Burgers turbulence: kinetic theory and complete integrability.

Govind Menon,
Division of Applied Mathematics,
Brown University.

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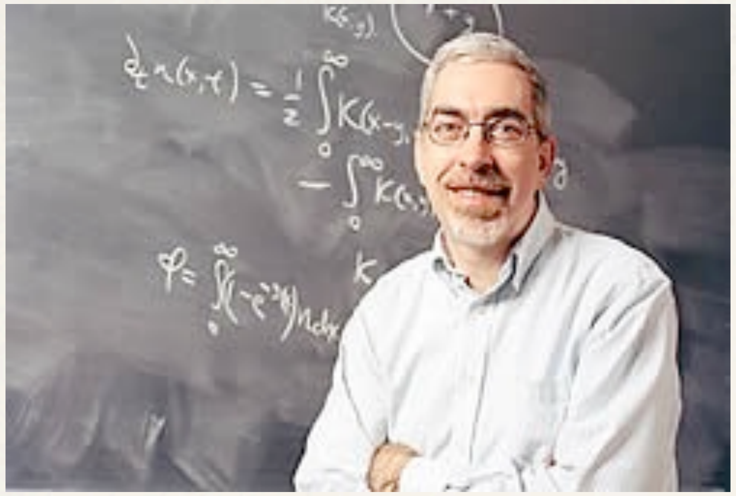
Joint work with:



Bob Pego,
Carnegie Mellon University

Ravi Srinivasan,
Univ. of Texas, Austin

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Burgers cartoon of turbulence

Solve the differential equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, t > 0,$$

with white noise as initial data.

J. M. Burgers (1929--1974).

Motivation

1) The pde is a caricature of the fundamental equations of fluid mechanics.

2) White noise as initial data seems reasonable....

3) Allows us to formulate precisely a statistical theory of turbulence, i.e. random processes that also solve equations of mechanics, even if in a vastly simplified setting ("Burgulence").

Some basic facts about Burgers equation

1) Global classical solutions do not exist.

2) Weak solutions are not unique.

3) There is a unique entropy solution, which is a vanishing viscosity limit (E. Hopf, 1950).

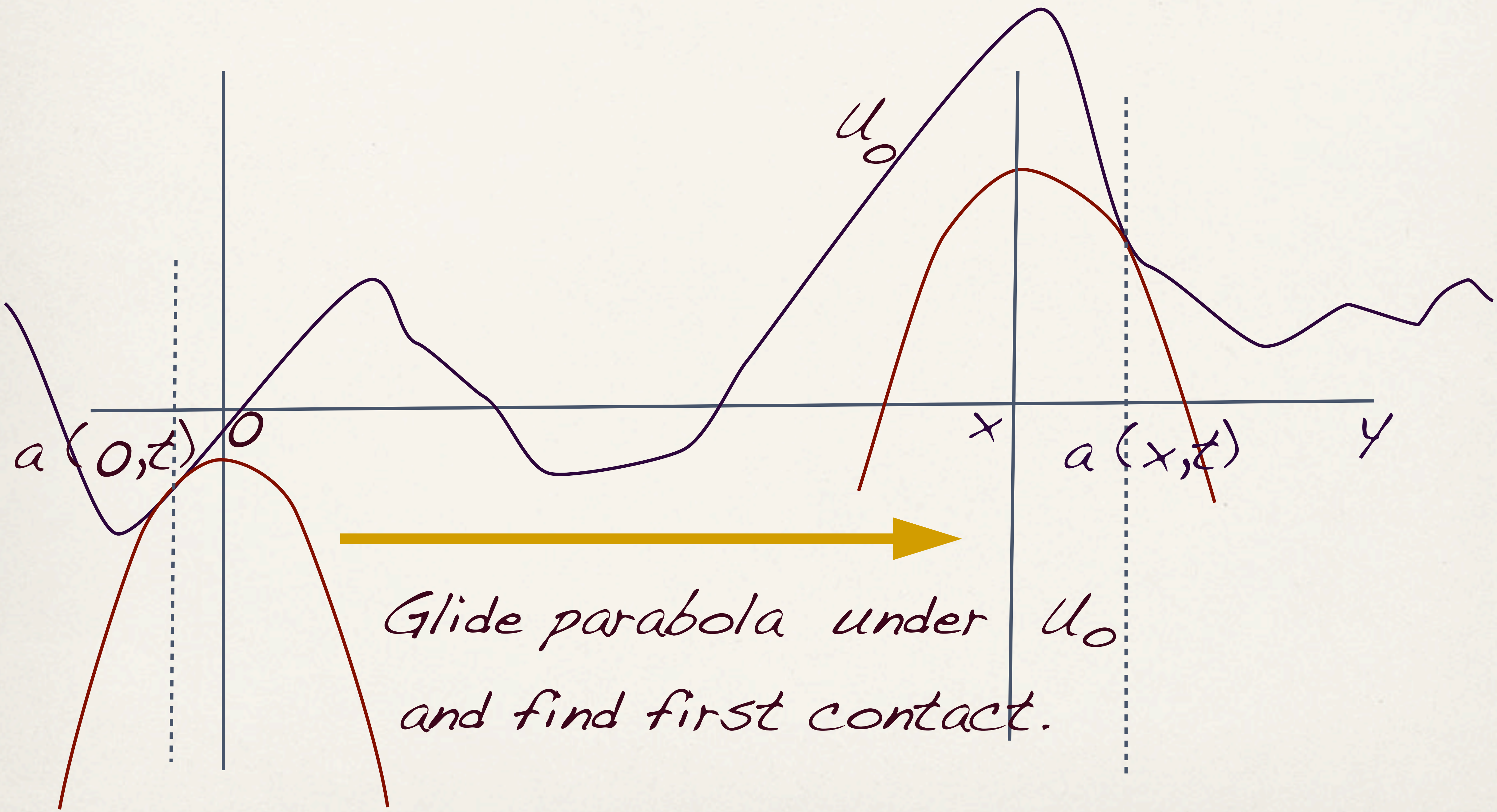
The unique entropy solution, or the Cole-Hopf solution, is given by a variational principle.

$$u(x,t) = \frac{x - a(x,t)}{t}$$

$$a(x,t) = \operatorname{argmin}_y^+ \left\{ U_0(y) + \frac{(x-y)^2}{2t} \right\}$$

