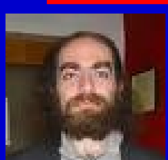
The P vs. NP problem

fficient computation, Internet security,
nd the limits of human knowledge

Avi Wigderson
Institute for Advanced Study

Clay Math Institute Millennium Problems - $1M each

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs. NP
- 
- ~~Poincaré Conjecture~~
- Riemann Hypothesis
- Yang-Mills Theory

Scientific / Mathematical/ Intellectual / Computational problems

NP: Problems we want to solve/understand

P: Problems we can solve/understand

P=NP? - limits on human knowledge

PLAN

- Computation is everywhere
- Algorithms: language of computation
- Efficient algorithms: P
- Efficient verification: NP
- NP-completeness
- Implications

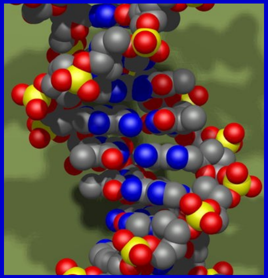
Computation

Mathematics

$X^n + Y^n = Z^n$

Computer

Computation



Physics

Biology

everywhere

Computation: every process which is a sequence of *simple, local* steps, that we want to perform, or understand

Variety of natural phenomena and intellectual challenges, each with an essential computational

1 month

input

2 pm

Fetal development

Weather evolution

3 months

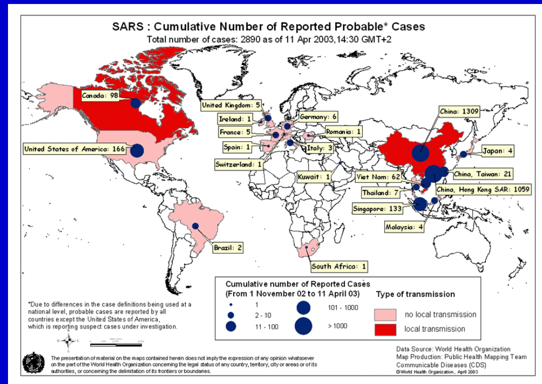
output

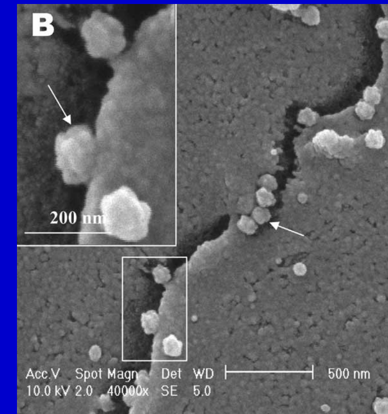
4 pm

Nature computes !

Can we simulate/predict?

4/11/03



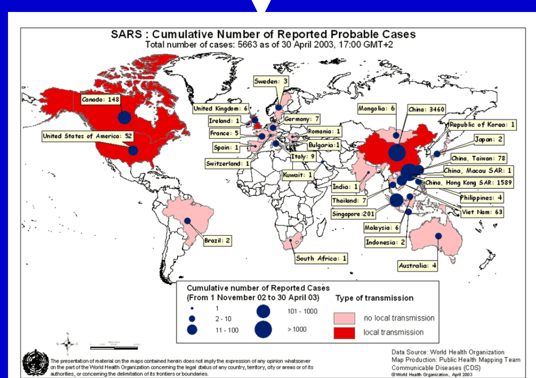


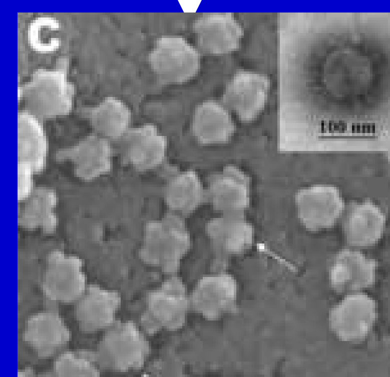
+15h

SARS infection
(in the world)

SARS infection
(in the cell)

4/30/03





24h

Will the epidemic spread, or die out?

$X^2 + Y^2 = Z^2$

$X^n + Y^n = Z^n \quad n>2$

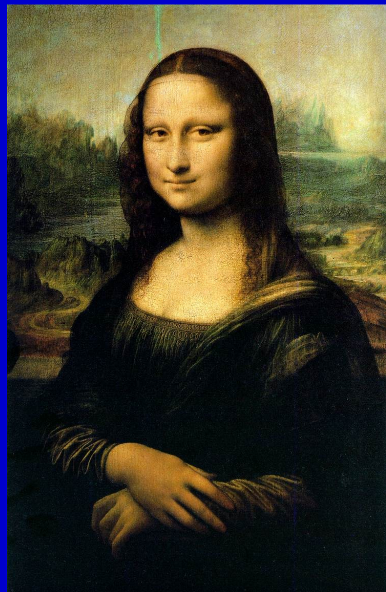
Solving equations

Proving theorems

$X=3 \; Y=4 \; Z=5$

Theorem: no solution!
Proof does not fit on this slide (200 pages)

Computations in Mathematics



Nearly 10,000 reported killed by China quake

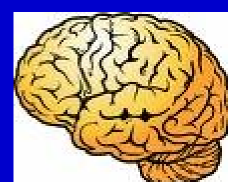
Rain hampering rescue efforts in worst-hit area

Nearly 900 children buried when a school building collapses, 50 bodies found

7.9 magnitude quake is felt throughout much of China

Face recognition

"Mona Lisa"



Emotional reactions

Sadness

The subconscious brain computes

Beauty from computation

Seashells compute

How to describe computation?

