Extremal Black Holes

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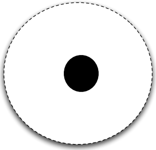
TIFR, August 2009

Black Holes

Black holes are objects of very large mass.

They are described as classical solutions of the equations of motion of general theory of relativity.

Their gravitational attraction is so large that even light cannot escape a black hole.



Horizon

A black hole is surrounded by an imaginary surface such that no object inside the surface can ever escape to the outside world.

This surface is called the event horizon.

To an outside observer the event horizon appears completely black since no light comes out of it.

Upon taking into account quantum corrections one finds that this picture of the black hole gets modified.

In its interaction with other objects a black hole behaves as a thermal object with definite temperature, entropy etc.

Hawking, Bekenstein 70's

In particular its entropy is given by the simple formula:

$$S_{BH} = A/(4G_N)$$

A: Area of the event horizon

G_N: Newton's gravitational constant

In conventional statistical mechanics, the entropy of a system has a microscopic explanation.

$$S_{stat} = \ln d_{micro}$$

d_{micro}: Number of quantum states (microstates) available to the system for a given set of macroscopic charges (e.g. total electric charge, energy etc.).

Question: Does the entropy of a black hole have a similar statistical interpretation?

We shall study these issues in the zero temperature (extremal) limit.

Since in this limit the black holes cease to Hawking radiate, the notion of degeneracy is better defined for these black holes.

Often, but not always, these black holes are supersymmetric, and hence are stable.

For a class of supersymmetric extremal black holes in string theory one can indeed find a microscopic explanation of the Bekenstein-Hawking entropy.

Strominger, Vafa

$$A/4G_N = \ln d_{micro}$$

d_{micro}: degeneracy of microstates

– usually calculated by considering a system of D-branes and other known objects in string theory carrying the same charges as the black hole, and then explicitly counting the number of states of this system.

This calculation does not make any direct reference to black holes.

This formula is quite remarkable since it relates a geometric quantity in space-time to a counting problem.

However the Bekenstein-Hawking formula is an approximate formula that holds in classical general theory of relativity.

– works well only when the charges carried by the black hole are large and hence the curvature at the horizon is small.

The calculation on the microscopic side also simplifies when the charges are large.

Instead of doing exact counting of quantum states, we can use approximate methods which gives the result for large charges.

Is it possible to test the $S_{BH} \leftrightarrow S_{stat}$ correspondence to better accuracy?

In order to address this issue we have to work on two fronts.

1. Count the number of microstates to greater accuracy.

2. Calculate black hole entropy to greater accuracy.

In this talk we shall describe the progress on both fronts.

Progess in microscopic counting

In a class of theories, known as N=4 and N=8 supersymmetric string theories in four dimensions, one now has a complete understanding of the microscopic degeneracies of supersymmetric black holes.

Typically such theories have multiple gauge fields.

$\Rightarrow$ the black hole is characterized by multiple charges, collectively denoted by $\vec{Q}$.

The degeneracy is expressed as a function $d_{micro}(\vec{Q})$ of the charges.

In these theories $d_{micro}(\vec{Q})$ is expressed as Fourier expansion coefficients of some well-known functions, *e.g.* Jacobi theta functions, Igusa cusp forms etc.

$\Rightarrow$ 'experimental data' to be explained by a 'theory of black holes'.

In the large charge limit these degeneracies agree with the exponential of the Bekenstein-Hawking entropy of black holes carrying the same set of charges

Example 1: Degeneracies of a class of supersymmetric states in type II string theory compactified on a six dimensional torus, as a function of two functions of charges: Δ and ℓ

Δ	ℓ	$\mathbf{d}_{\mathbf{micro}}$	$\ln \mathbf{d}_{\mathbf{micro}}$	$\mathbf{S}_{\mathbf{BH}} = \pi\sqrt{\Delta}$
36	1	85500	11.35627	18.85
36	3	85512	11.35641	18.85
112	1	18249586944	23.627408	33.25
112	2	18249601536	23.627409	33.25
120	1	51386683104	24.66264	34.41

In a systematic comparison we do not compare numbers, but compare the asymptotic expansions for large charges.

On the microscopic side we have a completely systematic algorithm for finding this asymptotic expansion for this special class of theories.

In the previous example we have, for large Δ:

$$\ln \mathbf{d}_{\mathbf{micro}}(\Delta, \ell) = \pi\sqrt{\Delta} - 2\ln\Delta + \cdots$$

$$\mathbf{d}_{\mathbf{micro}}(\Delta, \ell) - \mathbf{d}_{\mathbf{micro}}(\Delta, \ell = 1) = \exp\left[\pi\sqrt{\Delta}/s - 2\ln\Delta + \cdots\right]$$

s: The lowest integer > 1 which divides ℓ.

Example 2: Degeneracies of a class of supersymmetric states in heterotic string theory on a six dimensional torus as a function of two functions of charges D_1 and D_2

D_1	D_2	degeneracy d_{micro}	$\ln d_{micro}$	$S_{BH} = \pi\sqrt{D_1 D_2}$
2	2	50064	10.82	6.28
4	4	32861184	17.31	12.57
6	4	632078672	20.26	15.39
8	4	9337042944	22.96	17.77
10	4	113477152800	25.45	19.87

Again it is useful to examine the asymptotic expansion of the exact degeneracy formula for large charges.