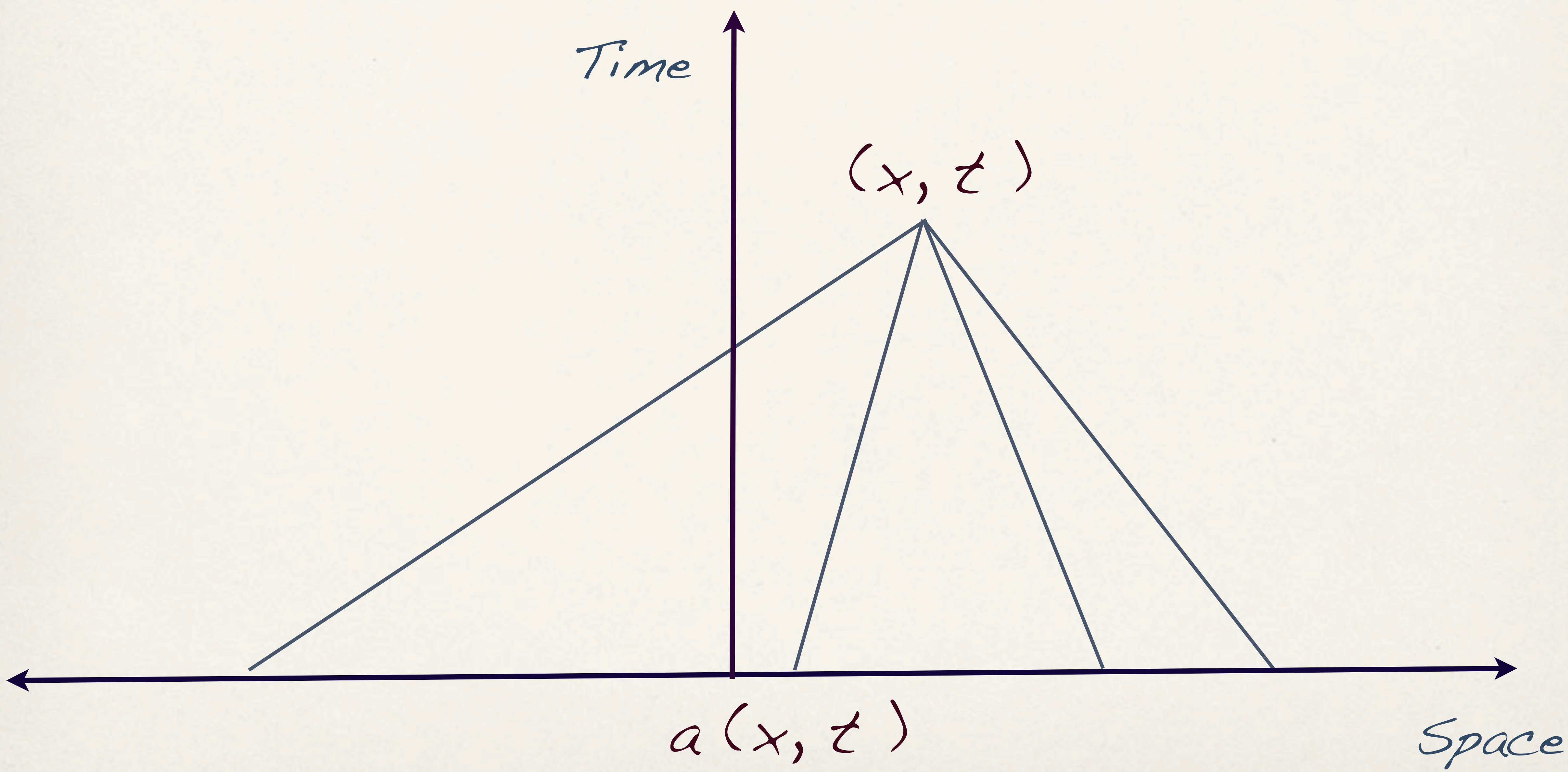
$$U_0(y) = \int_0^y u_0(s)\, ds$$

$u(x,t)$ is the velocity field. ψ is called the potential and $a(x,t)$ the inverse Lagrangian function. The variational principle is a geometric recipe that uses the potential.

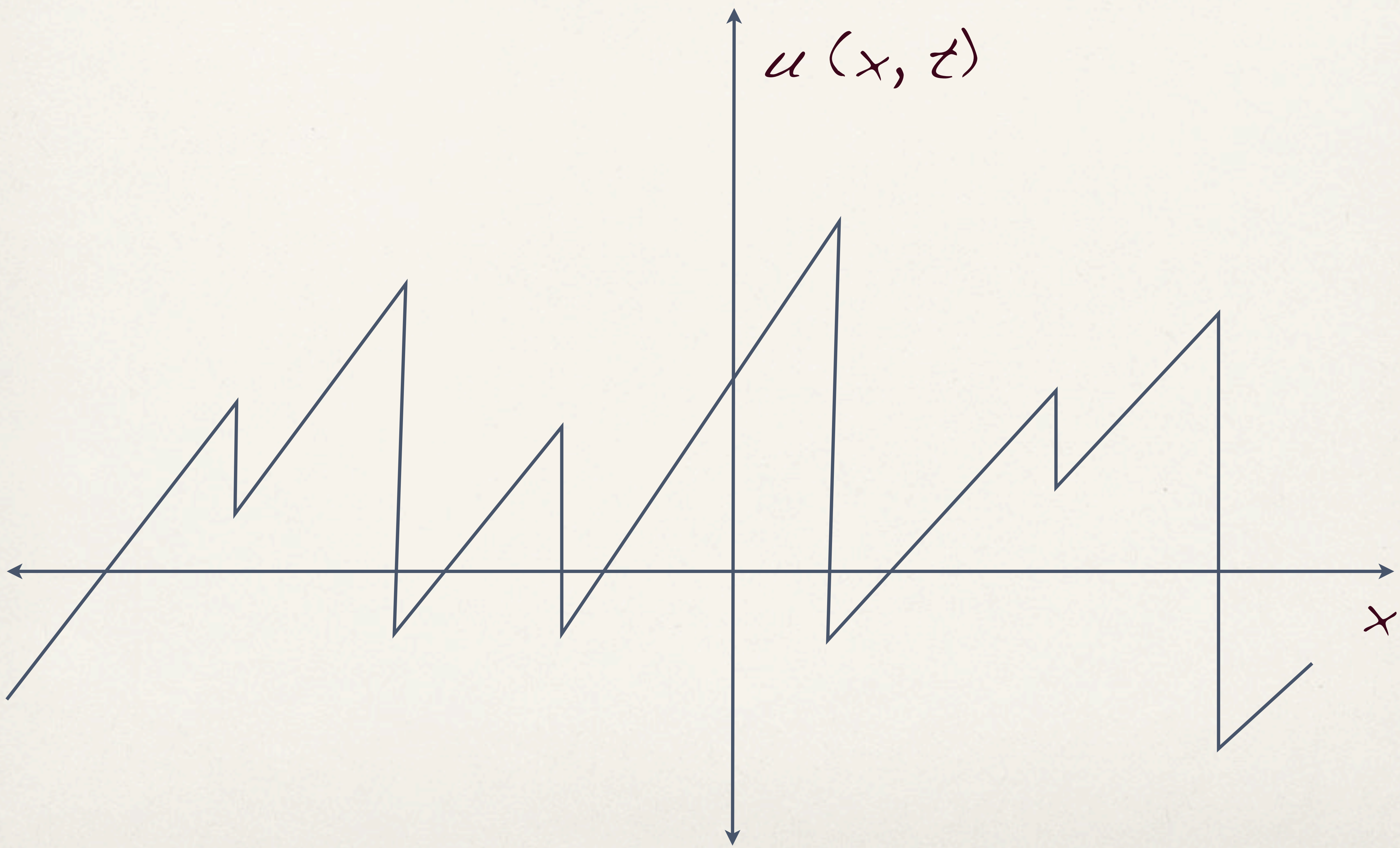


1) $a(x,t)$ gives the 'correct' characteristic through the point (x,t) in space-time.

