Knot theory for spatial graphs

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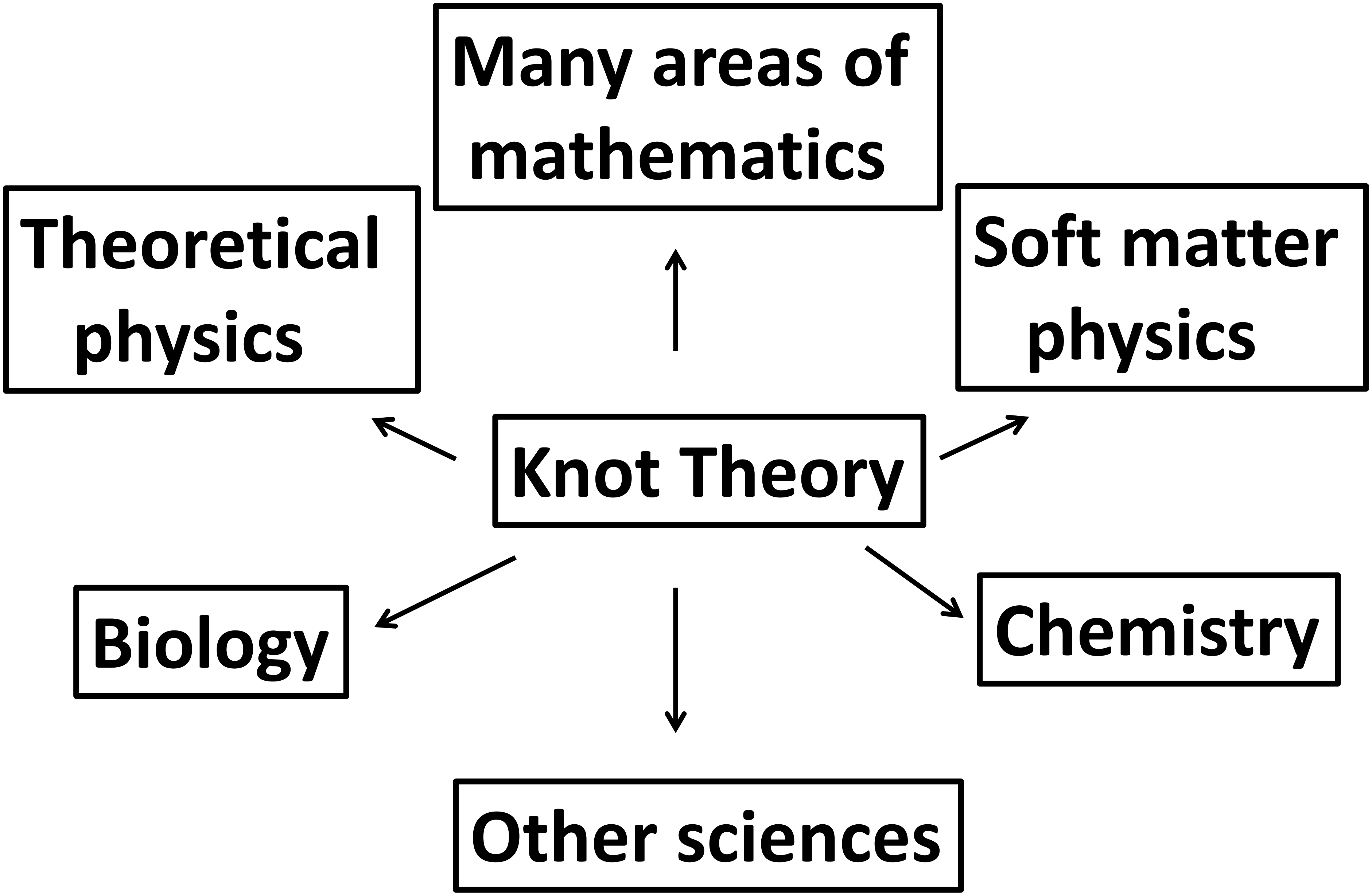

Many areas of mathematics
Theoretical physics
Soft matter physics
Knot Theory
Biology
Chemistry
Other sciences

Content

[Lecture 1]

Topology for spatial graphs without degree one vertices

[Lecture 2]

Unknotting notions on the spatial graphs

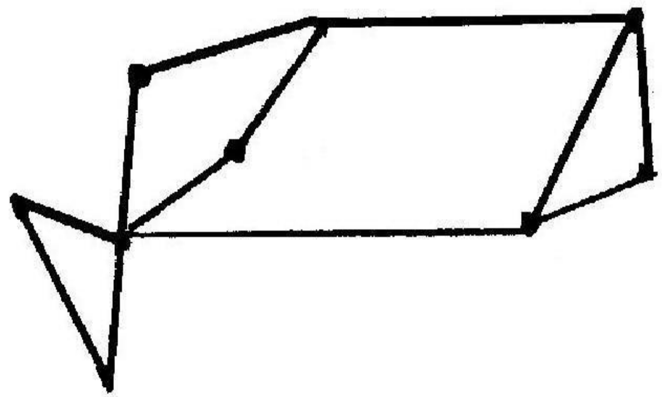
[Lecture 3]