For example, for $D_1 >> D_2$, we have

$$\ln d_{micro} = \pi\sqrt{D_1(D_2+8)} + \cdots$$

In order to explain the difference between $\ln d_{micro}$ and the Bekenstein-Hawking entropy we need to understand corrections to the Bekenstein-Hawking formula.

This is the problem we shall now address.

In string theory there are two types of corrections to the Bekenstein-Hawking formula.

1. Higher derivative corrections to the classical equations of motion of general relativity.

– These originate from the fact that strings are extended objects and not point particles.

2. Quantum corrections.

We would like to look for an exact formula for the black hole entropy taking into account both types of corrections.

These are necessary if we want to compute the black hole entropy away from the large charge limit.

Stringy corrections and quantum corrections are necessary for computing the black hole entropy away from the large charge limit.

One can make this statement more precise.

Typically a black hole in string theory is characterized by multiple charges.

Quantum corrections and stringy corrections are controlled by different combination of charges.

Depending on the values of the charges, either stringy corrections or quantum corrections or both may be important.

If we adjust the charges so that quantum corrections can be ignored and only stringy corrections are important, then we have an exact formula, due to Wald, that tells us how to compute corrections to the Bekenstein-Hawking entropy.

We can compare this with the result of microscopic computation.

– agrees in all cases which have been studied.

Example: Consider the black holes in heterotic string theory compactified on six dimensional torus (Ex. 2).

One finds that in the $D_1 \to \infty$ limit with D_2 fixed the quantum corrections to the black hole entropy can be ignored and hence Wald's formula should give the complete answer.

In this limit the different formulæ take the following forms

$$\text{microscopic}: \qquad \ln d_{micro} = \pi\sqrt{D_1(D_2+8)}$$

$$\text{Bekenstein}-\text{Hawking}: \qquad S_{BH} = \pi\sqrt{D_1 D_2}$$

$$\text{Wald}: \qquad S_{Wald} = \pi\sqrt{D_1(D_2+8)}$$

The last frontier: Understand quantum corrections to the extremal black hole entropy.

We shall now describe a proposal for systematically computing quantum corrections to the black hole entropy.

AdS_2 space plays a crucial role in this proposal.

What is AdS_2?

Take a three dimensional space labelled by coordinates (x, y, z) and metric

$$ds^2 = dx^2 - dy^2 - dz^2$$

AdS_2 may be regarded as a two dimensional Lorentzian space embedded in this 3-dimensional space via the relation:

$$x^2 - y^2 - z^2 = -a^2$$

a: some constant giving the radius of AdS_2.

This space has an SO(2,1) isometry.

$$x^2 - y^2 - z^2 = -a^2$$

Introduce independent coordinates (η, t):

$$x = a \sinh\eta \cosh t, \quad y = a \cosh\eta, \quad z = a \sinh\eta \sinh t$$

$$dx^2 - dy^2 - dz^2 = a^2(d\eta^2 - \sinh^2\eta \, dt^2)$$

Define: $r = \cosh\eta$

$$ds^2 = a^2 \left[\frac{dr^2}{r^2 - 1} - (r^2 - 1) dt^2 \right], \quad r \geq 1$$

Why AdS_2?

All known black holes develop an AdS_2 factor in their near horizon geometry in the extremal limit.

– time translation symmetry gets enhanced to $SO(2,1)$ in the near horizon limit.

Reissner-Nordstrom solution in $D = 4$:

$$\begin{aligned} ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\ &\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\ &\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Define

$$2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\,\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}$$

and take $\lambda \to 0$ limit keeping r, t fixed.

$$ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

AdS_2 $\times$ S^2

Postulate: Any extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.

– partially proved

Kunduri, Lucietti, Reall; Figueras, Kunduri, Lucietti, Rangamani

The full near horizon geometry takes the form $AdS_2 \times K$

K: some compact space.

Proposal:

The exact degeneracy of an extremal black hole is given by the path integral of string theory over the near horizon $AdS_2 \times K$ geometry of the black hole.

Consistency checks:

1. In the classical limit this reduces to the exponential of the Wald entropy.

2. This proposal follows naturally from the AdS/CFT correspondence.

Given this exact formula for black hole entropy, we should be able to compute systematic quantum corrections to the Wald's classical formula and compare these with the known microscopic results.

There has been some success but a detailed comparison is still underway.

During the next two lectures we shall elaborate on this proposal and carry out some tests.

Summary

1. String theory offers the possibility of testing the correspondence between black hole entropy and microscopic degeneracies far beyond the leading order.

2. On the microscopic side we now have a complete understanding of the degeneracies for a class of states in a class of theories.

3. On the black hole side we have a complete expression for the degeneracy in terms path integral of string theory over the near horizon geometry.