The language of Algorithms

Father of Computing

Alan Turing 1912-1954

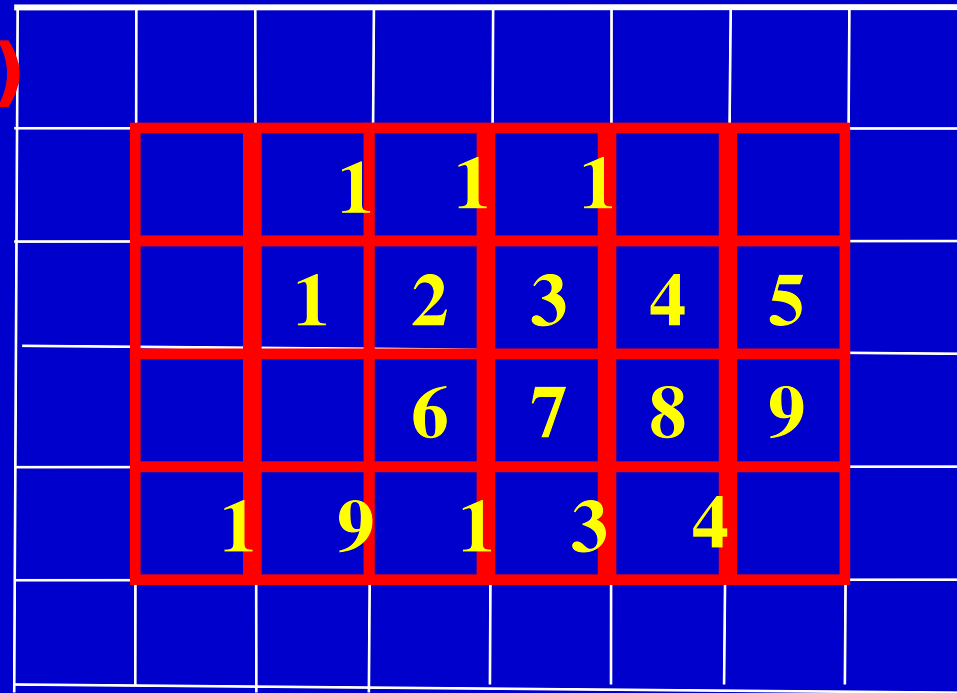
1936: “On computable numbers, with an application to the entscheindungsproblem”

- **Formal definition of algorithm (Turing machine)**
- **Seed of the computer revolution**
- **Church-Turing Thesis: everything that nature computes, can be emulated on a Turing machine**

ALGORITHM (informal)

Step-by-step, **local**,
simple, mechanical
rocedure.

Halts in **finite** time
or every input.



Example: Addition algorithm (informal)

1. Scan column. If empty, stop.
2. Add digits. Write answer, remember carry.
3. Move one column left, write carry.
4. Go to 1

Finite description vs. Infinite # inputs

Limits of Knowledge I

Unsolvable

Turing (& Godel): **Given a computer program, does it always halt?**

Mattiasevich: **Given an equation, does it have an integer solution?**

Conway: Given a

Solvable

When?

Computationa

Complexity

Theory

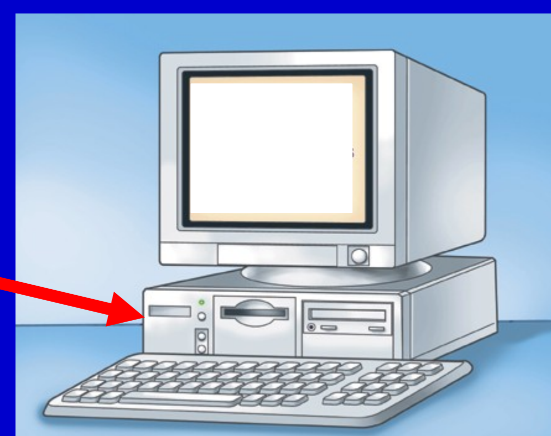
Efficiency of an algorithm -

asymptotic analysis:

Number of basic steps,

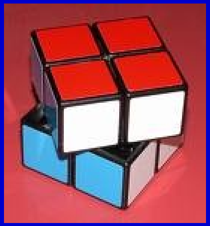
for larger and larger inputs.

input



Rubik’s cube

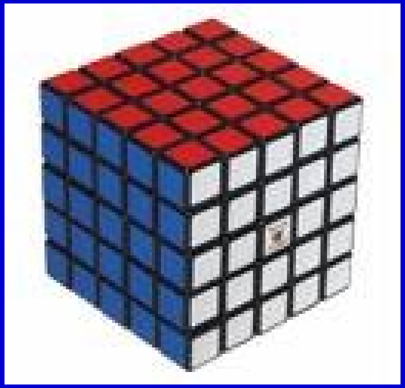
How many steps to solve..