Spatial graphs with degree one vertices attaching to a surface

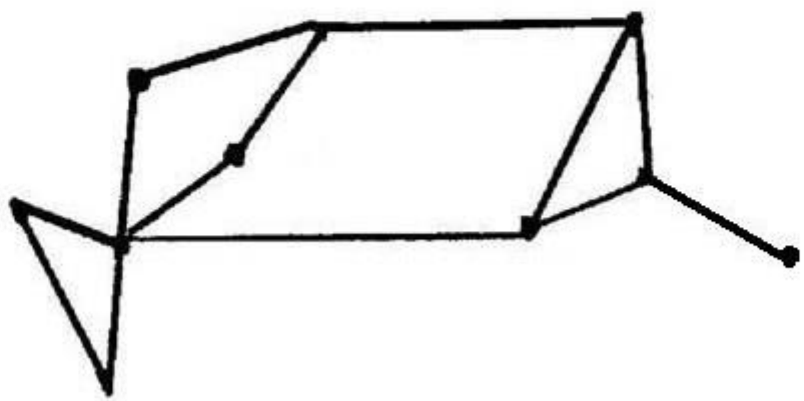
Knot theory for spatial graphs

[Lecture 1]
Topology for spatial graphs without degree one vertices

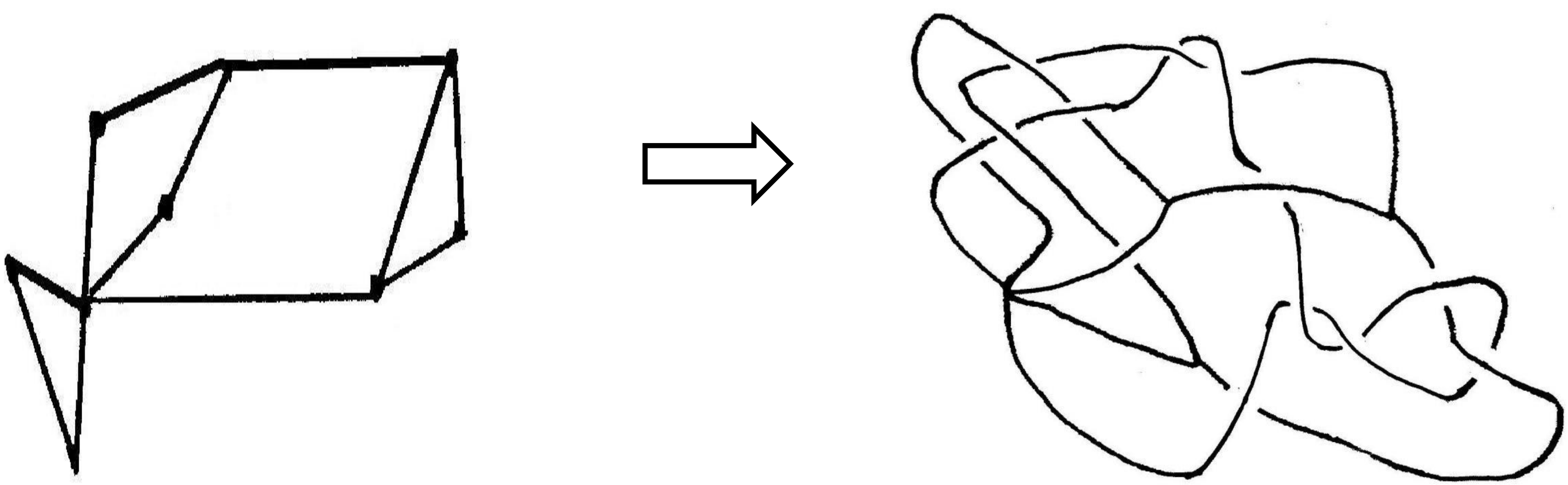
Let Γ = a finite graph with only vertices of degree ≧2.

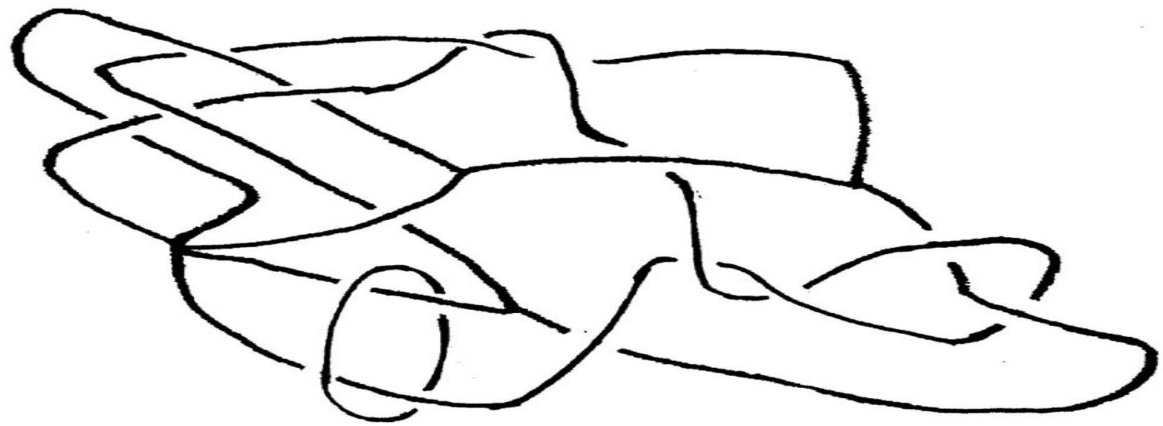


A finite graph with a degree one vertex:



<u>Definition.</u> **A <u>spatial graph</u> of Γ is the image G of an embedding Γ → R^3 which is sent to a polygonal graph in R^3 by a homeomorphism R^3 → R^3.**

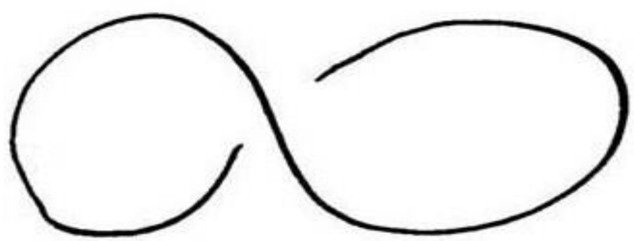




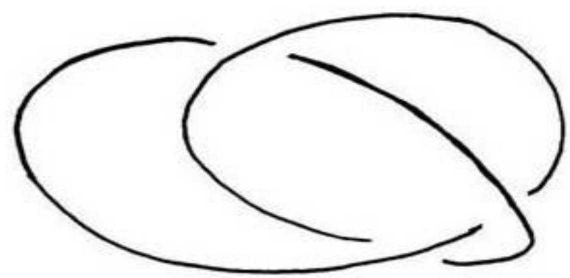
<u>Definition.</u>

A <u>diagram</u> $D=D_G$ of a spatial graph G in R^3 is an orthogonal projection image of G into a plane P with only double point singularities together with the upper-lower crossing information.

When Γ is a loop, G is called a knot , and it is trivial if it is the boundary of a disk.

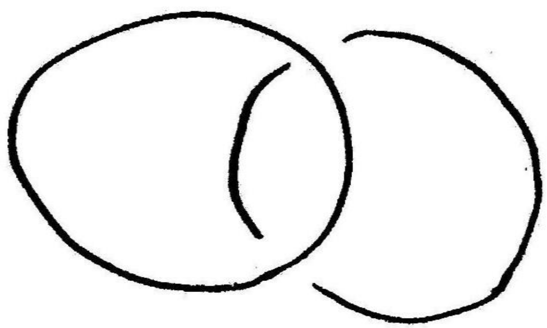


A trivial knot

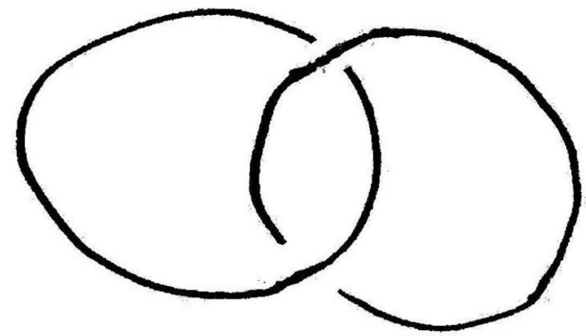


A non-trivial knot
(Trefoil knot)

When Γ is the disjoint union of finitely many loops, G is called a <u>link</u>, and it is <u>trivial</u> if it is the boundary of mutually disjoint disks.



A trivial link



A non-trivial link
(Hopf link)

Definition.

A spatial graph G is equivalent to a spatial graph G' if ∃ an orientation-preserving homeomorphism h: $R^3 \to R^3$ such that h(G)=G'.