4. Some checks have already been performed, but a complete comparison between the two sides is still underway.

Plan

1. Review of classical entropy of extremal black holes

2. A proposal for $d_{macro}(\vec{Q})$

3. Some exact results for $\mathbf{d_{micro}}(\vec{Q})$ in type IIB string theory on T^6.

4. Comparison of $\mathbf{d_{macro}}(\mathbf{Q})$ with $\mathbf{d_{micro}}(\vec{Q})$.

5. Black hole hair removal.

Postulate: An extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.

Regarding all other directions (including angular coordinates) as compact we can regard the near horizon geometry of an extremal black hole as

$$AdS_2 \times \text{a compact space (fibered over } AdS_2\text{)}$$

Note: Magnetic charges are encoded in the fluxes through the compact space.

Consider string theory in such a background containing two dimensional metric $g_{\mu\nu}$ and $U(1)$ gauge fields $A_\mu^{(i)}$ among other fields.

Consider the most general field configuration consistent with $SO(2,1)$ isometry (and other symmetries if any):

$$ds^2 \equiv g^{(2)}_{\mu\nu} dx^\mu dx^\nu = v \left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right)$$

$$F^{(i)}_{rt} = e_i, \quad \cdots\cdots\cdots$$

$\mathcal{L}^{(2)}(v, \vec{e}, \cdots)$**: The effective two dimensional Lagrangian density evaluated in this background.**

For black hole with electric charges $\{\vec{q}_i\}$, define

$$\mathcal{E}(\vec{q}, v, \vec{e}, \cdots) \equiv 2\pi \left(e_i\, q_i - v\, \mathcal{L}^{(2)} \right)$$

One finds that

1. All the near horizon parameters are obtained by extremizing $\mathcal{E}$ with respect to v, e_i and the other near horizon parameters.

2. $S_{wald}(\vec{q}) = \mathcal{E}$ at this extremum.

Thus in the classical limit

$$d_{macro}(\vec{q}) = \exp\left[S_{wald}(\vec{q})\right] = \exp\left[\mathcal{E}\right]$$

Applications

1. It can be used to give a proof of ‘attractor mechanism’ in any general higher derivative theory of gravity.

– In a given theory the entropy of an extremal black hole depends only on the quantized charges and not on any other asymptotic data e.g. the vev of the moduli scalars.

2. For spherically symmetric black holes it gives a simple algebraic method for computing the entropy.

We shall now try to find a generalization of this formula in the full quantum theory.

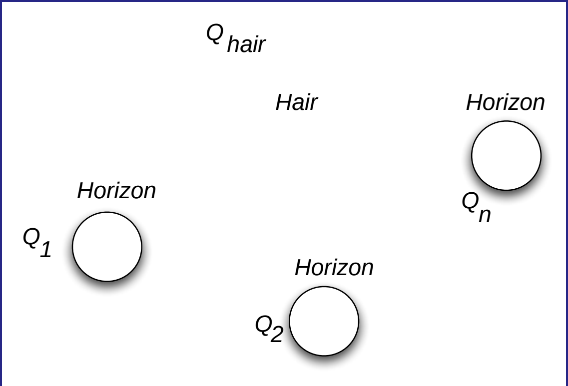
Take a macroscopic configuration of charge $\vec{Q}$ (includes both electric and magnetic charges)

In general such a configuration could involve an n centered black hole with charges $\vec{Q}_1, \cdots \vec{Q}_n$ and hair with charge $\vec{Q}_{hair}$.

Hair: smooth normalizable supersymmetric deformations of the black hole solution with support outside the horizon(s).

Example: If a black hole breaks some of the supersymmetries, then it should carry fermion zero modes associated with the broken supersymmetries.

These are part of the hair modes.



Q_{hair}
Hair
Horizon
Horizon
Q_1
Horizon
Q_2
Q_n

Proposal for $d_{macro}(\vec{Q})$:

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i+\vec{Q}_{hair}=\vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} d_{hair}(\vec{Q}_{hair};\{\vec{Q}_i\})$$

$d_{hor}(\vec{Q}_{hor})$: **contribution from the horizon with charge** $\vec{Q}_{hor}$

d_{hair}: **contribution from the hair of the** n**-centered black hole, with the horizons carrying charges** $\vec{Q}_1, \cdots \vec{Q}_n$**, and the hair carrying charge** $\vec{Q}_{hair}$**.**

Our main focus in this talk will be on $d_{hor}(\vec{Q})$**.**

Our goal: Find a macroscopic prescription for computing $d_{hor}(\vec{Q})$

To leading order in g_s but all orders in α', $d_{hor}(\vec{Q})$ is given by the exponential of the Wald entropy

– can be computed using the entropy function formalism described earlier.

We shall propose an expression for $d_{hor}(\vec{Q})$ in the full quantum theory as a path integral over the Euclidean continuation of the near horizon geometry.