2) $a(x,t)$ is increasing in x. Can only jump up.



3) As a consequence, $u(x,t)$ is of bounded variation. Jumps in a give rise to shocks in u. These correspond to 'double-touches' in the geometric principle.



Numerical experiments with white noise data.

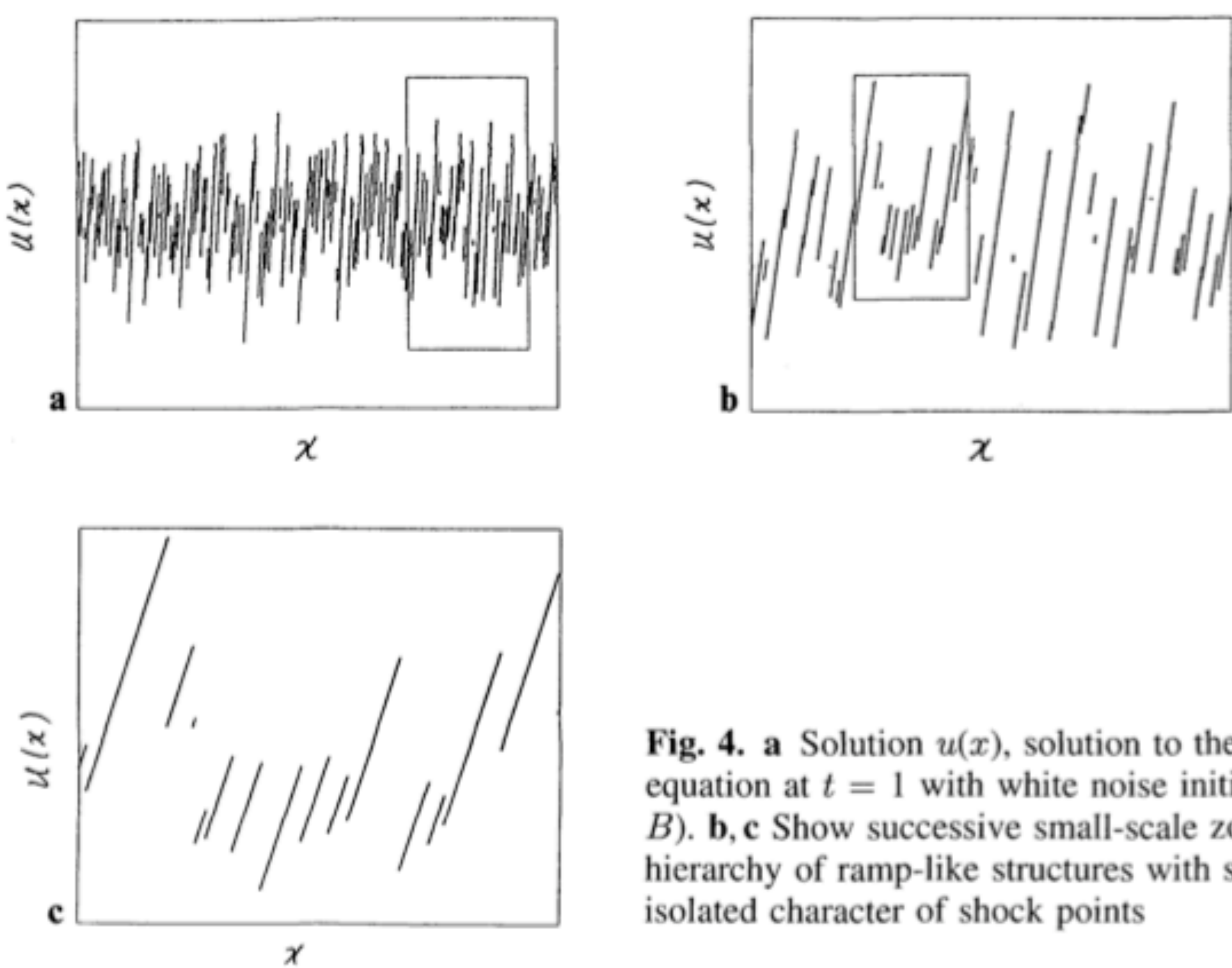


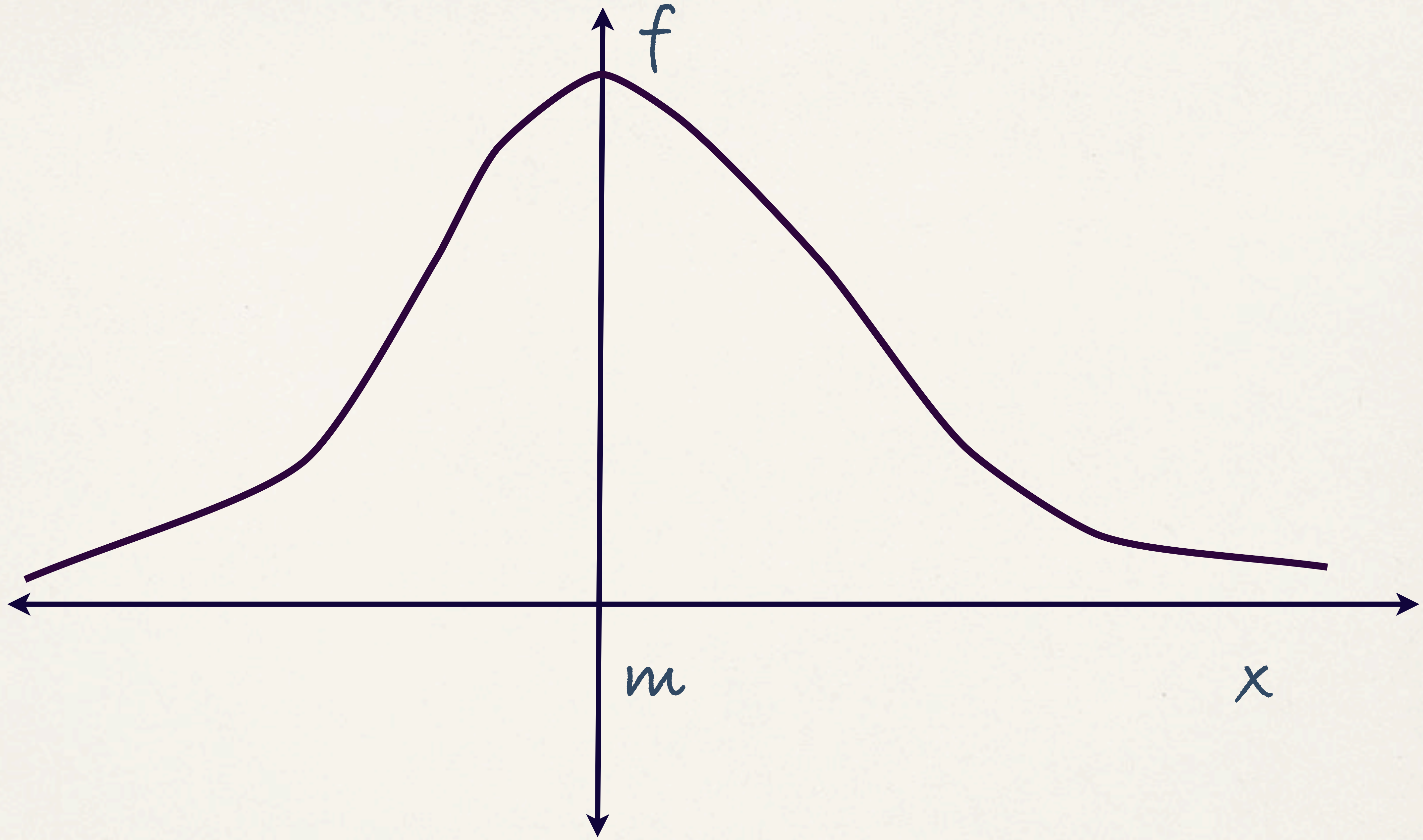
Fig. 4. a Solution $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type B). **b, c** Show successive small-scale zooming. Note the hierarchy of ramp-like structures with slope 1 and the isolated character of shock points

She, Aurell, Frisch, Commun. Math. Phys. 148, (1992) This problem has a remarkable exact solution.

P. Groeneboom, Brownian motion with a parabolic drift and Airy Functions, Prob.Th. Rel. Fields, 81, (1989).

L. Frachebourg, P. Martin, Exact statistical properties of the Burgers equation, J. Fluid Mech, 417, (2000).