Let [G] be the class of spatial graphs G' which are equivalent to G.

In a spatial graph G, ignore the degree 2 vertices.

Let v(G) be the set of vertices with degree ≧3 in G.

Fundamental topological problem on spatial graphs :

(1) Study what kinds of spatial graphs there are. List them up to equivalences.

(2) Determine whether two given spatial graphs of a graph Γ are equivalent or not.

This problem is a natural generalization of the fundamental problem of knot theory.

<u>Fundamental problem on knot theory :</u>

(1) Study what kinds of knots or links there are. List them up to equivalences.

(2) Determine whether two given knots or links are equivalent or not.

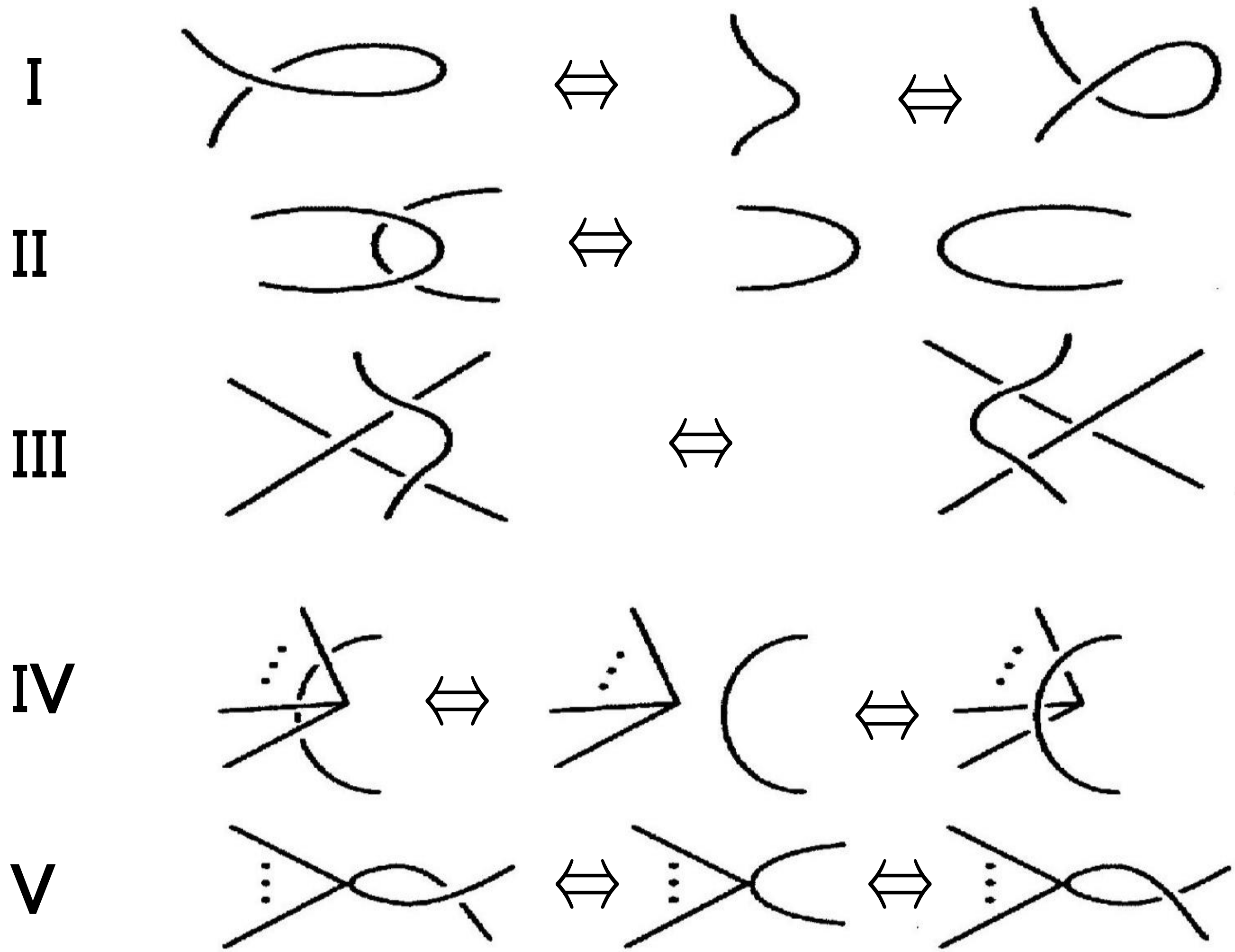
THEOREM 1.1 (Equivalence Theorem).

Explained in: [Kauffman,1989]

L. H. Kauffman, Invariants of graphs in three space, Trans. Amer. Math. Soc. 311(1989), 697-710.

G and G' are equivalent if and only if any diagram $D=D_G$ of G is deformed into any diagram $D'=D_{G'}$ of G' by a finite sequence of the generalized Reidemeister moves.

Generalized Reidemeister moves



Idea of the proof:

Let G and G' be equivalent spatial graphs.

Regard G and G' as polygonal graphs.

After some generalized Reidemeister moves on D_G and $D_{G'}$, we can assume that

∃ a homeomorphism $h:R^3 \to R^3$ such that $h(G)=G'$ and $h|B$=the identity for a 3-ball B containing v(G).

Thus, ∃ a one-parameter family of piecewise-linear homeomorphisms h_t:$R^3 \to R^3$ ($0 \leqq t \leqq 1$) such that $h_0=1$, $h_1(G)=G'$, $v(G)=v(G')$ and

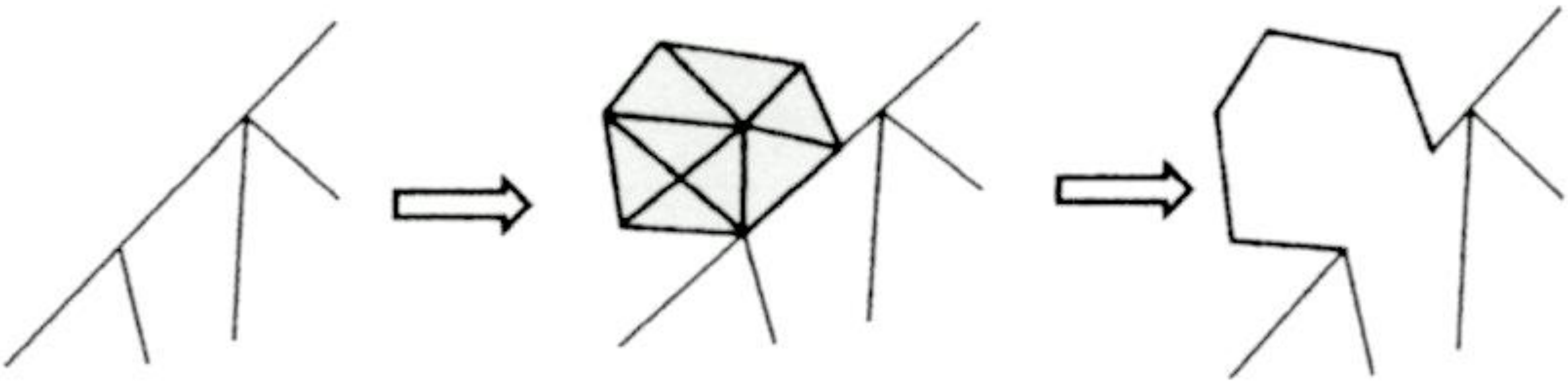
$$h_t|v(G) = \text{the identity } (0 \leqq t \leqq 1) .$$