2

3

4



5

…

Sudoku

How long does it take you to solve…

		8	6					
							6	
			4	8			2	3
		5		9				8
	4	9				2	1	
2				4		7		
3	6			2	9			
	1							
					5	1		

3

1			2	3	4			12		6				7	
		8				7			3			9	10	6	11
	12			10			1		13		11			14	
3			15	2			14				9			12	
13				8			10		12	2		1	15		
	11	7	6				16				15			5	13
			10		5	15			4		8			11	
16			5	9	12			1						8	
	2						13			12	5	8			3
	13			15		3			14	8		16			
5	8			1				2				13	9	15	
		12	4		6	16		13			7				5
	3			12				6			4	11			16
	7			16		5		14			1			2	
11	1	15	9			13			2				14		
	14				11		2			13	3	5			12

4

Sudoku

	y				b		a	c	x	n			h			t		f	l			d	e	
	t		s	u	j	h	v				d	q	c			o	k		b	n		a	w	p
w	h	e	m	a	n		l	u	k	p		r	y		s	x	d	q	c	o	j		i	b
b		j		p	s			t				i	m		v	n	g	h	a	q		r	x	y
x	o	l	d		i	p			r	e			f		u	j	w	y	m		h		s	c
	w	q	u	j			i		e		x	b	o	m	a			n	h	k	c	s		
n	c				w	x	u	s	f		q		l			e	m	k	v				j	a
a	i		x	f	c	l			m		v	k	w		q			j		d	g		b	h
s			v		h	k	p	o	b	u	f	j	n				t		d	i	m		r	q
	b	d		m	r	v			j		h	p			o	g	y	w			t		u	
y	p		e	l	a	m		v	h	o	b		x	i	t	s	q	u	w	g	r	c	d	k
	q	g	j		e		s	r		h	c				f	k			x		y	l	a	o
		u	t	k		n	o		l		r	m	q	y		b	a	v	j		i	p	h	
	x	r		w	p		y	k	i		l	e	j					m		t	q	v		u
	s		n	b	q	c		g	w	k	a	u	t	p	y		o		r	x		j	m	
j	n	s	q	v	x	y	h		u	t	p	o	g	l	m		f	d			w	i	k	r
u		w	b	t	l	e	r	p	o	m		c	d	f	k	v					s	q		
d			h		m	s	c	f		q	j		k	n	g	w		b		l	v	u		e
			o	e	d	i	k	n	q		w		u		j	a	l			h		b	p	m
l	k				v	j	t	w		a	s	h					u	r	q	c	d	f		n
			g	d	y	r	w			c		l	i		n	p	v	a	f	e			q	
	v	x	p	o		t	b			d	n	f			w			g		s	a	h	y	i
i		k	w	c	g	q	x	h					a	u	l	d	e		s			m	f	v
		a	y	r		d	f	e	n	x	k		s	h			b		u		p			
q	l		f	s			m	i	v			w			h		x	t	y				c	d

Efficiency of the addition algorithm

1. Add digits. Write answer, retain carry.
2. Move one column left, write carry.
3. Scan column. If empty, stop.
4. Go to 1

6 basic steps per column

5 DIGI TS
30 STEPS

12345
+6789

10 DIG
60 STEPS

123456789
+987654321

20 DIGI TS
120 STEPS

72635273545786043726
+53827484732625435473

50 DIGI TS
300 STEPS

47563739203487456438992305757328576452364568456465744576
2

Is there a faster algorithm?

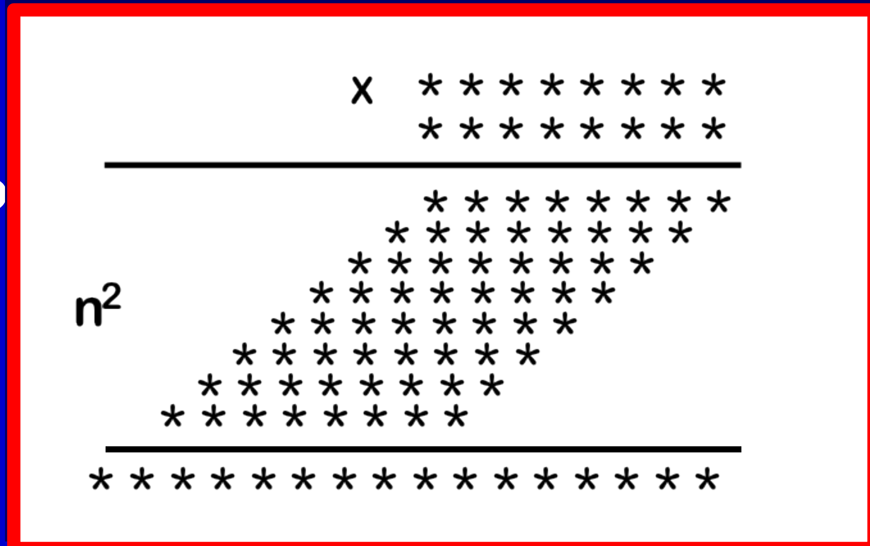
No!

Solving is as fast as reading the input

N DIGIT
6N STEPS

ficiency of the multiplication algorithm

Grade-school multiply algorithm



5 DIGI TS 12345

25 STEPS x6789

10 D

123456789

100

x

987654321

72635273545786043726

400 STEPS x53827484732625435473

50 DIGI TS 47563739203487456438992305757328576452364568456465744576

2500 STEPS 98656092843467546234868431987543210979832865874134653472

I there a faster algorithm? **Yes!**

N DIGIT in a

sec

But not as fast as addition

Efficiency of a factoring algorithm

? × ? = 147,573,952,588,676,412,927

Find **nontrivial factors of a number A**

N DIGITS

$10^{N/2}$ STEPS

Brute force **factoring algorithm**

Input: A

- **For B = 2,3,…√A do:**
- **If B divides A, return B, A/B**

Very slow! 1000 digits → sun will die before finishing

Is there a faster algorithm?