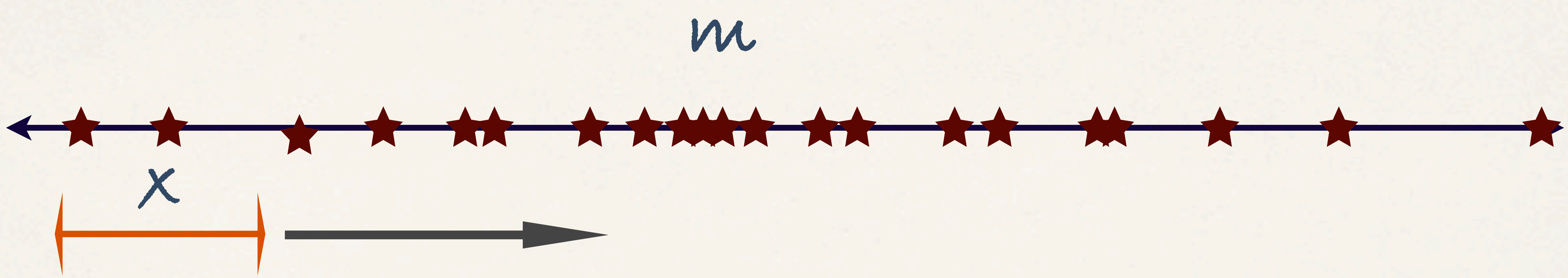
Estimating the mode (Chernoff, 1964)



Given n samples from a unimodal distribution,

how do we estimate the <u>mode</u> m?

Naive "binning" strategy given n samples:



$N_n(x)$ = number of points in bins centered at x.

Guess (estimator) : $m_n = \text{argmax } N_n(x)$.

The estimator converges to the true mode as n increases. This is as expected.

The surprise is that the difference $m_n - m$ has large fluctuations. Precisely:

$$n^{1/3}(m_n - m) \xrightarrow{\mathcal{L}} \text{argmax}_s \left(U_0(s) - \frac{s^2}{2} \right),$$

where U_0 is a two-sided Brownian motion.

Initially thought to be a curiosity, this example is representative of a large number of limit theorems for estimators near <u>extrema</u>.

Kim and Pollard, "Cube-root fluctuations", Ann. Stat., (1990).