$\longrightarrow$ **Quantum entropy function**

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right)$$
$$F^{(i)}_{rt} = e_i$$

Euclidean continuation:

$$t = -i\theta, \quad r = \cosh\eta, \quad 0 \le \eta < \infty$$

This gives

$$\begin{aligned} ds^2 &= v\left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \quad \rightarrow \theta \equiv \theta + 2\pi, \\ F^{(i)}_{\theta\eta} &= ie_i \sinh\eta \end{aligned}$$

$$\rightarrow \quad A^{(i)}_{\theta} = -i\, e_i\,(\cosh\eta - 1) = -i\, e_i\,(r-1)\,.$$

Proposal for the quantum entropy function $d_{hor}(\vec{q})$

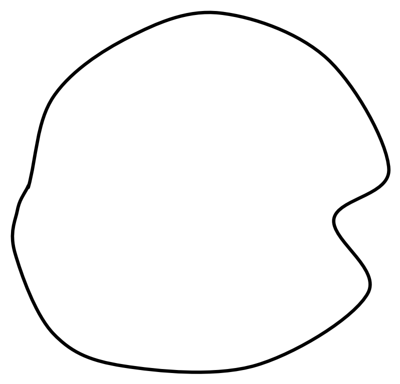
$$d_{hor}(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

$\langle\ \rangle_{AdS_2}$: path integral over various fields of string theory on euclidean global $AdS_2 \times K$.

$\oint$: a closed contour at the boundary of AdS_2.

'finite': Infrared finite part of the amplitude.

We need to regularize the infinite volume of AdS_2 by putting a cut-off $r \leq r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.



Cut-off: $r \leq r_0 f(\theta)$ **for some smooth periodic function** $f(\theta)$.

The superscript '*finite*' refers to the finite part of the amplitude defined by expressing it as

$$e^{CL} \times \textbf{finite part}$$

L: **length of the boundary of** AdS_2.

C: **A constant**

The definition can be shown to be independent of the choice of $f(\theta)$.

We shall work with $f(\theta) = 1$.

The role of

$$\exp[-iq_i \oint d\theta \, A_\theta^{(i)}]$$

We could absorb this into the boundary terms in the action.

However we have displayed it explicitly since it plays a special role.

It is the only term in the boundary action that involves the gauge field and not its field strength.

Why do we need this term?

In AdS_d the Maxwell's equation has two solutions in the asymptotic region:

$A_\theta^{(i)} \sim r^{-d+3}$: electric field mode

$A_\theta^{(i)} \sim$ constant: constant mode

Thus for $d \geq 4$ the constant mode of the gauge field is dominant at infinity.

We fix the constant mode by a boundary condition and integrate over the electric field mode.

However for $d = 2$,

Electric field mode: $A_\theta^{(i)} \sim r$

Constant mode: $A_\theta^{(i)} \sim$ constant

Thus the electric field mode is dominant

$\rightarrow$ we must work in a sector with fixed asymptotic electric field i.e. fixed charge, and allow the constant mode to fluctuate.

However now the extremization of the action no longer gives the classical equations of motion.

The variation of the action contains boundary terms proportional to $\delta A_\theta^{(i)}$ which are no longer constrained to vanish by boundary condition.

$\rightarrow$ **we need to add new boundary term in the action to cancel the boundary terms proportional to $\delta A_\theta^{(i)}$.**

The $\exp[-iq_i \oint d\theta\, A_\theta^{(i)}]$ precisely achieves this task.

$$d_{hor}(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

We shall try to justify this proposal by showing that

1. In the classical limit

$$\ln d_{hor}(\vec{q}) \to S_{wald}(\vec{q})$$

2. One can give a formal proof of this using AdS/CFT **correspondence.**

Classical limit:

$$\left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}$$

In the classical limit this reduces to

$$e^{-\mathcal{S}} \exp[-iq_i \oint d\theta\, A_\theta^{(i)cl}]$$

$$A_\theta^{(i)cl} = -i\, e_i\, (r-1)$$

$\mathcal{S}$ = **Euclidean action** $= \mathcal{S}_{bulk} + \mathcal{S}_{boundary}$

$$\mathcal{S}_{bulk} = -\int_1^{r_0} dr\,\sqrt{\det g}\,d\theta\,\mathcal{L}^{(2)} = -(r_0-1)\,2\pi v\,\mathcal{L}^{(2)}$$

$$-iq_i \oint d\theta\,A_\theta^{(i)cl} = -2\pi\,\vec{q}\cdot\vec{e}\,(r_0-1)$$

$$\mathcal{S}_{boundary} = -2\pi\,K\,r_0 + \mathcal{O}(r_0^{-1})$$

K**: some constant which depends on the details of the boundary terms.**

The length of the boundary is

$$L = 2\pi\sqrt{v}\,r_0 + \mathcal{O}(r_0^{-1})\,.$$

This gives

$$\left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}$$
$$= \left[e^{L(v\,\mathcal{L}^{(2)}+K-\vec{e}\cdot\vec{q})/\sqrt{v}+2\pi(\vec{e}\cdot\vec{q}-v\,\mathcal{L}^{(2)})+\mathcal{O}(r_0^{-1})}\right]$$

Extracting the finite part we get

$$d_{hor}(\vec{q}) \simeq \exp\left[2\pi(\vec{e}\cdot\vec{q} - v\,\mathcal{L}^{(2)})\right] = \exp\left[S_{wald}(\vec{q})\right]$$

Note: A change in the boundary action changes K but the finite part is insensitive to such a change.