Then, for example, by

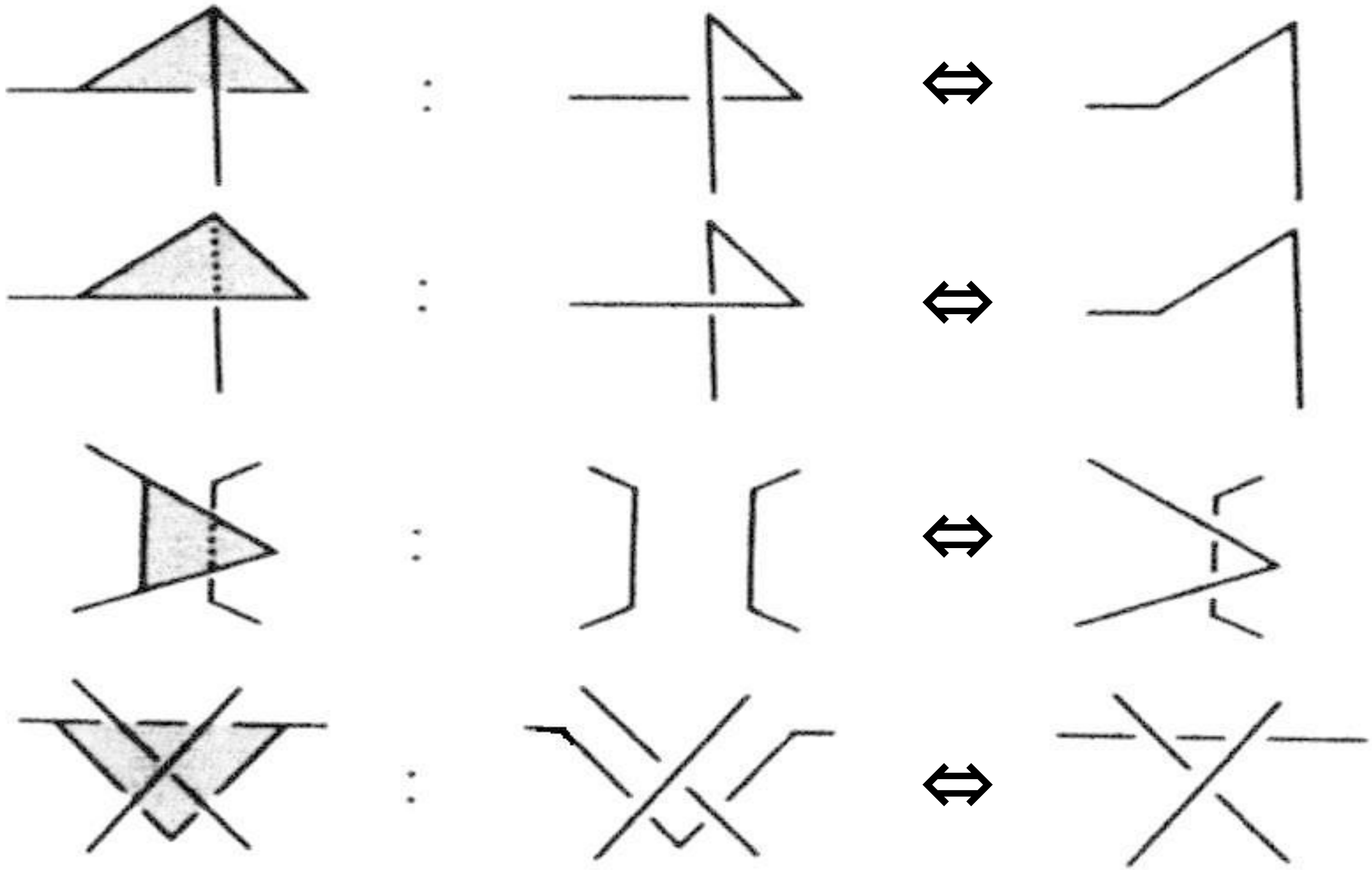
[Kamada-Kawauchi-Matumoto, 2001]

S. Kamada, A. Kawauchi and T. Matumoto, Combinatorial moves on ambient isotopic submanifolds in a manifold, J. Math. Soc. Japan 53(2001),321-331

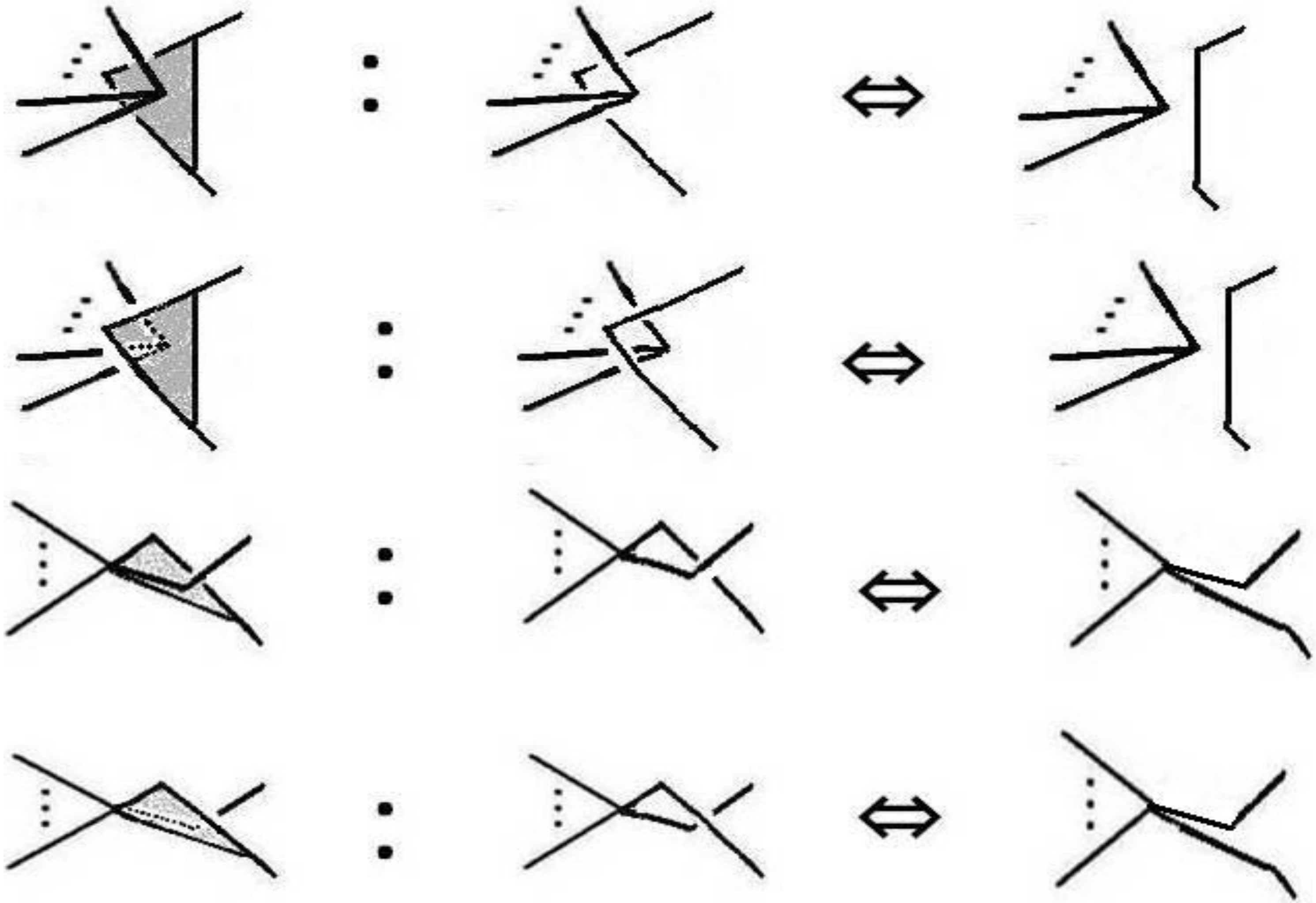
we see that G' is obtained from G by a finite number of cellular moves, that is, a combination of a finite number of 2-simplex moves.



