Nonlinearity in Lagrangian data assimilation and hybrid particle Kalman filter

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Skew-product structure of the LaDA problem

- Recalling notation from Elaine's talk: combine
  - the prognostic variables (collectively denoted by x_v) and
  - the positions of the drifters (denoted by x_d)

  into the state vector:

  $$x = (x_v, x_d)^T$$

  gives the following skew-product structure of the dynamical model:

  $$\frac{dx_v}{dt} = m_v(x_v)\,, \qquad \frac{dx_d}{dt} = m_d(x_v, x_d) = V(x_d, x_v)\,,$$

  where V is the velocity of the fluid flow at the point x_d.
- *If the only observations are drifter locations*, then the observations at time t can be written as

  $$y(t) = Hx(t) + \eta$$

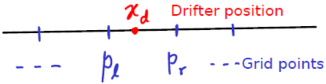
  where $x = (x_v, x_d)$, and $H = [0 \;\; I]$ is just a projection;

Drifter model m_d is always nonlinear

Two main cases are the following:

- When using discretized velocity field, $x_v = (\dots, v_l, v_r, \dots)$; So velocity V at the position of drifter $x_d \in [p_l, p_r]$ is obtained by some interpolation $\implies$ at least quadratic non-linearity:

$$V(x_d, x_v) \propto (p_r - x_d)v_l + (x_d - p_l)v_r$$



- When using spectral methods, $x_v = (\dots, v_1, v_2, \dots)$ containing the Fourier modes of velocity $\implies$

$$V(x_d, x_v) \propto v_1 e^{ik_1 x_d} + v_2 e^{ik_2 x_d} + \dots$$

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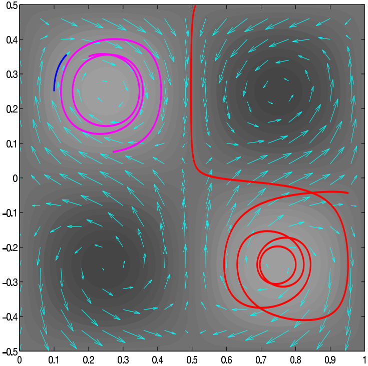
Linear shallow water equations with Lagrangian data

For two dimensional velocity (u, v) and height h fields:

$$\frac{\partial u}{\partial t} = v - \frac{\partial h}{\partial s_1},$$
$$\frac{\partial v}{\partial t} = -u - \frac{\partial h}{\partial s_2},$$
$$\frac{\partial h}{\partial t} = -\frac{\partial u}{\partial s_1} - \frac{\partial v}{\partial s_2},$$

Height and velocity fields



We seek periodic solutions on $\mathbb{R}^2$ in u, h, specifically, the following Fourier modes:

$$u(s_1, s_2, t) = -2\pi l \sin(2\pi k s_1)\cos(2\pi l s_2)u_0 + \cos(2\pi m s_2)u_1(t)$$
$$v(s_1, s_2, t) = 2\pi k \cos(2\pi k s_1)\sin(2\pi l s_2)u_0 + \cos(2\pi m s_2)v_1(t)$$
$$h(s_1, s_2, t) = \sin(2\pi k s_1)\sin(2\pi l s_2)u_0 + \sin(2\pi m s_2)h_1(t)$$

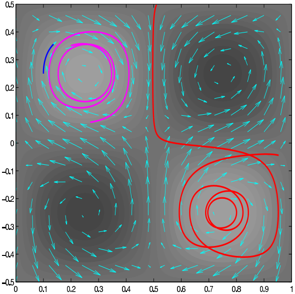
Linear shallow water equations with Lagrangian data

The amplitudes satisfy the following:

$$\dot{u}_0 = 0, \qquad \dot{u}_1 = v_1,$$
$$\dot{v}_1 = -u_1 \quad -2\pi m h_1, \quad \dot{h}_1 = 2\pi m v_1$$

The observations are the positions of the drifters that satisfy:

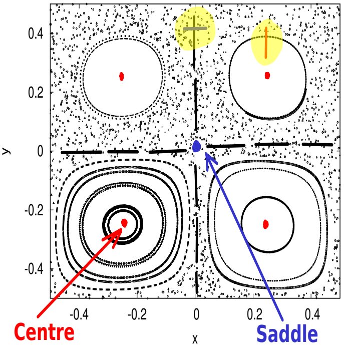
$$\dot{s}_1(t) = u(s_1(t), s_2(t), t), \quad \dot{s}_2(t) \quad = v(s_1(t), s_2(t), t)$$



- Observations of drifter positions alone: Lagrangian data assimilation
- Main points of interest: this flow has
  - Nonlinear centre with shear (differential rotation) around it and
  - The unstable fixed points have chaotic regions near the separatrices
  - Velocity field is coupled to an additional variable (height)