AdS_2/CFT_1 correspondence

Euclidean AdS_2 is the Poincare disk.

$\rightarrow$ its boundary is a circle of circumference L.

Thus AdS/CFT correspondence $\rightarrow$

$$\left\langle \exp[-iq_i \oint d\theta \, A_\theta^{(i)}] \right\rangle_{AdS_2} = Z_{CFT_1} = Tr \, e^{-LH}$$

Tr**: trace over states of CFT_1**

H**: Hamiltonian of dual quantum mechanics**

Thus we have, for large L,

$$\left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2} = Tr\, e^{-LH} = d_{CFT}(\vec{q})e^{-E_0 L}\,.$$

E_0 , $d_{CFT}(\vec{q})$: **ground state energy, degeneracy**

Taking the finite part we get

$$d_{hor}(\vec{q}) = d_{CFT}(\vec{q})$$

Note: In the more conventional units we take the length of the boundary to be finite, but scale energies by L.

Only the ground states of the CFT survive.

What can we say about CFT_1?

It should be identified as the infrared limit of the quantum mechanics associated with the microscopic description of the black hole, after stripping off the hair contribution.

Thus d_{CFT} together with the hair contribution should give us the microscopic degeneracies.

– agrees with our proposal.

Summary of the proposal

$$d_{micro}(\vec{Q}) = d_{macro}(\vec{Q})$$

$d_{macro}(\vec{Q})$ is given by the formula

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i+\vec{Q}_{hair}=\vec{Q}}} \left\{\prod_{i=1}^n d_{hor}(\vec{Q}_i)\right\} d_{hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

$$d_{hor}(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta\, A_\theta^{(i)}] \right\rangle_{AdS_2}^{finite}$$

Degeneracy or Index?

Often in the microscopic theory we compute the index rather than degeneracy.

– protected against quantum corrections.

e.g. **in** $D = 4$ **we calculate the helicity trace index**

$$B_{2n} = \frac{(-1)^n}{(2n)!} Tr\left[(-1)^{2h}(2h)^{2n}\right]$$

Tr**: trace over states of fixed charges but different** $J^3 \equiv h$

$4n$**: Number of broken SUSY generators.**

Thus on the black hole side also we should compute the index.

$$(-1)^{2h}(2h)^{2n} = (-1)^{2h_{hor}+2h_{hair}}(2h_{hor} + 2h_{hair})^{2n}$$

The $(2h)^{2n}$ factor is needed to absorb the fermion zero modes associated with broken SUSY.

For a black hole solution these zero modes form part of hair degrees of freedom.

Thus if we expand

$$(2h)^{2n} = \left(2\sum_i h_{i,hor} + 2\,h_{hair}\right)^{2n}$$

in a binomial expansion, then only the $(2\,h_{hair})^{2n}$ term will survive.

Thus B_{2n} for the black hole takes the form

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i+\vec{Q}_{hair}=\vec{Q}}} \left\{\prod_{i=1}^n I_{hor}(\vec{Q}_i)\right\} B_{2n;hair}(\vec{Q}_{hair};\{\vec{Q}_i\})$$

I_{hor}: Witten index associated with the horizon

Note: In this formula $\vec{Q}$ no longer contains J^3.

Since in $D = 4$ the black hole horizons always have $h = 0$ we get

$$I_{hor}(\vec{Q}_{hor}) = d_{hor}(\vec{Q}_{hor})$$

This gives the following formula for the index on the macroscopic side

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} B_{2n;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

– can be computed using quantum entropy function.

We shall now compare the macroscopic index computed using this prescription with the microscopic index in a specific example.

Consider type IIB string theory on $T^4 \times S^1 \times \widetilde{S}^1$

We shall focus on a special class of states in this theory consisting of

D5/D3/D1 branes wrapped on 4/2/0 cycles of $T^4 \times (S^1 \text{ or } \widetilde{S}^1)$

$\vec{Q}$: D-brane charges wrapped on 4/2/0 cycles of $T^4 \times \widetilde{S}^1$

$\vec{P}$: D-brane charges wrapped on 4/2/0 cycles of $T^4 \times S^1$

$\vec{Q}$ and $\vec{P}$ are each 8 dimensional vectors.

Note: The $\vec{Q}$ used earlier now stands for $(\vec{Q}, \vec{P})$

These states are 1/8 BPS i.e. they break 28 of the 32 supersymmetries.

$B_{14}(\vec{Q}, \vec{P})$: microscopic 14th helicity trace index of 1/8 BPS states carrying charges $(\vec{Q}, \vec{P})$.

In this case only single centered black holes contribute to the index.

Furthermore for these black holes the only hair degrees of freedom are expected to be the fermion zero modes associated with broken supersymmetry.