A cellular move



2-simples moves on I, II, III



2-simples moves on IV, V

By a slight leaning of the plane P used for the orthogonal projection $p_a : R^3 \rightarrow P$, any diagram D of G is deformed into any diagram D' of G' by a finite sequence of the generalized Reidemeister moves.

This completes the proof of Theorem 1.1 (Equivalence Theorem).//

Let $[D_G]$ be the class of diagrams obtained from a diagram D of G by the generalized Reidemeister moves.

Then $[G] \Leftrightarrow [D_G]$

One basic problem on spatial graph theory is to ask a relationship to knot theory.

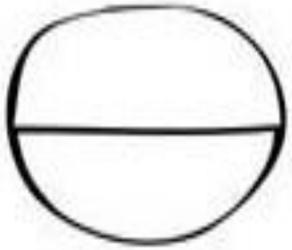
Definition.

A constituent knot (or a constituent link, resp.) of a spatial graph G is a knot (or link, resp.) contained in G.

<u>Proposition.</u>

If two spatial graphs G^* and G are equivalent, then there is a graph-isomorphism $f : G^* \to G$ such that every constituent knot or link L^* of G^* is equivalent to the corresponding constituent knot or link $f(L^*)$ of G.

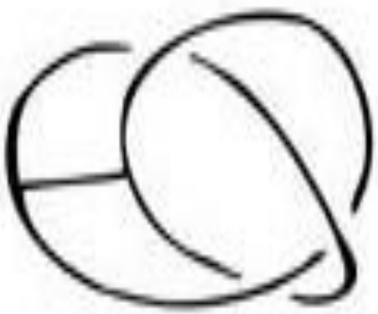
Examples on θ-curves:



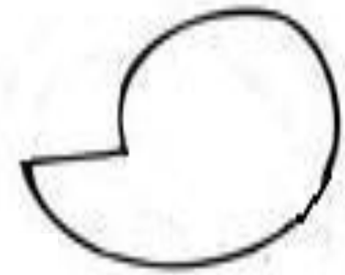
$\Rightarrow$

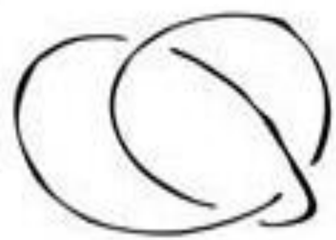
Trivial θ-curve **The constituent knots**

$\not\Downarrow$



$\Rightarrow$

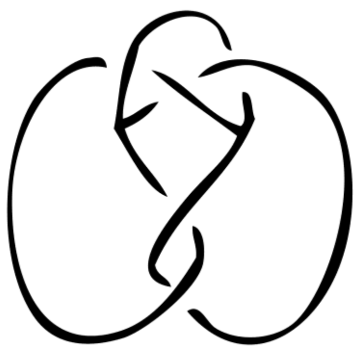




(A non-trivial knot)

A knotted θ-curve **The constituent knots**

Kinoshita's θ-curve is known to be non-equivalent to a trivial θ-curve, but it has only trivial constituent knots :



⇒

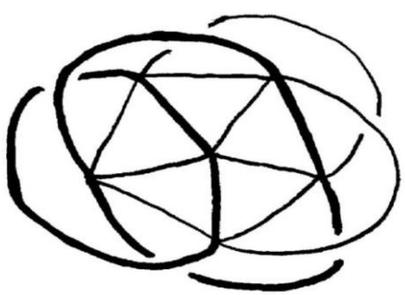
Kinoshita's θ-curve

The constituent knots are all trivial.

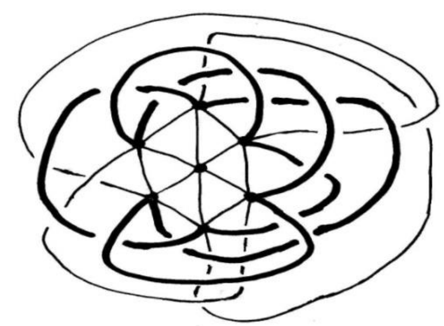
Conway-Gordon Theorem.

J. H. Conway and C. McA. Gordon, Knots and links in spatial graphs, J. Graph Theory7(1983), 445-453.

Every spatial 6-complete graph K_6 contains a non-trivial constituent link.
Every spatial 7-complete graph K_7 contains a non-trivial constituent knot.