Yes, but still extremely slow!

Which problems are hard to solve?

Addition & Multiplication: Easy

Is Factoring hard ?

Finding efficient algorithms, or proving that no such algorithms exist:
Bread and butter of our field

Cobham, Edmonds
Rabin ~1965

The class P

All problems having an efficient (polynomial time, e.g. n, n^2) algorithm

like Addition and Multiplication

Many practical interesting problems in P

Efficient algorithms –

Drivers of invention & industry

Who were

Edison ? Marconi ? Guttenberg ? Stevenson ?

Light bulb Radio Printing press Steam engine

Shortest path

Dijkstra 1959


```
define Dijkstra(Graph G, Node s)
        S := {}
        Q := Nodes(G)
        while not empty(Q)
            u := extractMin( Q )
            S := S ∪ u
            for each node v in neighbors( u )
                if d(u) + w(u,v) < d(v) then
                d(v) := d(u) + w(u,v)
                pi(v) := u
```

Distance (Delhi, Bangalore)

Path (Delhi, Bangalore)

Network flows

Internet routing

Dynamic Programming

Pattern matching

Knuth-Morris-Pratt

Boyer-Moore 1977

Spell checking

Text pr ng

Genome

Molecular Biology

se

Text CAUCGCGCUUCGC

Pattern CGC

```
algorithm kmp_search:
  input: T (text), P (pattern sought)
  define variables:
    m ← 0, i ← 0, M (the table)
  while m + i is less than length of T, do:
    if P[i] = T[m + i], let i ← i + 1
      if i = length of P then return m
    otherwise, let m ← m + i - M[i],
      if i > 0  let  i ← M[i]
```

Text CAUCGCGCUUCGC

Location X X X

Fast Fourier Transform (FFT)

Cooley-Tukey 1965

Gauss 1805

T(0), T(1), T(2), ….T(N)

RECURSIVE-FFT(a)

$n \leftarrow length[a]$
if $n = 1$
  then return a
$\omega_n \leftarrow e^{2\pi i/n}$
$\omega \leftarrow 1$
$a^{[0]} \leftarrow (a_0, a_2, \ldots, a_{n-2})$
$a^{[1]} \leftarrow (a_1, a_3, \ldots, a_{n-1})$
$y^{[0]} \leftarrow$ RECURSIVE-FFT$(a^{[0]})$
$y^{[1]} \leftarrow$ RECURSIVE-FFT$(a^{[1]})$
for $k \leftarrow 0$ **to** $n/2 - 1$
  do $y_k \leftarrow y_k^{[0]} + \omega\, y_k^{[1]}$
    $y_{k+(n/2)} \leftarrow y_k^{[0]} - \omega\, y_k^{[1]}$
    $\omega \leftarrow \omega\, \omega_n$
return y

$$T_N(x) = \sum_{n=0}^{N} a_n \cos(nx) + \mathrm{i} \sum_{n=0}^{N} a_n \sin(nx)$$

Audio processing

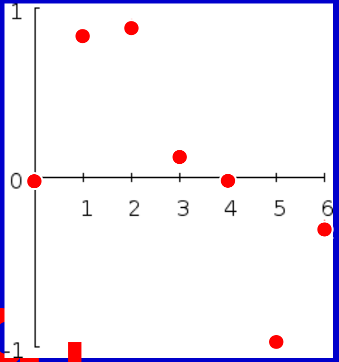
Image processing

Tomography, MRI

Fast multiplication

Quantum algorithms

Error correction



Reed-Solomon decoding

Petersen 60

Berlekamp-Massey 6

INPUT: a binary sequence S = $S_O, S_1, S_2, \ldots S_n$.

OUTPUT: the complexity L (S) of S, 0< L (S) < N.

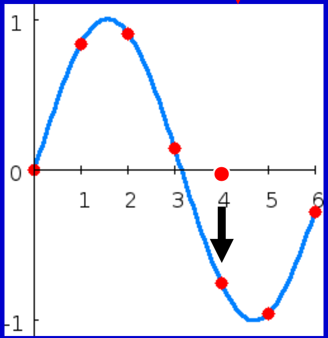
1. Initialization: C(D):=l, L:=O m:=-l, B{D):=l, N:=O.
2. While (N < n) do the following:
   - 2.1 Compute the next discrepancy d.
     d:= ($S_N + \Sigma\ c_i S_{Ni}$) mod 2.
   - 2.2 If d = 1 then do the following:
     T (D):=C (D), C (D):=C(D)+B(D)·D^{Nm} .
     If L < N/2 then L:=N+l-L, m:=N, B(B):=T (D).
   - 2.3 N:=N+l.
3. Return(L) .