A first glimpse at Groeneboom's solution

The one-point distribution of u at time 1 has density

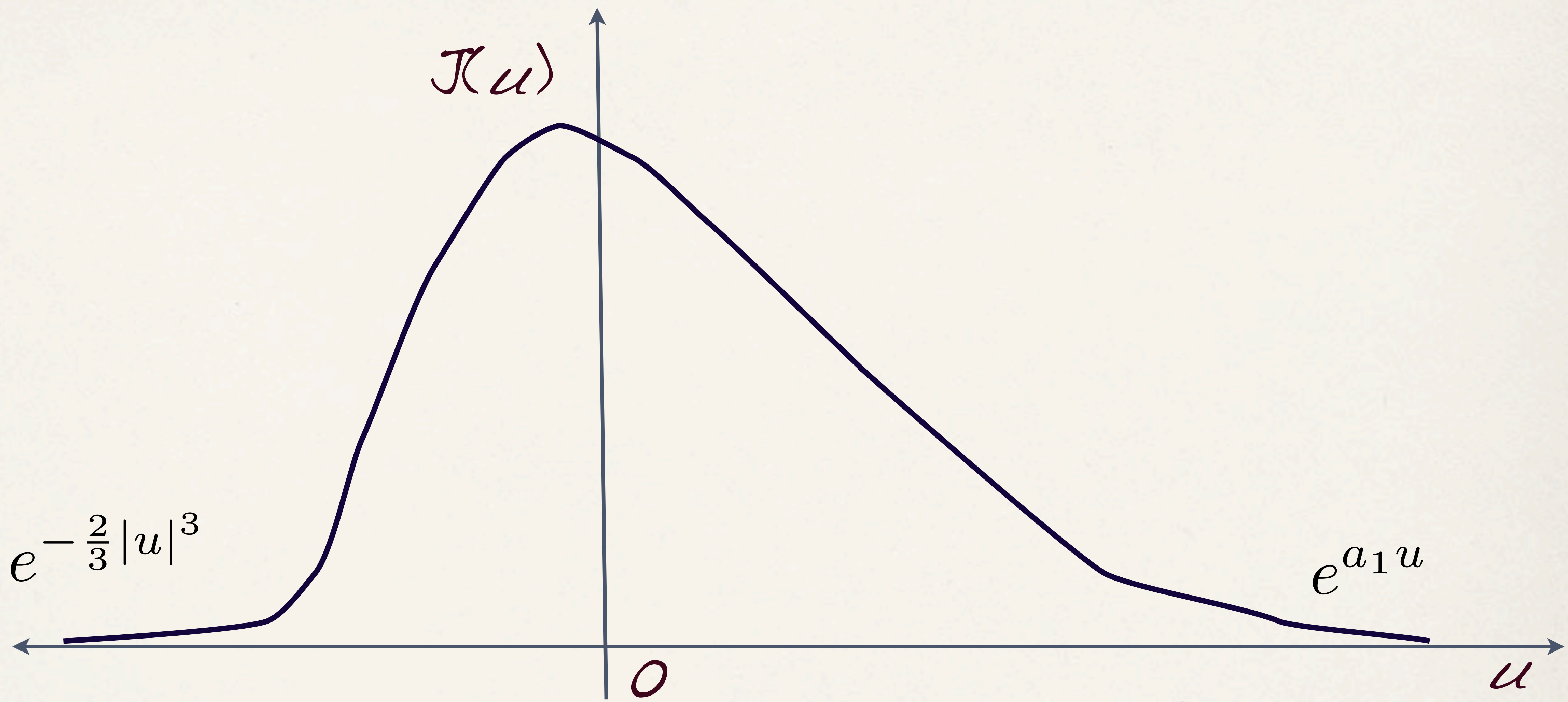
$$p(u) = J(u)J(-u), \quad u \in \mathbb{R}.$$

The function J has an explicit Laplace transform

$$\int_{-\infty}^{\infty} e^{-qu} J(u)\, du = \frac{1}{Ai(q)},$$

where $Ai(q)$ is the (first) Airy function.

Classical Tauberian theorems yield asymptotics of J.



a_1 is the first zero of the Airy function.

$$p(u) = J(u)J(-u) \sim e^{-\frac{2}{3}|u|^3}, \qquad u \to \infty.$$

The general question

Let f be convex. What can we say about the statistics of the entropy solution to

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0,$$

when the initial data is random?

More precisely, how do we describe the n point statistics for $u(x,t)$ and how does this relate to the coalescence of shocks?

Markov processes and their generators

A Markov process is characterized by its transition semigroup Q and generator $\mathcal{A}$. For suitable test functions, we have

$$\mathcal{A}\varphi = \lim_{h \downarrow 0} \frac{Q_h \varphi - \varphi}{h}.$$

For Markov processes, the n point distribution factors into 2 point distributions.

Generators of spectrally negative Markov processes

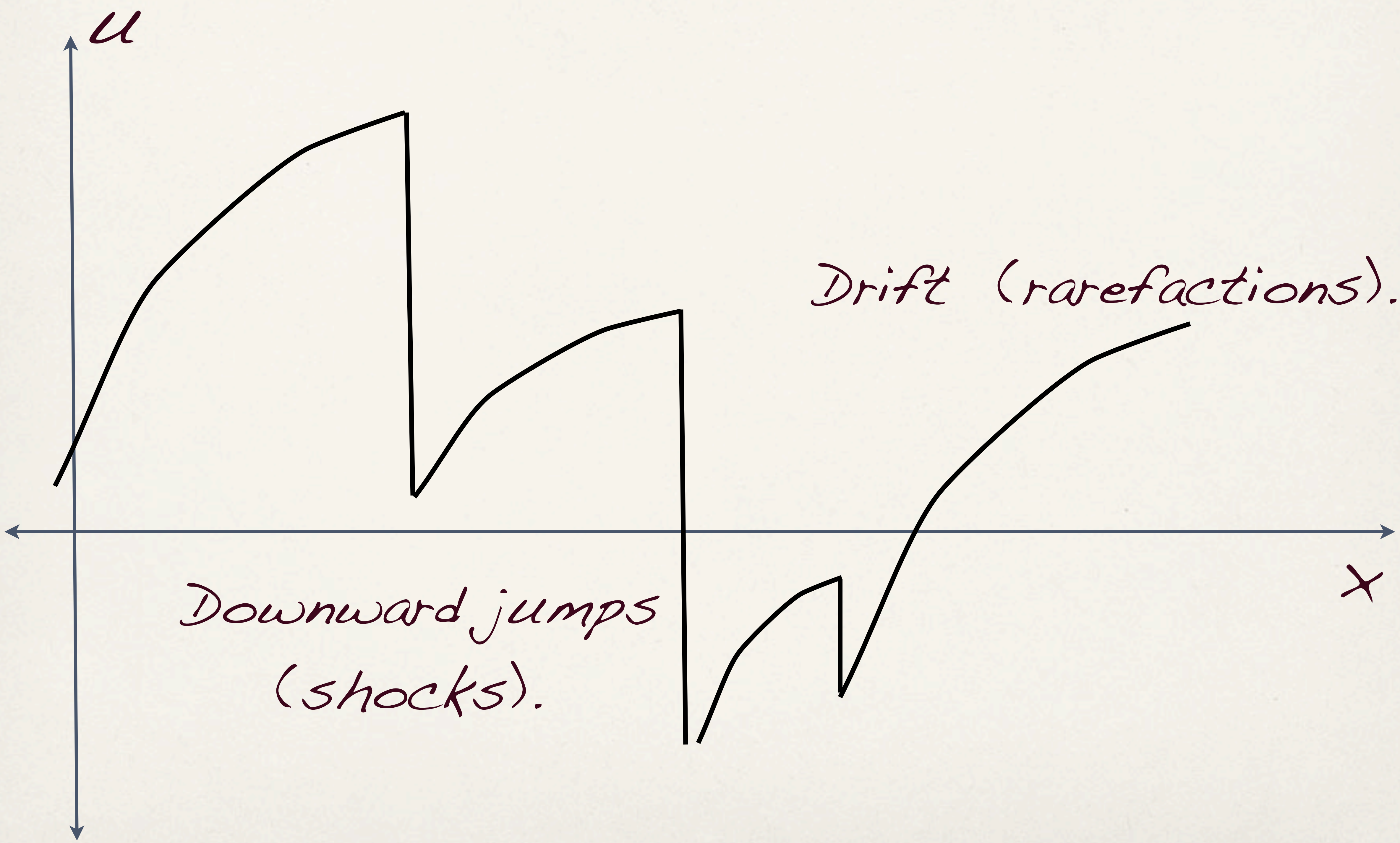
A Markov process with BV sample paths and only downward jumps has an infinitesimal generator

$$\mathcal{A}\varphi(u) = \underbrace{b(u)}\,\varphi'(u) + \int_{-\infty}^{u} \underbrace{n(u,v)}\,\left(\varphi(v) - \varphi(u)\right)\,dv.$$