Linear shallow water equations with Lagrangian data

Poincaré plot of drifter trajectories



A few more properties of the drifter dynamics:

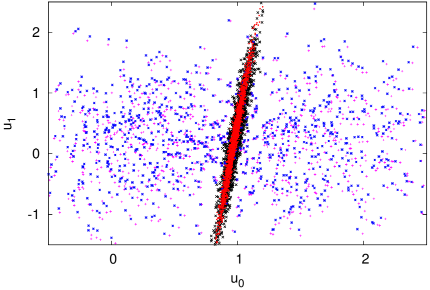
- No attractor (the unperturbed flow is Hamiltonian)
- Some regions with regular trajectories (periodic / quasi-periodic)
- Some regions with chaotic trajectories

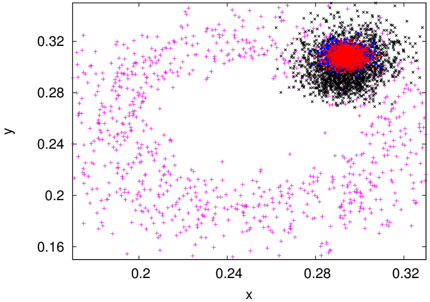
In the case of the model above, the velocity flow itself:

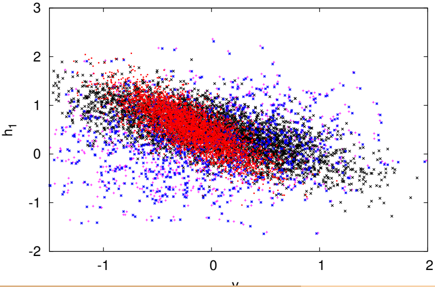
- has no attractor or chaotic dynamics
- is purely periodic

Thus, the nonlinearity is entirely in the drifter dynamics, which are the observed variables.

An extreme, but not uncommon, effect of nonlinearity







- Particle filters may provide a possible way to capture such extreme non-Gaussianity.
- But there are well-known problems of degeneracy of weights, which only worsens due to the chaotic nature of the system.

Chaotic trajectories also affect the particle filter

- The particles that have low likelihood have
  - "poor" drifter trajectory, which is in turn because
  - the velocity flow itself is far from the true flow.
- Thus a "importance sampling" step will be to sample the flow itself using the observations.
- But the flow is usually high dimensional: thus we use the ensemble Kalman filter for the flow alone.
- The nonlinearity can be captured using the weights of the drifters.

Hybrid Grid-Particle Filter

H. Salman, *Q. J. R. Meteorol. Soc.*, vol. 134, pp. 1539-1550, 2008

Decompose the joint distribution for flow x^F, x^D and approximate by using particles for the flow variables:

$$p(x^F, x^D) = p(x^D|x^F)p(x^F)$$
$$\approx \frac{1}{N_e}\sum_{i=1}^{N_e} \delta(x^F - x_i^F)\phi_i(x^D)$$

where $\phi_i(x^D) = p(x^D|x_i^F)$.

Hybrid Grid-Particle Filter

- Particle filter on flow variables x^F
- Exact probability density function on drifter variables x^D
- When observation is available, drifter pdf is updated via Bayes' Rule: $\phi_i(x^D)^a = \phi_i(x^D)^f p(y|x_D^f)$
- PF weights are defined in terms of $\phi_i(x^D)$, so they are updated implicitly when drifter pdf is updated: $w_i = \int \phi_i(x^D) dx^D$

$$\left\{ \begin{matrix} x_1^F \\ \phi_1(x^D) \\ w_1 \end{matrix} \right\} \quad \left\{ \begin{matrix} x_2^F \\ \phi_2(x^D) \\ w_2 \end{matrix} \right\} \quad \dots \quad \left\{ \begin{matrix} x_{Ne}^F \\ \phi_{Ne}(x^D) \\ w_{Ne} \end{matrix} \right\}$$

Hybrid Grid-Particle Filter

Main disadvantage of grid-particle filter:

- Solving exact drifter pdf evolutions can be computationally intensive

Our contribution: make a further approximation

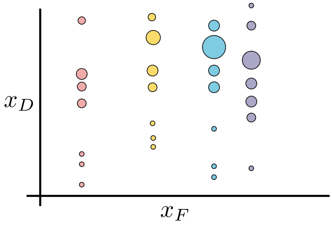
$$p(x^F, x^D) = p(x^D|x^F)p(x^F)$$
$$\approx \frac{1}{N_e}\sum_{i=1}^{N_e}\sum_{j=1}^{M} w_{i,j}\delta(x^D - x^D_{i,j})\delta(x^F - x^F_i)$$

ie, replace exact drifter distribution on previous slide with weighted ensemble.