$\rightarrow \qquad \vec{Q}_{hair} = 0, \qquad B_{14;hair} = 1$

Thus $B_{14}(\vec{Q}, \vec{P}) = d_{hor}(\vec{Q}, \vec{P})$

$\rightarrow$ **quantum entropy function directly computes the index.**

Duality symmetries

Full duality group: $E_{7(7)}(\mathbb{Z})$

T-duality group: $SO(6,6;\mathbb{Z})$

A subgroup of $SO(6,6;\mathbb{Z})$ **which preserves the constraint we have imposed on the D-branes is given by**

$$SO(4,4;\mathbb{Z}) \times SL(2,\mathbb{Z})$$

$SO(4,4;\mathbb{Z})$**: T-dualty group of** T^4

$SL(2,\mathbb{Z})$**: global diffeomorphism of** $S^1 \times \widetilde{S}^1$

Intersection form of 4/2/0 forms on T^4 defines $SO(4,4;\mathbb{Z})$ invariant inner products

$$Q^2, \qquad P^2, \qquad Q\cdot P$$

Define:

$$\ell_1 \equiv \mathrm{gcd}\{Q_i P_j - Q_j P_i\}$$

$$\ell_2 = \mathrm{gcd}(Q^2/2, P^2/2, Q\cdot P)$$

For states with $\mathrm{gcd}(\ell_1, \ell_2) = 1$

$$B_{14}(\vec{Q}, \vec{P}) = (-1)^{Q\cdot P+1} \sum_{s|\ell_1\ell_2} s\, \widehat{c}(\Delta(Q,P)/s^2)$$

$$\Delta(Q,P) = Q^2 P^2 - (Q\cdot P)^2$$

$$-\vartheta_1(z|\tau)^2\, \eta(\tau)^{-6} \equiv \sum_{k,l} \widehat{c}(4k - l^2)\, e^{2\pi i(k\tau + lz)}$$

For large Δ:

$$\widehat{c}(\Delta) \sim (-1)^{\Delta+1} \exp(\pi\sqrt{\Delta} - 2\ln\Delta + \cdots)$$

$$B_{14}(\vec{Q}, \vec{P}) = (-1)^{Q\cdot P+1} \sum_{s|\ell_1\ell_2} s\,\widehat{c}(\Delta(Q,P)/s^2)$$

The sth term $\sim \exp(\pi\sqrt{\Delta}/s - 2\ln\Delta + \cdots)$

$\rightarrow$ **the $s = 1$ term dominates for large charges.**

Leading macroscopic entropy: $\pi\sqrt{\Delta}$

Our goal: Use quantum entropy function to understand the general microscopic formula for B_{14}.

For simplicity of presentation we shall consider a restricted class of states for which $\ell_2 = 1$.

In this case

$$B_{14}(\vec{Q},\vec{P}) = (-1)^{Q\cdot P+1} \sum_{s|\ell_1} s\,\widehat{c}(\Delta(Q,P)/s^2)$$

Using $SL(2,\mathbb{Z})$ transformation we can bring any charge vector of this form to

$$(Q,P) = (\ell_1 Q_0, P_0), \quad \mathrm{gcd}\{Q_{0i}P_{0j} - Q_{0j}P_{0i}\} = 1$$

We shall proceed with this choice.

The near horizon geometry

$$T^4 \times S^1 \times \widetilde{S}^1 \times AdS_2 \times S^2$$

$$ds^2 = v\left(\frac{dr^2}{r^2-1} + (r^2-1)\,d\theta^2\right) + \frac{R^2}{\tau_2}\left|dx^4 + \tau dx^5\right|^2 + w(d\psi^2 + \sin^2\psi d\phi^2) + \widehat{g}_{mn}du^m du^n$$

v, w, R: **real constants**

$\tau = \tau_1 + i\tau_2$: **a complex constant** $\in$ **UHP**

$\widehat{g}_{mn}$: **metric on** T^4

x^4, x^5: **coordinates along** $\widetilde{S}^1, S^1$

There are also background RR fluxes.

$\vec{Q}$: **represent RR fluxes through the cycles of T^4 times the 3-cycle spanned by (x^5, ψ, ϕ).**

$\vec{P}$: **represent RR fluxes through the cycles of T^4 times the 3-cycle spanned by (x^4, ψ, ϕ).**

The classical contribution to $d_{hor}(\vec{Q}, \vec{P})$ from this saddle point is exponential of the Wald entropy:

$$\exp\left[\pi\sqrt{\Delta}\right]$$

IIB coupling constant at the horizon $\sim$ charge^{-1}

Quantum corrections computed via path integral over AdS_2 should have the form:

$$\exp[\pi\sqrt{\Delta} + \sum_{n\geq 0} b_n \Delta^{-n/2}]$$

b_n: n-loop contribution

Structure of quantum corrections computed via path integral over AdS_2:

$$\exp[\pi\sqrt{\Delta} + \sum_{n\geq 0} b_n \, \Delta^{-n/2}]$$

Compare this with the asymptotic expansion of $\widehat{c}(\Delta)$.

$$\widehat{c}(\Delta) = \exp[\pi\sqrt{\Delta} - 2\ln\Delta + \sum_{n\geq 1} c_n \Delta^{-n/2}]$$

– gives explicit prediction for n-loop contribution to string theory partition function in $AdS_2 \times S^2 \times T^6$.