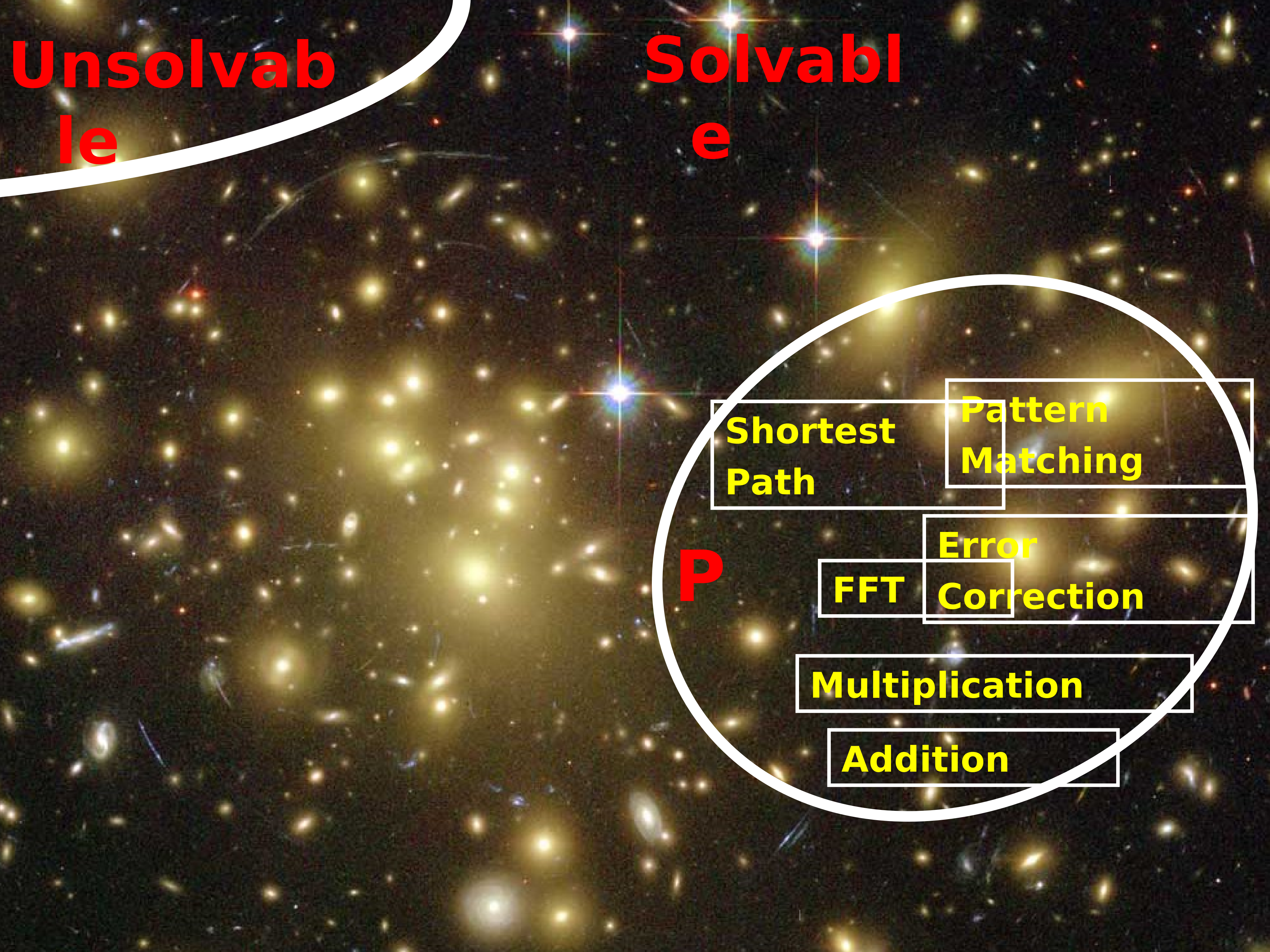
CDs

DVDs

Satellite communica

Cell phone communication




Unsolvab
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P
Shortest
Path
Pattern
Matching
Error
Correction
FFT
Multiplication
Addition

Cobham, Edmonds
Rabin ~1965

The class P

All problems having an efficient (polynomial time) algorithm

Many interesting problems in P

Are all interesting problems in P?
What are "interesting" problems?

Search problems

Short Path: FIND short path from Princeton to LA

Pattern Matching: FIND CGC in CAUCGCGCUUCGC

Easy

Hard

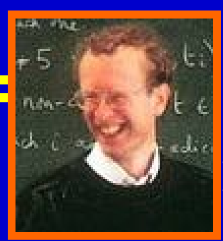
What is common to all these problems?

In all, solutions are easy to check & verify!

Factoring: FIND factors of 147,573,952,588,676,412,927

= 193,707,721 × 761,838,257,287

Theorem Proving: F … **200 page proof of the** … **mann** … **s”**



Lemma…Proof…Lemma..Proof..

