergence theorem and the strain-displacement relations, we can restate the integral as

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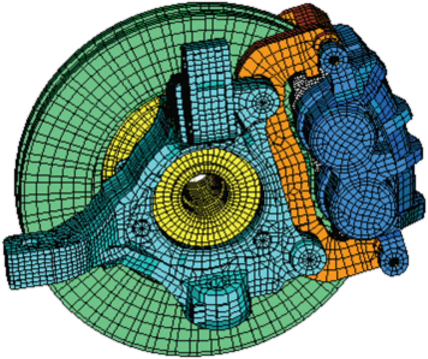
- This is not an energy balance, it is merely a restatement of equilibrium!

How is a finite-element solution implemented?



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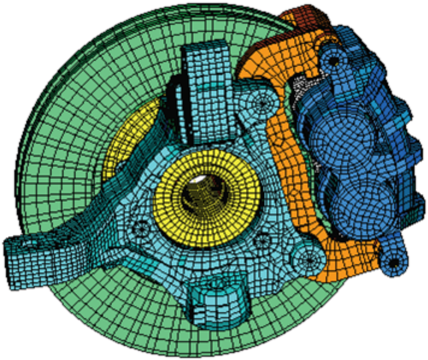

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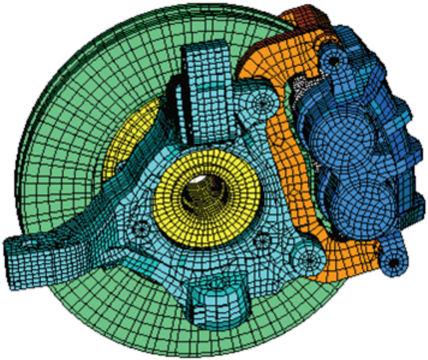

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- Nodes are typically points on the element boundaries

Displacement at any point in the solid is expressed in terms of nodal displacements

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$$\mathbf{u}(x_1, x_2, x_3) = \sum_{i=1}^{3} N^i(x_1, x_2, x_3)\mathbf{u}^i$$

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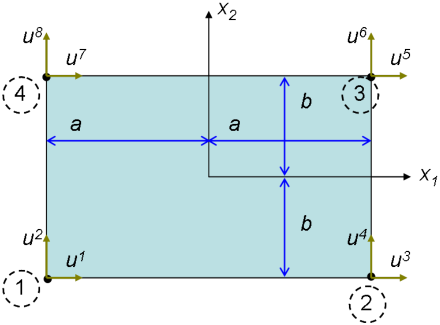
- For full 3D analysis, ($3 \times \#$ of nodes) unknowns

Linear shape functions are the simplest



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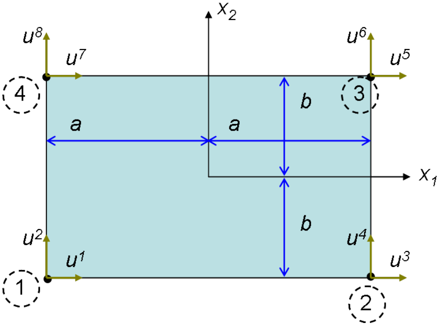

$$
\begin{aligned}
u_1(x_1, x_2) &= \sum_{i=1}^{4} N^i(x_1, x_2) u^{2*i-1} \\
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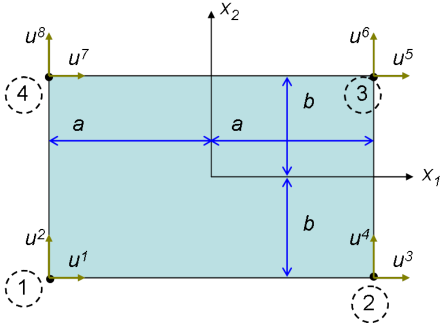

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$$N^1 = \frac{(a-x_1)(b-x_2)}{4ab} \qquad N^2 = \frac{(a+x_1)(b-x_2)}{4ab}$$

$$N^3 = \frac{(a+x_1)(b+x_2)}{4ab} \qquad N^4 = \frac{(a-x_1)(b+x_2)}{4ab}$$

Displacements are written in vector form

$$\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left[\begin{array}{cccccccc} N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 & 0 \\ 0 & N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 \end{array} \right] \left\{ \begin{array}{c} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \\ u^6 \\ u^7 \\ u^8 \end{array} \right\}$$

Derivatives of shape functions relate strains to displacements

Recall strain-displacement relations:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

In vector form,

$$\boldsymbol{\epsilon} = \left\{ \begin{array}{c} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{array} \right\} = \mathbf{[B]\{u^i\}}$$

where

$$\mathbf{[B]} = \left[\begin{array}{cc} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ (1/2)\ \partial/\partial x_1 & (1/2)\ \partial/\partial x_2 \end{array} \right] \mathbf{[N]}$$

Matrix of elastic constants relates stress components to strain components

Recall Hooke's law:

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right]$$

In matrix form, we can write this as

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\epsilon}\} = [\mathbf{D}][\mathbf{B}]\{\mathbf{u^i}\}$$

where **[D]** is a matrix of elastic constants

Substitution into PVW gives element equilibrium equations

Recall the PVW:

$$\int_V \epsilon_{ij}^* \sigma_{ij} dV = \int_S F_i u_j^* dS$$

In matrix form,

$$\int_V \{\boldsymbol{\epsilon}^*\}^T \{\boldsymbol{\sigma}\} dV = \int_S \{\mathbf{u}^*\}^T \{\mathbf{F}\} dS$$

$$[\mathbf{K}]\{\mathbf{u^i}\} = \{R\}$$

where

$$[\mathbf{K}] = \int_V [\mathbf{B}]^{\mathbf{T}} [\mathbf{D}][\mathbf{B}] dV \text{ and } \{\mathbf{R}\} = \int_S [\mathbf{N}]^T \{\mathbf{F}\} dS$$

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- Using the finite-element matrices, strains and stresses are then computed

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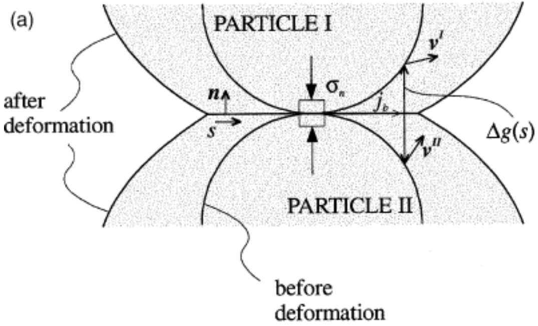
- Fatigue and fracture studies

A finite-element formulation was used to study hot-isostatic pressing of powders



- Bulk deformation: Elasticity + power-law creep
- Interparticle mass diffusion
- Surface diffusion

Modeling the interaction between densification mechanisms in powder compaction (2001), S. J. Subramanian and P. Sofronis, Int. J. Sol. Struct., 38, 7899-7918

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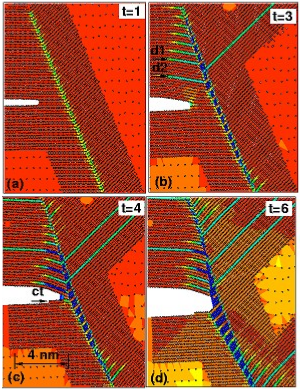
- http://www.qcmethod.com: a good resource

Quasi-continuum method used to study crack propagation in Ni



Reference: *Quasicontinuum simulation of fracture at the atomic scale (1998)*, R. Miller, E. B. Tadmor, R. Phillips and M. Ortiz, Modelling and Simulation in Materials Science and Engineering, vol. 6, 607-638

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