Drift at level u. | *Jumps from u to v.*

(rarefactions) | *(shocks)*

Typical profile of solutions to a scalar conservation law (not just Burgers)



Closure theorem (Srinivasan, Ph.D thesis 2009)*

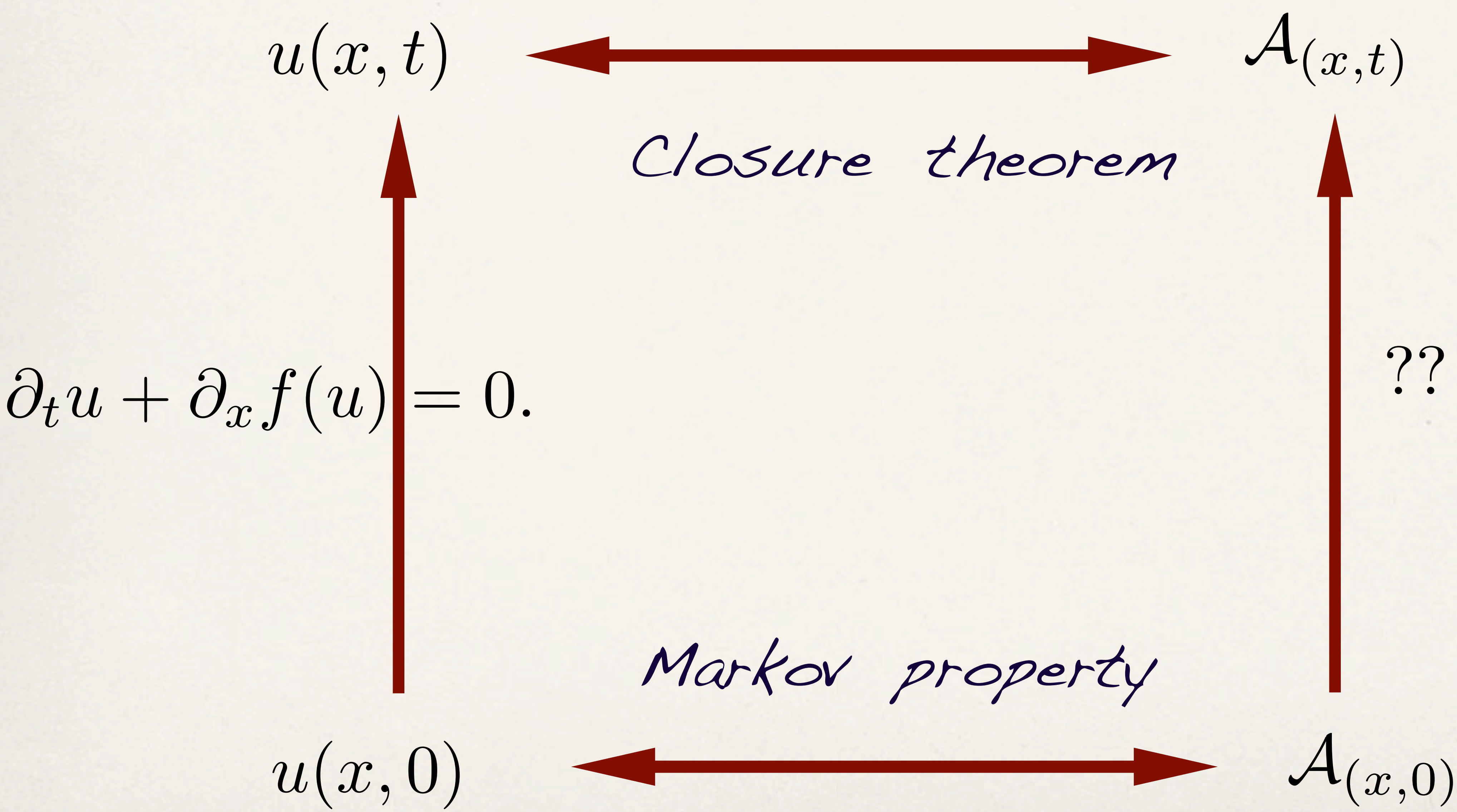
$$\partial_t u + \partial_x f(u) = 0.$$ f strictly convex.

Thm. 1. Assume the initial velocity $u(x,0)$ is a Markov process with only downward jumps.

Then so is the solution $u(x,t)$ for every $t>0$.

* Here by the term closure we mean that this class of random processes is preserved by the entropy solution.

Since the process is Markov, it has an infinitesimal generator that depends on (x,t). Conceptually, we have the following picture.



The "generator" in time

First recall the definition of the generator:

$$\mathcal{A}\varphi(u) = b(u)\varphi'(u) + \int_{-\infty}^{u} n(u,v)\left(\varphi(v) - \varphi(u)\right)\,dv.$$

Now define an associated operator (here f is the flux function in the scalar conservation law):

$$\mathcal{B}\varphi(u) = -f'(u)b(u)\varphi'(u) - \int_{-\infty}^{u} \frac{f(v) - f(u)}{v-u} n(u,v)\left(\varphi(v) - \varphi(u)\right)\,dv.$$

The backward Kolmogorov equations

There is a backward equation associated to every Markov process. Since we have a two-parameter process, we obtain two backward equations.

$$\partial_x \varphi + \mathcal{A}\varphi = 0, \qquad \text{and} \qquad \partial_t \varphi + \mathcal{B}\varphi = 0.$$

The Lax equation

Since we have semigroups in x and t, we ask for compatibility between these semigroups, i.e.

$$\partial_t \partial_x \varphi = \partial_x \partial_t \varphi$$

If this holds for a large enough class of functions we obtain the Lax equation

$$\partial_t \mathcal{A} - \partial_x \mathcal{B} = [\mathcal{A}, \mathcal{B}].$$

The Lax equation may also be derived by:

2. Elementary arguments of kinetic theory and the evolution of a single shock and rarefaction wave.

3. Vol'pert's BV chain rule, the Markov property and an unjustified interchange of limits.

4. Hopf's method: a formal evolution equation for the Fourier transform of the law of $u(x,t)$.

Main results for stationary processes.

$u(x,t)$ ⟷ (Closure theorem) ⟷ $\mathcal{A}(t)$ ⟷ (Characterization of generator) ⟷ (b_t, n_t)

$\partial_t u + \partial_x f(u) = 0.$

$\dot{\mathcal{A}} = [\mathcal{A}, \mathcal{B}]$

Kinetic equations.