Sudoku: FIND solution of

		8	6					
							6	
			4	8			2	3
		5		9				8
	4	9				2	1	
2				4		7		
3	6			2	9			
	1							
					5	1		

9	2	8	6	1	3	4	5	7
4	7	3	9	5	2	8	6	1
1	5	6	4	8	7	9	2	3
7	3	5	2	9	1	6	4	8
6	4	9	7	3	8	2	1	5
2	8	1	5	4	6	7	3	9
3	6	7	1	2	9	5	8	4
5	1	2	8	7	4	3	9	6
8	9	4	3	6	5	1	7	2

The class NP- problems like **FIND**: needle in a haystack

May be **hard** to *find* Always **easy** to *verify* ook & Levin 1971

ödel 1956

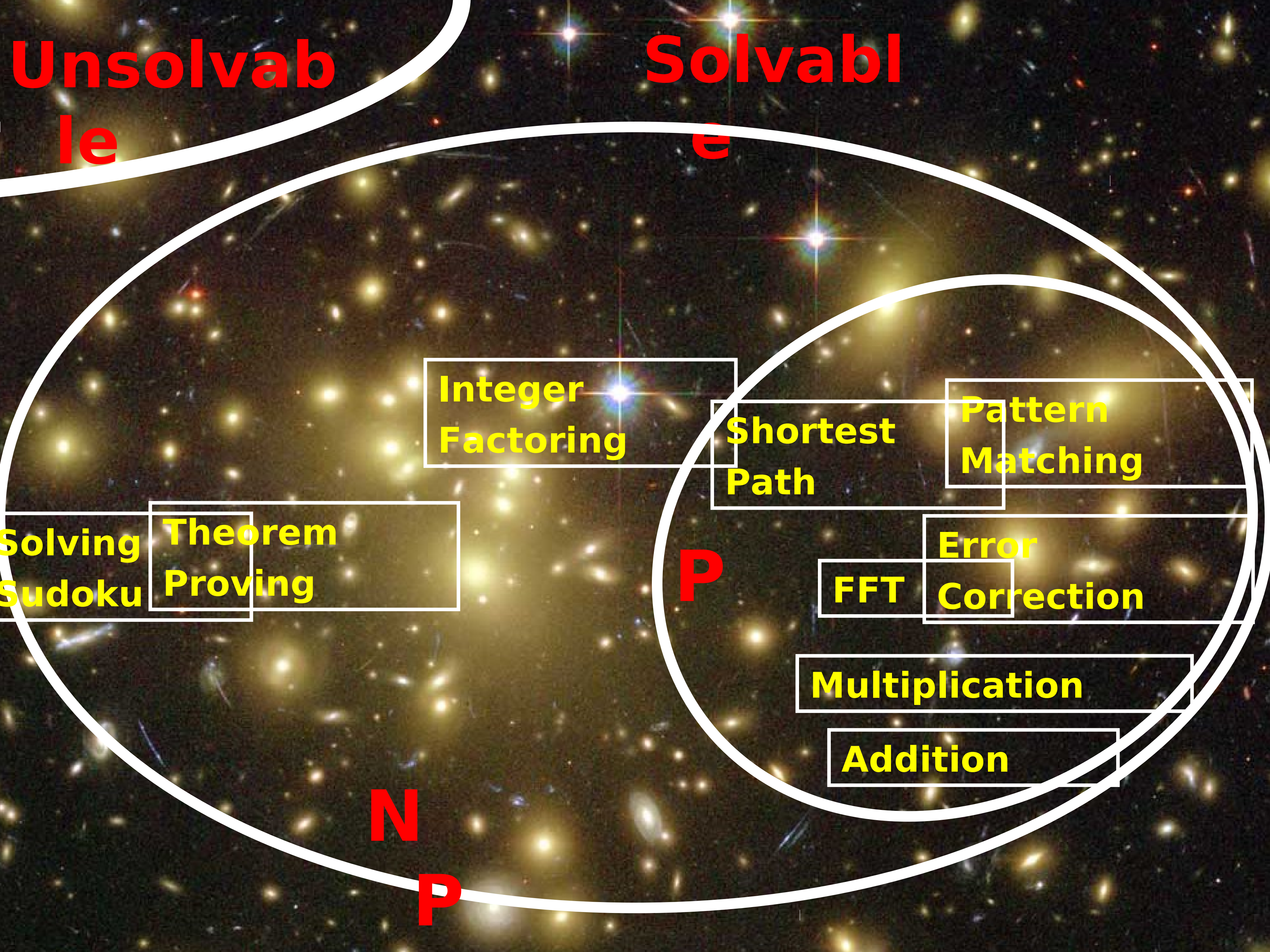
The class NP

All problems having efficient verification

algorithms of given solutions

For every such problem, finding a solution (of length n) takes ≤ 2^n steps: try all possible solutions & verify each.

Can we do better than “brute force” ?


Unsolvable
Solvable
Integer Factoring
Shortest Path
Pattern Matching
Solving Sudoku
Theorem Proving
P
FFT
Error Correction
Multiplication
Addition
NP

P versus NP

P: Problems for which solutions can be efficiently *found*

NP: Problems for which solutions can be efficiently *verified*

Conjecture: P ≠ NP

[finding is much harder than verification]

"P=NP?" is a central question of math, science & technology !!!

What is in NP?

Mathematician: Given a statement, ***find*** a proof

Scientist: Given data on some phenomena,

find a theory explaining it.

Engineer: Given constraints (size,weight,energy)

find a design (bridge, medicine, phone)

In many intellectual challenges, *verifying* that we found a good solution is an easy task !

niversality: NP-completeness

Are SuDoku,Theorem Proving,Factoring hard?

These problems are intimately related!!