Can one explicitly compute these loop corrections on the AdS_2 side and verify the microscopic predictions?

– work in progress.

$$B_{14}(\vec{Q}, \vec{P}) = (-1)^{Q\cdot P+1} \sum_{s|\ell_1} s\, \widehat{c}(\Delta(Q,P)/s^2)$$

We shall try to use quantum entropy function to understand the origin of the $s > 1$ terms.

Claim: The $s > 1$ terms are generated by new saddle points in the path integral.

Consistency check:

1. The saddle point must exist only when $\ell_1/s \in \mathbb{Z}$.

2. The asymptotic field configuration of the saddle point must match that of the original near horizon geometry.

3. The classical contribution from the saddle point must be given by $\exp\left(\pi\sqrt{\Delta}/s\right)$.

Consider an orbifold of the leading saddle point by the transformation

$$\theta \to \theta + 2\pi/s, \quad \phi \to \phi + 2\pi/s, \quad x^5 \to x^5 + 2\pi/s$$

At $r = 1$ ($\eta = 0$) the shift in θ is irrelevant.

$\to$ **the identification is** $(\phi, x^5) \equiv (\phi + 2\pi/s, x^5 + 2\pi/s)$**.**

Thus the RR flux $\vec{Q}$ through the cycle at $r = 1$, spanned by (x^5, ψ, ϕ) times a cycle of T^4, gets divided by s.

Flux quantization $\to$ the orbifold is well defined only if $\vec{Q}$ is divisible by s, i.e. if

$$\ell_1/s \in \mathbb{Z}$$

We shall now show that this orbifold has the same asymptotic behaviour as the original background and hence represents an admissible saddle point of the path integral.

Denoting the (r, θ, ϕ, x^5) coordinates of the orbifold by $(\widetilde{r}, \widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5)$ we get the new metric

$$
\begin{aligned}
ds^2 &= v\left(\frac{d\widetilde{r}^2}{\widetilde{r}^2-1} + (\widetilde{r}^2-1)\, d\widetilde{\theta}^2\right) \\
&\quad + \frac{R^2}{\tau_2}\left|dx^4 + \tau d\widetilde{x}^5\right|^2 \\
&\quad + w(d\psi^2 + \sin^2\psi\, d\widetilde{\phi}^2) + \widehat{g}_{mn} du^m du^n
\end{aligned}
$$

$$(\widetilde{\theta} + 2\pi/s, \widetilde{\phi} + 2\pi/s, \widetilde{x}^5 + 2\pi/s) \equiv (\widetilde{\theta}, \widetilde{\phi}, \widetilde{x}^5)$$

Define

$$\theta = s\widetilde{\theta}, \quad r = \widetilde{r}/s, \quad \phi = \widetilde{\phi} - \widetilde{\theta}, \quad x^5 = \widetilde{x}^5 - \widetilde{\theta}$$

Then

$$\begin{aligned} ds^2 &= v\left(\frac{dr^2}{r^2 - s^{-2}} + (r^2 - s^{-2})\, d\theta^2\right) \\ &+ \frac{R^2}{\tau_2}\left|dx^4 + \tau(dx^5 + s^{-1}d\theta)\right|^2 + \widehat{g}_{mn} du^m du^n \\ &+ w(d\psi^2 + \sin^2\psi(d\phi + s^{-1}d\theta)^2) \end{aligned}$$

$$(\theta + 2\pi, \phi, x^5) \equiv (\theta, \phi, x^5)$$

This has the same asymptotic behaviour as the original saddle point and hence is an admissible saddle point.

Its contribution to $d_{hor}(\vec{Q}, \vec{P})$ in the classical limit is given by

$$\exp[S_{wald}/s] = \exp\left[\pi\sqrt{\Delta}/s\right]$$

This is the same behaviour as of $\widehat{c}(\Delta/s^2)$.

Thus this saddle point is the ideal candidate for the contribution $\widehat{c}(\Delta/s^2)$ in the microscopic formula.

Furthermore it exists iff $s|\ell_1$, as the case for the term $\widehat{c}(\Delta/s^2)$ in the microscopic formula.

A possible approach to computing the full path integral around each saddle point:

Using the su(1,1|2) supersymmetry of the background one can argue that the path integral receives contribution only from configurations which are invariant under certain supersymmetry transformatons.

Hope: Using this we should be able to collapse the path integral around each saddle point into a finite dimensional integral which can then be evaluated explicitly.

Black hole hair removal:

– a consistency check for the formula for $B_{2n;macro}(\vec{Q})$

$$\sum_{n} \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^{n} \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^{n} d_{hor}(\vec{Q}_i) \right\} B_{2n;hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

Suppose two black holes have identical near horizon geometry but different asymptotic geometries.

Suppose further that we know the appropriate index for both these black holes from microscopic analysis, and can compute the hair contribution for both the black holes.

Then by stripping off the hair contribution we can get the 'microscopic result' for $d_{hor}(\vec{Q})$ for both the black holes.