$u(x,0)$ ⟷ $\mathcal{A}(0)$ ⟷ (b_0, n_0)

The kinetic equation for clustering (Burgers)

$$\dot{b} = -b^2, \qquad \partial_t n(u,v,t) = \underbrace{D(b,n)}_{\text{Drift}} + \underbrace{Q(n,n)}_{\text{Collisions}}.$$

$$D(b,n) = \left(\frac{u-v}{2}\right)\left(b(u)\partial_u n - \partial_v(b(v)n)\right)$$

$$Q(n,n) = \frac{u-v}{2}\int_v^u n(u,w)n(w,v)\,dv \quad \longleftarrow \text{Birth}$$

$$-n(u,v)\int_{-\infty}^v n(v,w)\left(\frac{u-w}{2}\right)dw$$

$$-n(u,v)\int_{-\infty}^u n(u,w)\left(\frac{w-v}{2}\right)dw$$

Death

Groeneboom's solution (Burgers with white noise)

$$b(u,t) = \frac{1}{t}, \quad n(u,v,t) = \frac{1}{t^{1/3}} n_* \left(\frac{u}{t^{2/3}}, \frac{v}{t^{2/3}} \right).$$

The jump density factorizes into:

$$n_*(u,v) = \frac{J(v)}{J(u)} K(u-v),$$

where J and K have Laplace transforms:

$$j(q) = \frac{1}{Ai(q)}, \qquad k(q) = -2 \frac{d^2}{dq^2} \log(Ai(q)).$$

More on Groeneboom's solution

In order to verify that this is a solution we need to use some interesting identities. These are best written in terms of the variable $e = j'/j$. Then

$$e' = -q + e^2,$$

← Riccati eqn.

$$k' = -2(1 - ek),$$

$$k''' = 6kk' + 4qk' + 2k.$$

These yield three moment identities, such as

$K * J(x) = x^2 J - J'$ and some amazing cancellations.

The Painleve property

In fact, e is the first Airy solution to Painleve 2.

$$w'' = 2w^3 + 2wq + \frac{1}{2}.$$

Self-similar solutions to several completely integrable systems (Kdv, NLS, Sine-Gordon) can be expressed in terms of Painleve transcendents. They also appear in famous 'solvable' problems in mathematical physics such as random matrix theory and the 2-D Ising model.

Complete integrability.

Hamiltonian systems

The basic example is:

$$\dot{x} = J\nabla H, \quad x \in \mathbb{R}^{2n},$$

where J is the symplectic matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth Hamiltonian.

Liouville's theorem

The Poisson bracket of two Hamiltonians is

$$\{H, F\} = \langle \nabla H, J \nabla F \rangle \, .$$

The Hamiltonian vector fields associated to H and F commute if the bracket vanishes.

Liouville's theorem: The flow of H is integrable if we have n non-degenerate Hamiltonians that commute with H.

Examples

Classical (1800's):

1) Geodesic flow on ellipsoids in 3D (Jacobi)

2) Particle constrained on a sphere (C. Neumann)

Modern (1968+):

Toda lattice, KdV, NLS, sine-Gordon, KP,...

Unifying themes: Lax pairs, algebraic structure, analytic techniques, and many "miracles".

Some surprising links (Moser, 1980)

Geodesic flow on ellipsoids

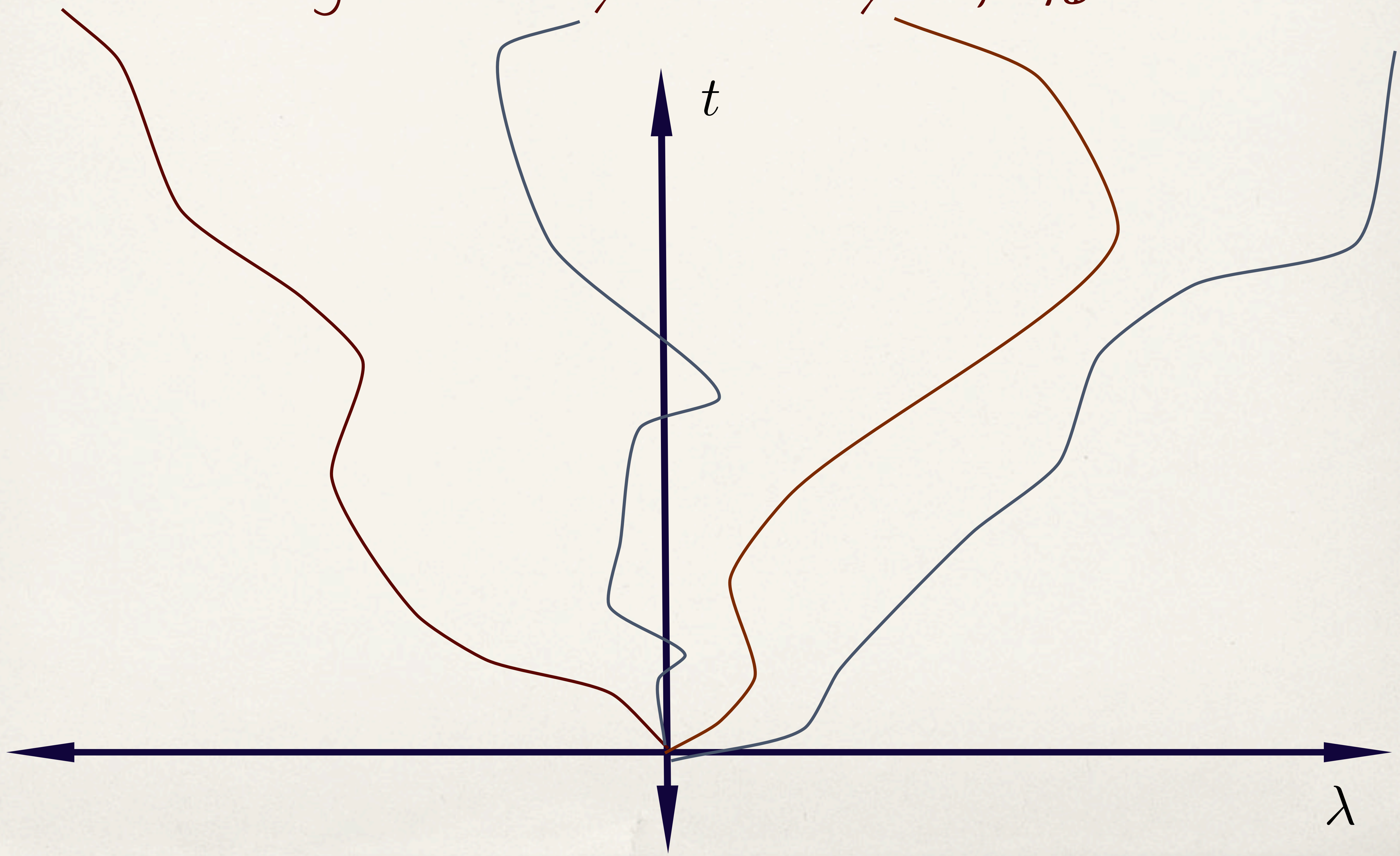
== Constrained motion on spheres

== KdV with spatially periodic data.