Theorem: If SuDoku is easy then

- **Theorem proving is easy**
- **Factoring is easy**

Proof: SuDoku is NP-complete

SuDoku solver can solve any NP problem

P=NP iff SuDoku has an efficient algorithm

niversality: NP-completeness

NP-complete problems:

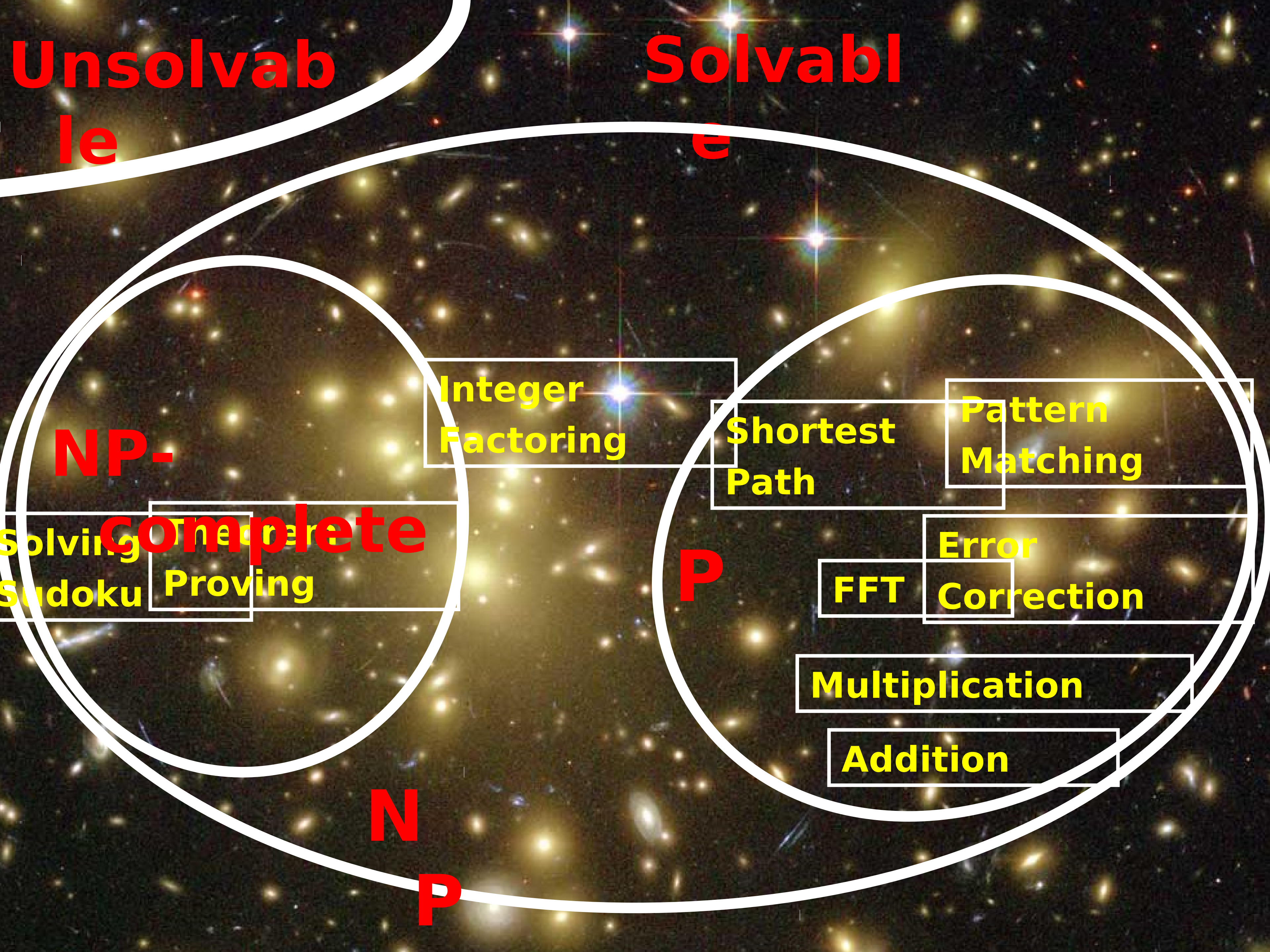
If one is easy, then all are!

If one is hard, then all are!

SuDoku: NP-complete

Thm proving: NP-complete

Integer factoring: we don’t know


Unsolvab
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niversality: NP-completeness

NP-complete problems:

If one is easy, then all are!

If one is hard, then all are!

SuDoku: NP-complete

Thm proving: NP-complete

Integer factoring: we don’t know

Thousands of NP-complete problems known in Math, Biology, Physics, Economics,….

Protein Engineering vol. 7 no. 9 pp. 1059-1068, 1994

The protein threading problem with sequence amino acid interaction preferences is NP-complete

Richard H. Lathrop

Economic Theory vol. 23, no. 2 , pp. 445-454, 2004

Finding a Nash equilibrium in spatial games is NP-complete

R. Baron, J. Durieu, H. Haller and P. Solal

[math.GR] arXiv:0802.3839v1

Quadratic equations over free groups are NP-complete

O. Kharlampovich, I.G. Lysenok, A G Myasnikov, N. Touikan

NP-completeness: sign of structural "nastiness".