They must agree.

We consider two single centered black holes in type IIB string theory compactified on $T^4 \times S^1$:

1. Rotating charged black hole carrying Q_5 units of D5-brane charge along $T^4 \times S^1$, Q_1 units of D1-brane charge along S^1, n units of momentum along S^1 and equal angular momentum J along the two transverse planes.

$\rightarrow$ a BMPV black hole. Breckenridge, Myers, Peet, Vafa

2. The same black hole with transverse space Taub-NUT.

$\rightarrow$ a four dimensional black hole.

Gauntlett, Gutowski, Hull, Pakis, Reall

These two black holes have identical near horizon geometries but different index and different $B_{2n;hair}$

For both black holes one can identify the hair modes and calculate their contribution to the index.

Using the knowledge of the microscopic index, and the contribution from the hair, one can determine d_{hor}.

One finds that d_{hor} computed by stripping off $B_{2n;hair}$ from the index gives the same result for both.

Banerjee, Mandal. A.S.; Jatkar, A.S., Srivastava

Partition functions

Note that both the BMPV black hole and the four dimensional black hole are characterized by four quantum numbers Q_1, Q_5, n and J.

The degeneracy depends only on n, J and the combination $N \equiv Q_5(Q_1 - Q_5)$.

Thus in the microscopic analysis we can set $Q_5 = 1$ and analyze the partition function $Z(\rho, \sigma, v)$.

(ρ, σ, v): conjugate to (n, Q_1, J).

Result:

$$Z_{5D}(\rho,\sigma,v) = e^{-2\pi i\rho-2\pi i\sigma} \prod_{\substack{k,l,j\in\mathbf{Z}\\ k\geq 1, l\geq 0}} \left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)}$$
$$\times \prod_{l\geq 1}\left\{\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right.$$
$$\left.\left(1-e^{2\pi i l\rho}\right)^{4}\right\}(-1)\left(e^{\pi i v}-e^{-\pi i v}\right)^2 .$$

$$Z_{4D}(\rho,\sigma,v) = -e^{-2\pi i\rho-2\pi i\sigma-2\pi i v}$$
$$\prod_{\substack{k,l,j\in\mathbf{Z}\\ k,l\geq 0, j<0 \text{ for } k=l=0}} \left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} .$$

Dijkgraaf, Verlinde, Verlinde

The coefficients c(n) are defined via

$$8\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]=\sum_{j,n\in\mathbf{Z}} c(4n-j^2)\, e^{2\pi i n\tau+2\pi i j z}$$

The starting point of both the four and five dimensional black holes is the elliptic genus of symmetric product of T^4's, describing the degeneracies associated with the relative motion between the D1 and D5-branes.

Dijkgraaf, Moore, Verlinde, Verlinde

Z_{5D} and Z_{4D} are obtained by multiplying it by the partition function associated with the additional degrees of freedom of the system.

Task

1. Calculate the partition function Z_{5D}^{hair} associated with the hair degrees of freedom of the 5D black hole.

2. Calculate the partition function Z_{4D}^{hair} associated with the hair degrees of freedom of the 4D black hole.

Compare Z_{5D}/Z_{5D}^{hair} with Z_{4D}/Z_{4D}^{hair}.

Hair removal

Hair of five dimensional black hole:

1. Normalizable plane wave like excitations of the gravitino.

– characterized by four independent functions of $(t+y)$

t: time   y: coordinate along S^1

2. Some additional fermion zero modes associated with broken supersymmetry.

All these deformations have been constructed explicitly as classical supersymmetric solutions of the supergravity equations of motion.

Result for the hair partition function:

$$Z_{5D}^{hair} = (e^{\pi i v} - e^{-\pi i v})^4 \prod_{l \geq 1} (1 - e^{2\pi i l \rho})^4 \,.$$

Hair of four dimensional black hole:

1. Normalizable plane wave of gravitons describing transverse oscillation of the system.

– characterized by 3 independent functions of $(t + y)$

2. Plane wave like excitations of the self-dual 2-form fields associated with the normalizable harmonic 2-form of the Taub-NUT space.

– characterized by 21 independent functions of $(t + y)$

3. Normalizable plane wave like excitations of the gravitino.

– characterized by 4 independent functions of $(t + y)$

4. Some additional fermion zero modes.

$$Z_{4D}^{hair}(\rho, \sigma, v) = \prod_{l=1}^{\infty} \left[\left(1 - e^{2\pi i l \rho} \right)^{-20}\right.$$

$$\mathbf{Z_{5D}}/\mathbf{Z_{5D}^{hair}} = -e^{-2\pi i\rho-2\pi i\sigma}\left(e^{\pi i v}-e^{-\pi i v}\right)^{-2} \prod_{\substack{k,l,j\in\mathbf{Z}\\ k\geq 1, l\geq 0}} \left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \left\{\prod_{l\geq 1}\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right\}$$

$\mathbf{Z_{4D}}/\mathbf{Z_{4D}^{hair}} =$ **same as above**

Thus the two results match, as is expected from identification of the near horizon geometries of the two black holes.