Hybrid PF-EnKF

- EnKF on high-dimensional flow state x^F
- PF on low-dimensional, highly nonlinear Lagrangian coordinates x^D

Ensemble: $\{x_i^F, x_{i,j}^D, w_{i,j}\}_{i=1\ldots N_e,\, j=1\ldots M}$



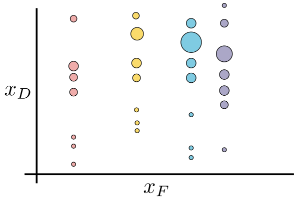
Update weights via standard particle filter update, and at resampling times, update x^F according to EnKF analysis.

Hybrid PF-Kalman filter update

- Update the flow particles (high dimensional, less nonlinear) using EnKF update step, but keep the same weights:

$$x_i^{F,a} = x_i^{F,f} + K(y - \bar{x}_i^{D,f})$$

where $\bar{x}_i^{D,f}$ is the average of $x_{i,j}^{D,f}$. This gives $\{x_i^{F,a}, w_i^{F,f}\}$.



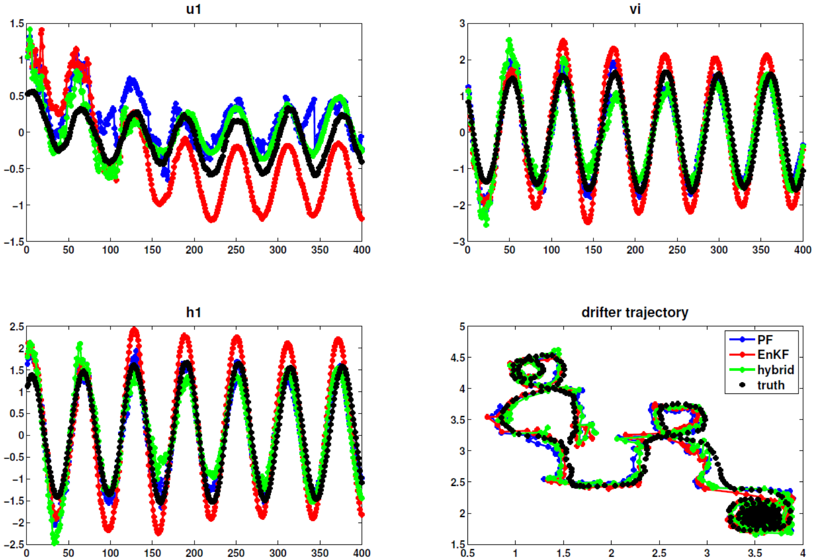
- Resample (only the flow part) from the above distribution.
- Update the weights of the drifter particles:

$$w_{i,j}^{D,a} = w_{i,j}^{D,f} p(y|w_{i,j}^{D,f})$$

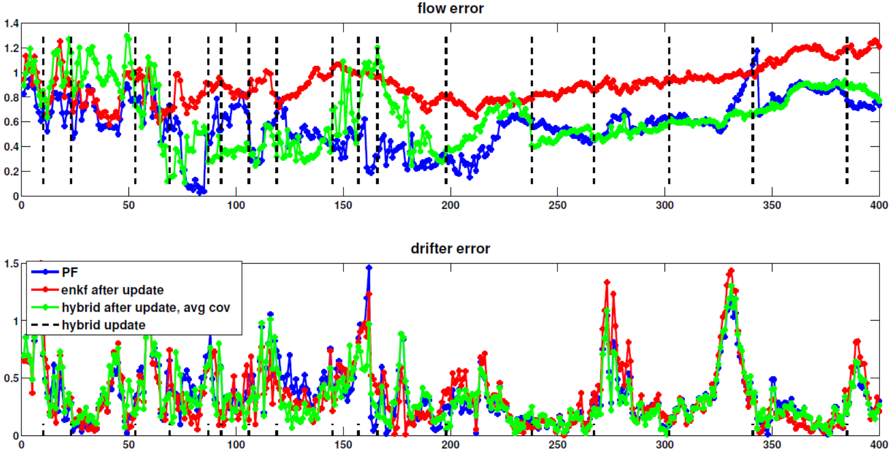
This gives $\{x_{i,j}^{D,f}, w_{i,j}^{D,a}\}$.

- Resample (only the drifter part) from the above distribution.

Long trajectory, frequent observations



Long trajectory, frequent observations - errors



Summary

- Lagrangian data assimilation provides a unique system in which to test assimilation algorithms
- The skew-product nature of the dynamics, and the highly nonlinear nature of the Lagrangian drifters suggests different treatment of the velocity flow and the drifter position.
- The hybrid particle - Kalman filter is an attempt to capture nonlinearity and to overcome the problem is degeneracy of weights.