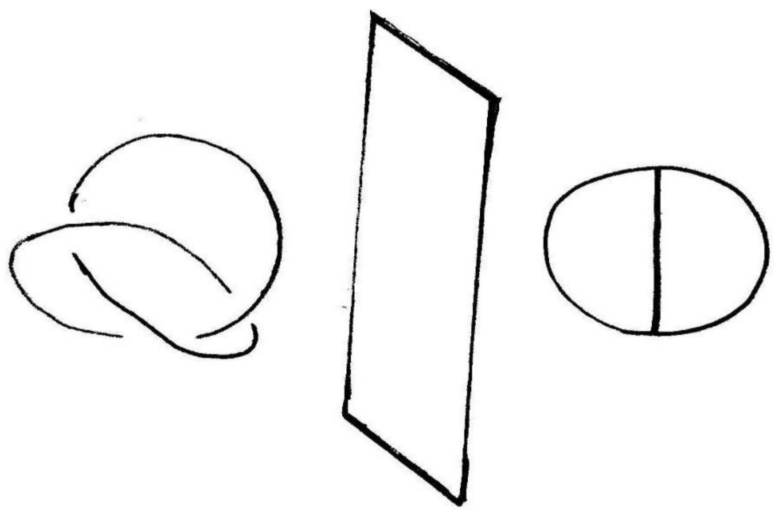
A spatial graph of K_6



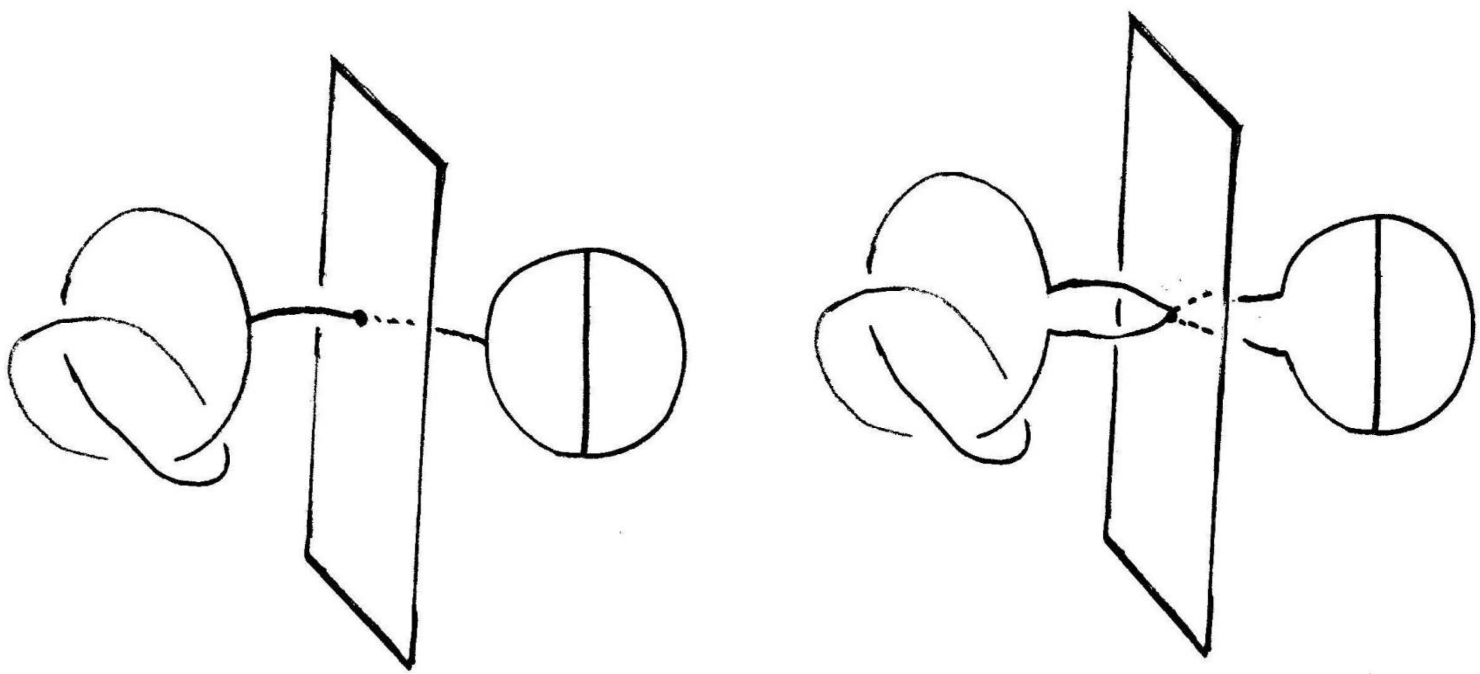
A spatial graph of K_7

Definition. A spatial graph G without degree one vertices is **prime** if G is not equivalent to any spatial graph G’ in the following cases (0)-(2):

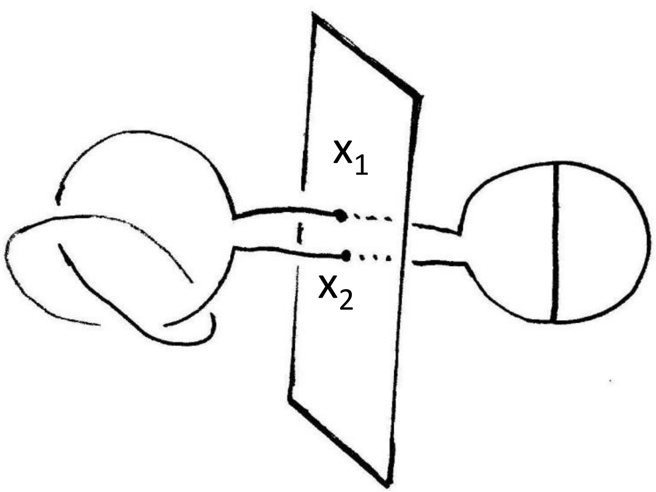
(0) There is a plane which separates G’ into two spatial graphs.

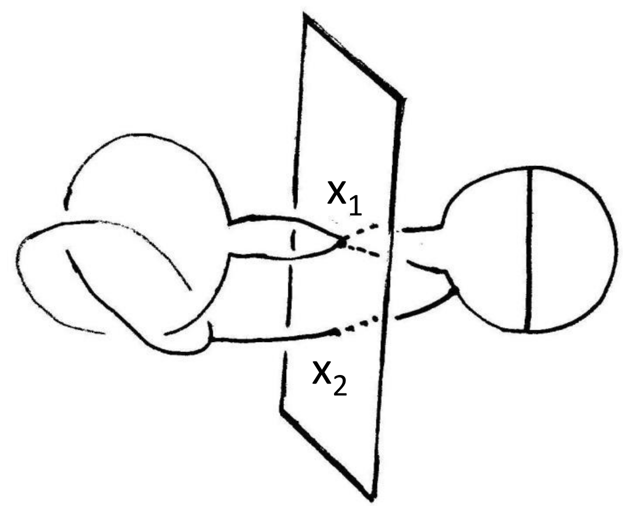


(1) There is a plane meeting G' in one point which separates G' into two spatial graphs.



(2) There is a plane meeting G' in two points x_1, x_2 which separates G' into two spatial graphs G_1', G_2' such that none of $G_i' \cup [x_1, x_2]$ (i=1,2) is a trivial knot.





The following shows that *Spatial Graph Theory is much harder than Knot Theory:*

<u>THEOREM 1.2.</u>

For every spatial graph G except knots and links , ∃ an infinite family of prime spatial graphs G* (up to equivalences) with a graph-isomorphism f : $G^* \to G$ such that every constituent knot or link L* of G* is equivalent to the corresponding constituent knot or link f(L*) of G.

To explain Theorem 1.2, we introduce <u>topological imitation theory</u>.