== means there are symplectic transformations from one model to the other and explicit solutions.

Random matrices, Burgers turbulence, and complete integrability.

Suppose $M(t)$ is a real, symmetric $n \times n$ matrix with standard, independent Brownian motions as upper triangular entries. The eigenvalues of $M(t)$ act like repelling unit charges driven by noise (Dyson, 1962).



Wigner's semicircle law and Burgers equation

The Cauchy transform of the empirical measure

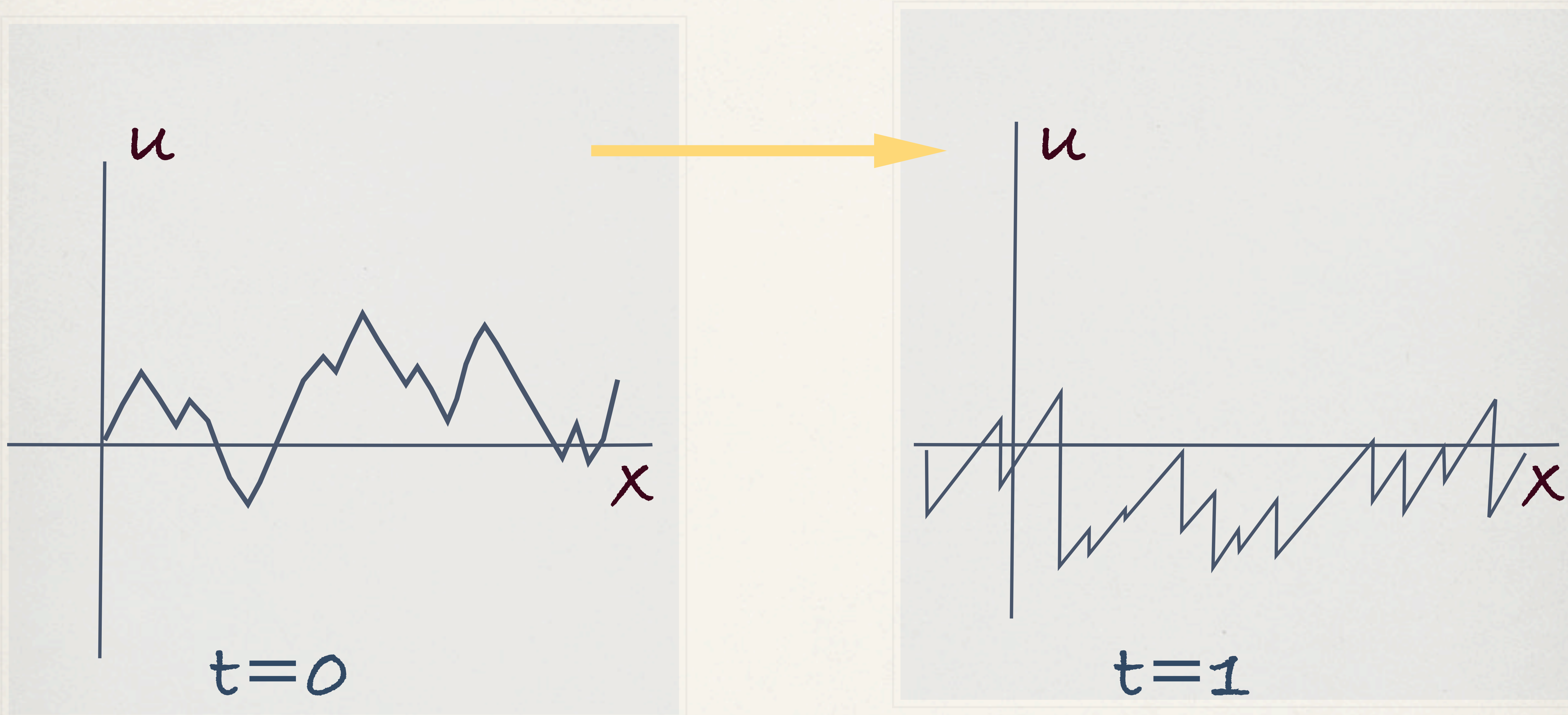
$$g_n(t,z) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{z-\lambda_k(t)},$$

converges (after rescaling by $\sqrt{n}$) to

$$g_t + g g_z = 0, \quad g_0(z) = \frac{1}{z}.$$

$g(z,t)$ is the Cauchy transform of semicircle law.

Brownian motion as initial velocity



Initial data is a one sided Brownian motion (for simplicity).

We then study $u(x,t) - u(0,t)$.

Generators of Levy processes

For Levy processes the transition probabilities are independent of the state, and we may use Fourier analysis. That is, we consider exponentials as test functions and find

$$\mathcal{A} e^{qs} = \psi(q)\, e^{qs}, \qquad q > 0.$$

For example, if $u(x)$ is a Brownian motion,

$$\psi(q) = \frac{q^2}{2}.$$

The Lax equation for Levy processes

We compute the commutator to obtain

$$\mathcal{A}(t)\; e^{qs} = \psi(q,t)\; e^{qs}, \quad q > 0.$$

$$[\mathcal{A}, \mathcal{B}]\; e^{qs} = (-\psi \partial_q \psi)\;\; e^{qs}.$$

$$\partial_t \psi + \psi \partial_q \psi = 0, \quad q, t > 0.$$

Complete integrability

The solution to Burgers equation in the spectral variable q never forms shocks. In fact, it is more natural to write it in characteristics as:

$$\frac{dq}{dt} = \psi, \quad \frac{d\psi}{dt} = 0.$$

But this is a completely integrable Hamiltonian system in action-angle variables....

Burgers turbulence and the semicircle law

$$g_t + g g_z = 0.$$

$$\psi_t + \psi\psi_q = 0.$$

$$g_0(z) = \frac{1}{z}$$

$$\psi_0(q) = q^2.$$

These are linked by the change of variables

$$z = 2 + \frac{1}{q}$$

$$\frac{g(z\sqrt{t}, t)}{\sqrt{t}} = \frac{\psi(q/t, t)}{q/t}$$

Much remains to be done...

1) "Framework issues"

   a) Symplectic structure of flow of measures.

   b) Well-posedness of Lax equation.

2) "Computational issues"

   a) Inverse scattering/ Riemann-Hilbert problems.

   b) Connections to Tracy-Widom laws.