Potential guide to better models and

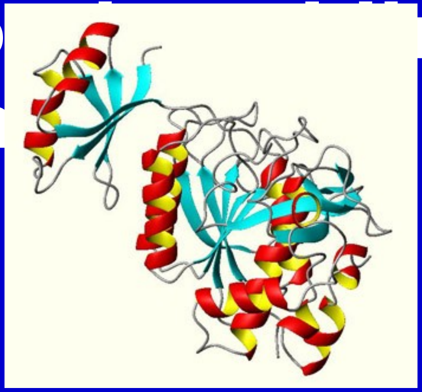
P ≠ NP as a law of nature

The following problems are NP-complete

Biology: Minimum energy **Physics**: Minimum

Pro ng s

Foa



Economics: Nash Equilibrium in strategic games

What is efficient computation?

efficiently?

Church-Turing Thesis:

Every *reasonable* process, can be simulated by a Turing machine

- Adding random bits

Theorem [Blum-Micali, Yao, Nisan-Wigderson, Impagliazzo-Wigderson]

If "P≠NP", randomness add no power!

- Adding quantum bits

Theorem [Shor]

Positive consequences of P≠NP

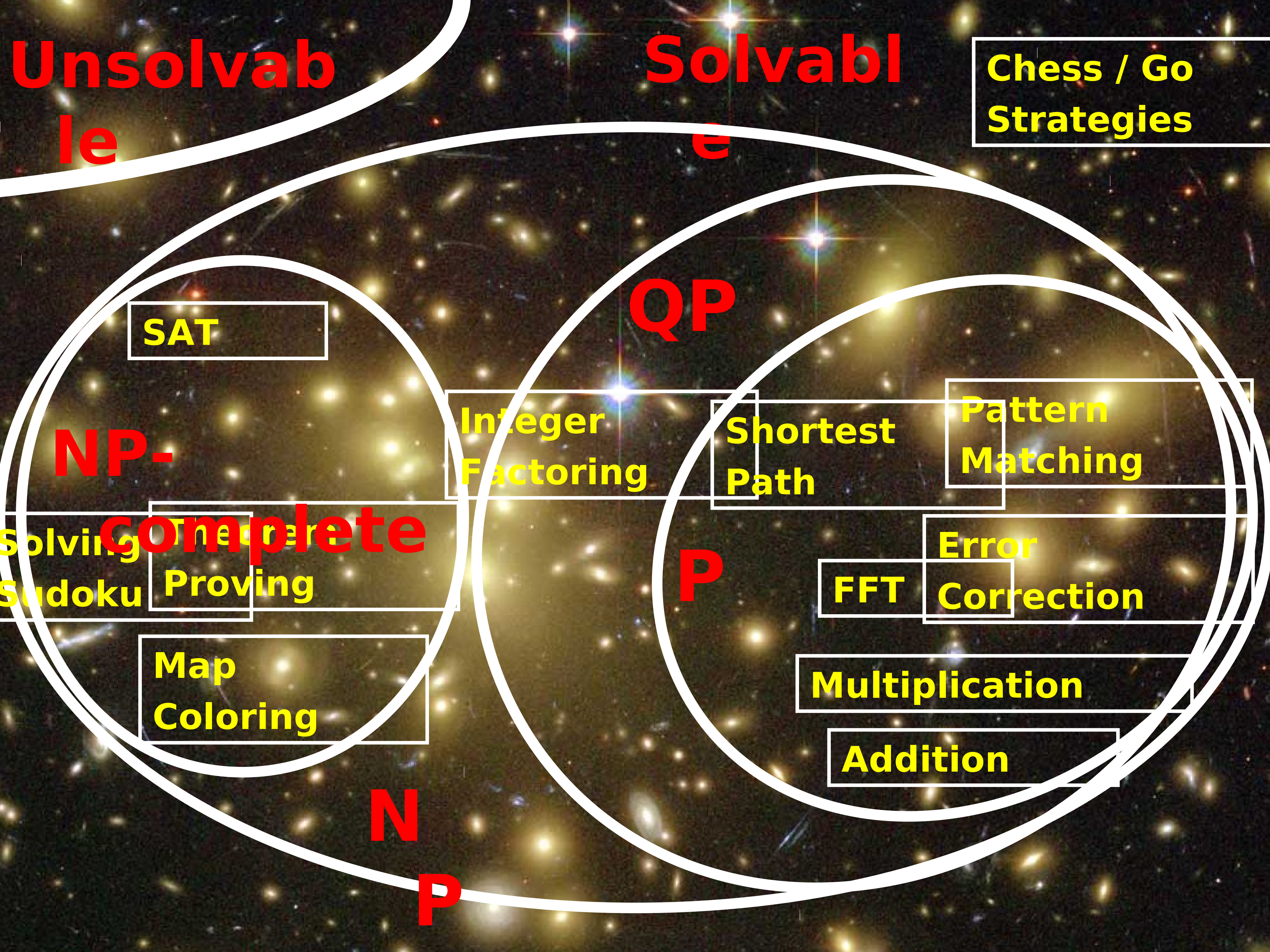
P≠NP Some of the problems we want to solve are hard. Are hard problems useful?

Cryptography: If **Factoring** is hard then:

- Encryption
- Electronic commerce
- Digital signatures
- On-line shopping

Things we didn't cover

- How to prove NP-completeness
- Attempts to prove P≠NP and restricted lower bounds
- Other resources (space, parallelism communication) and complexity classes
- Other modes of computation (average-case, approximate,…)
- ……


Unsolvab
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Chess / Go
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QP
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