<u>Definition.</u> Let $S^3 = R^3 \cup \{\infty\}$. $I=[-1,1]$.

A map $q: (S^3, G^*) \to (S^3, G)$ is a <u>normal imitation</u> if:

$$q: (S^3, G^*) \xrightarrow{\cong} Fix(\alpha) \subset (S^3, G) \times I \xrightarrow{proj} (S^3, G)$$

for an involution α on $(S^3, G) \times I = (S^3 \times I, G \times I)$ such that

$$\alpha(x,t)=(x,-t) \text{ for } \forall (x,t) \in S^3 \times \partial I \cup N(G) \times I,$$

where $N(G)$ is a regular neighborhood of G in S^3.

Properties.

Let $q: (S^3, G^*) \rightarrow (S^3, G)$ be a normal imitation, and $N(G)$ a normal regular neighborhood.

Then:

(0) $N(G^*)=q^{-1}N(G)$ is a regular neighborhood of G^* with $q|_{N(G^*)} : N(G^*) \rightarrow N(G)$ a homeomorphism and $q(E(G^*))= E(G)$ for the exteriors $E(G^*)=cl(S^3-N(G^*))$ and $E(G) =cl(S^3-N(G))$.

(1) The map $q_1 : (S^3, G_1^*) \to (S^3, G_1)$ defined for $\forall$ graph G_1 in N(G) and $G_1^* = q^{-1}(G_1)$ is a normal imitation.

(2) $Link_{S^3}(L^*) = Link_{S^3}(L)$ for $\forall$ oriented 2-component link L in N(G) and $L^* = q^{-1}(L)$.

(3) The homomorphism

$$q_{\#}: \pi_1(S^3 - G^*) \rightarrow \pi_1(S^3 - G)$$

is an epimorphism whose kernel is a perfect group: $\mathrm{Ker}\, q_{\#} = [\mathrm{Ker}\, q_{\#}, \mathrm{Ker}\, q_{\#}]$.

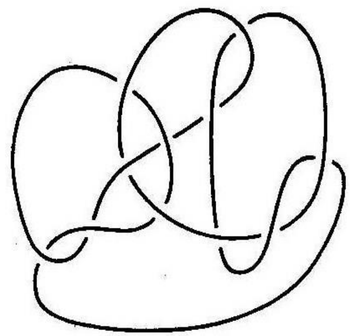
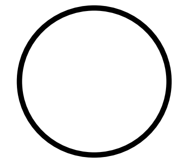
(4) For normal imitations

$q: (S^3, G^*) \rightarrow (S^3, G)$ and $q^*: (S^3, G^{**}) \rightarrow (S^3, G^*)$,

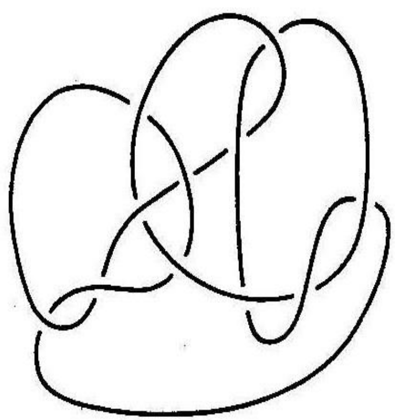
$\exists$ a normal imitation

$$q^{**}: (S^3, G^{**}) \rightarrow (S^3, G).$$

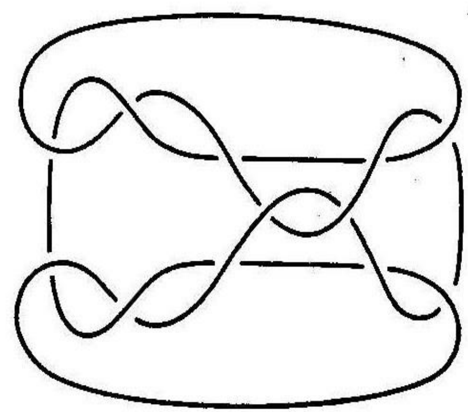
<u>Example of an imitation of a trivial knot</u>:

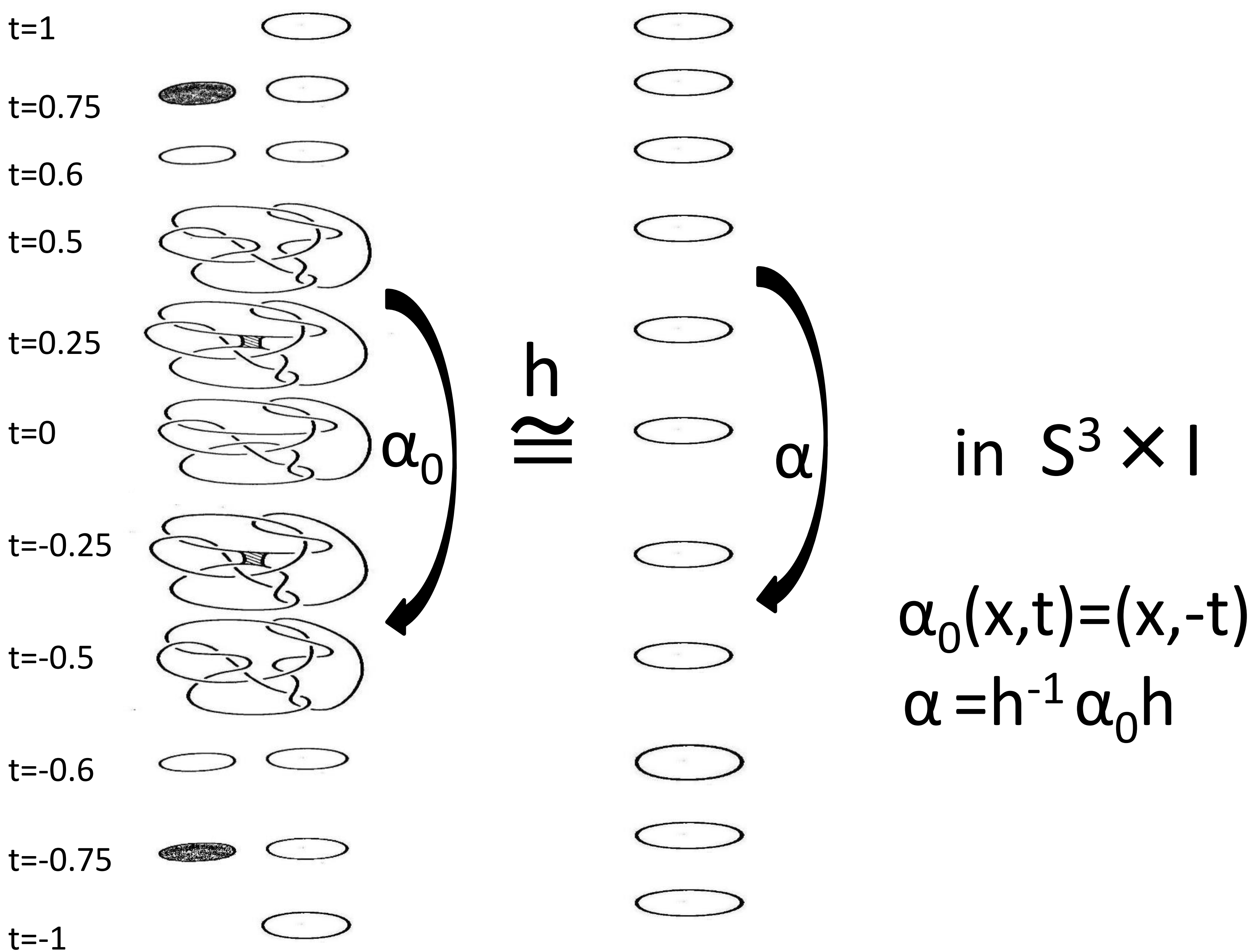
q: $(S^3,$  $) \to (S^3,$  $)$

▪ **Kinoshita-Terasaka knot (discovered in 1957):**



$\cong$





t=1
t=0.75
t=0.6
t=0.5
t=0.25
t=0
t=-0.25
t=-0.5
t=-0.6
t=-0.75
t=-1
α_0
h
$\cong$
α
in $S^3 \times I$
$\alpha_0(x,t)=(x,-t)$
$\alpha = h^{-1}\alpha_0 h$

Definition. A normal imitation $q: (S^3, G^*) \to (S^3, G)$ is homotopy-trivial if $\exists$ a 1-parameter family $\{q_s\}_{0 \leqq s \leqq 1}$ of normal imitations $q_s: (S^3, G^*) \to (S^3, G)$ such that $q_0 = q$ and q_1 is a homeomorphism.

Definition. A normal imitation $q: (S^3, G^*) \to (S^3, G)$ is an AID imitation if $q|_{(S^3, G^* - \alpha^*)}: (S^3, G^* - \alpha^*) \to (S^3, G - \alpha)$ is homotopy-trivial for $\forall$ edges α, α^* of G, G^* with $q(\alpha^*) = \alpha$.

Existence Theorem (of an AID imtation).

A. Kawauchi, Almost identical imitations of (3,1)-dimensional manifold pairs, Osaka J. Math. 26(1989),743-758.

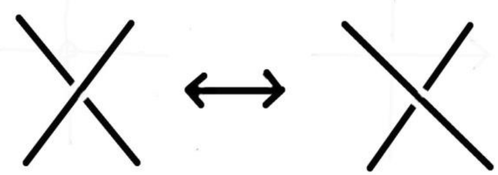
For ∀ spatial graph G, ∃ an infinite family of prime spatial graphs G* (up to equivalences) with an AID imitation q: $(S^3,G^*)\rightarrow(S^3,G)$.

Theorem 1.2 is a direct consequence of this theorem (Imitation Existence Theorem).

Corollary to Existence Theorem.

A. Kawauchi, Almost identical link imitations and the skein polynomial, Knots 90, Walter de Gruyter, 1992, 465-476.

For $\forall$ spatial graph G, $\exists$ an infinite family of prime spatial graphs G^* (up to equivalences) with an AID imitation q: $(S^3,G^*)\rightarrow(S^3,G)$ such that G is obtained from G^* by one crossing change.

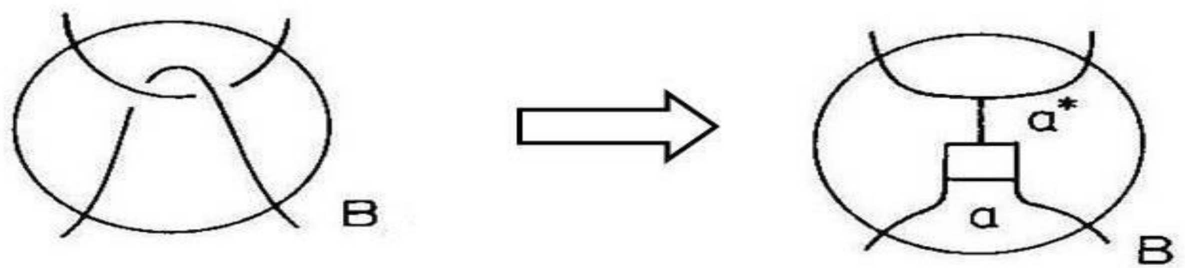
Here, one crossing change: 

Proof of Corollary to Existence Theorem.

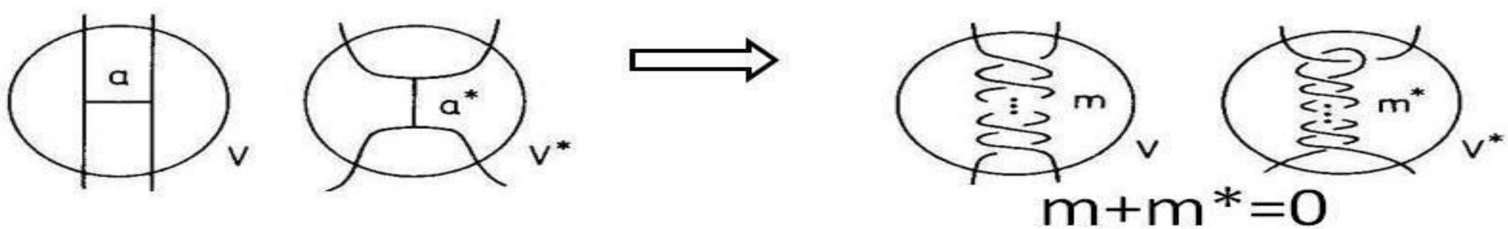
Assume that G has the left part of (1) where a crossing change gives a spatial graph equivalent to G.

Let G' be the spatial graph obtained from G by replacing the the left part of (1) with the right part of (1).

(1)



(2)

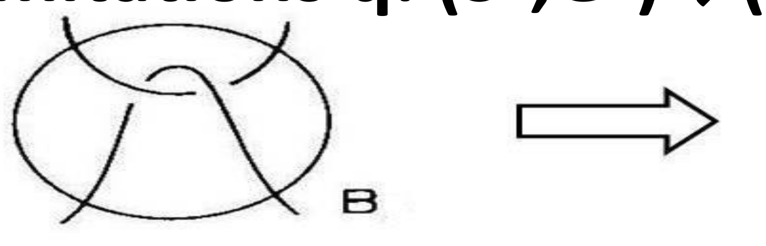


By Existence Theorem, ∃ an infinite family of prime spatial graphs G'^* (up to equivalences) with an AID imitation

$$q': (S^3, G'^*) \rightarrow (S^3, G').$$

and then replace the left parts of G' of (2) with the right parts of (2). If $|m|$ and $|m^*|$ with $m+m^*=0$ are taken sufficiently large, ∃ desired AID imitations $q: (S^3, G^*) \rightarrow (S^3, G)$ are obtained. //

(1)



(2)