Multiscale Dynamics and Information: Dimensional Reduction and Data Assimilation

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Data Assimilation in High-Dimensional Chaotic Systems

Objectives are to develop efficient methods and algorithms for

- the assimilation of data in high-dimensional multi-scale systems, and
- steering the measurement process towards "information rich" data locations.

$\downarrow\downarrow\downarrow$: efficient methods for assimilation : $\downarrow\downarrow\downarrow$

Dimensional reduction and Nonlinear filtering

- Dimensional reduction in nonlinear filtering: – issues in high-dimensions
- Construct and validate homogenized particle filters with Optimal nudging

$\downarrow\downarrow\downarrow$: information rich data locations :$\downarrow\downarrow\downarrow$

Sensor Control

- Dynamically steer the measurement processes (use Finite-Time LE)
- Control that minimize the loss attributed to the estimation procedure

Lecture 3: Homogenized Hybrid Particle Filter (HHPF)

"curse of dimensionality" is partially resolved by the Homogenized Filtering Equations

$$d\rho_t^0(\varphi) = \rho_t^0\left(\bar{\mathcal{L}}\varphi\right)dt + \rho_t^0\left(\bar{h}\varphi\right)dY_t^\varepsilon, \quad \rho_0^0(\varphi) = \mathbb{E}\left[\varphi\left(X_0^0\right)\right]$$

Particle method

- Numerical approximation of π^0 or ρ^0
- (weighted) particles to represent (conditional) density

Main Idea: The solution of the nonlinear filtering equation is approximated by a system of N particles with varying weights

$$U_N^\varepsilon(t) = \sum_{j=1}^{N} w^{x_t^{\varepsilon,j}} \delta_{x_t^{\varepsilon,j}},$$

where $\{x_t^{\varepsilon,j}; 1 \leq j \leq N\}$ is a set of random particles which evolve according to system dynamics and $w^{x_t^{\varepsilon,j}}$s are the corresponding weights.

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Homogenized Hybrid Particle filter [7]

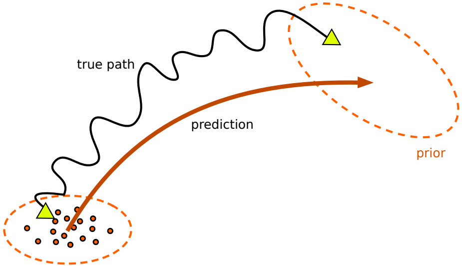

Figure: Initial condition

Zakai equation: $\underbrace{d\bar{u}^{\varepsilon}(x,t) = \bar{\mathcal{L}}\bar{u}^{\varepsilon}(x,t)dt}_{\text{Fokker-Planck equation}} + \bar{h}^{*}(x,t)\bar{u}^{\varepsilon}(x,t)dY_t^{\varepsilon}$

Weight update: $w_{t+\delta t}^{\varepsilon,i} = \exp\left\{\int_t^{t+\delta t} \bar{h}^{*}(x_s^i, s)dY_s^{\varepsilon} - \frac{1}{2}\int_t^{t+\delta t} \left\|\bar{h}(x_s^i, s)\right\|^2 ds\right\}$

$\bar{w}_{t+\delta t}^{\varepsilon,i} = \frac{w_{t+\delta t}^{\varepsilon,i}}{\sum_{i=1}^{N_s} w_{t+\delta t}^{\varepsilon,i}}$

[7] J. H. Park, N. Sri Namachchivaya, H. Yeong, Particle Filters in a multi-scale environment: Homogenized hybrid particle filter. J. Appl. Mech., 78(6), 2011

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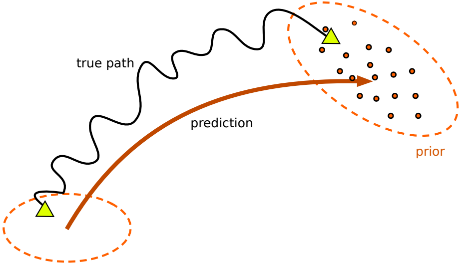

Figure: Particles propagation

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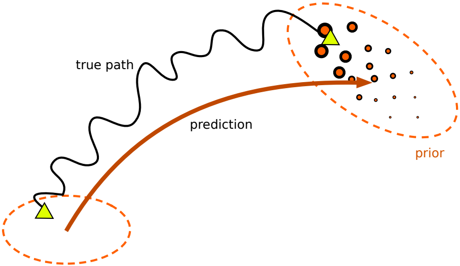

Figure: Weights update

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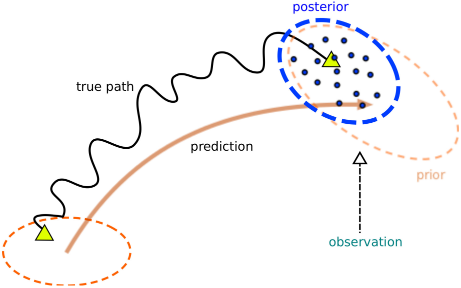

Figure: Resample → Conditional density

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$$\dot{Z}_t^\varepsilon = -\frac{1}{\varepsilon}(Z_t^\varepsilon - X_t^\varepsilon) + \frac{1}{\sqrt{\varepsilon}}\dot{W}_t, \quad Z_0^\varepsilon = z_0 \tag{34a}$$

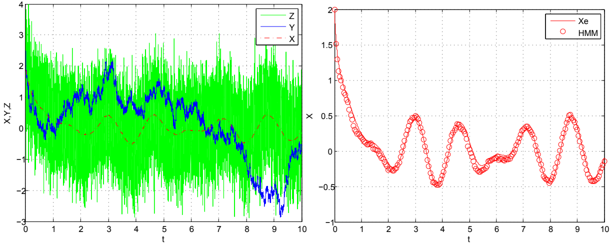
$$\dot{X}_t^\varepsilon = -(Z_t^\varepsilon)^3 + \sin(\pi t) + \cos(\sqrt{2}\pi t), \quad X_0^\varepsilon = x_0 \tag{34b}$$

with the observation

$$Y_t^\varepsilon = \frac{1}{2}(X_t^\varepsilon)^2 + B_t$$

Homogenized signal

$$\dot{X}_t^0 = -(X_t^0)^3 - \frac{3}{2}\sigma^2 X_t^0 + \sin(\pi t) + \cos(\sqrt{2}\pi t), \quad X_0^0 = x. \tag{35a}$$



* Jun H. Park, N. Sri Namachchivaya and Hoong Chieh Yeong "Particle Filters in a Multiscale Environment: Homogenized Hybrid Particle Filter (HHPF)," *Journal of Applied Mechanics*, Vol. 78(6), 2011, pp. 61001-1 - 61001-10.

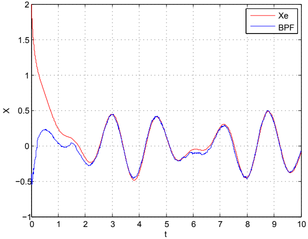


Figure: PF -Full System:447Sec and Slow variable is approximated by the $N = 500$ particles

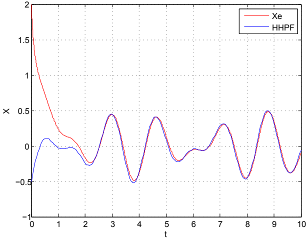


Figure: HHPF:17Sec and Slow variable is approximated by the $N = 500$ particles

We compare the performance of HHPF algorithm with that of the standard particle filter (PF) on the following multi-scale system.

$$\dot{X}_t^\varepsilon = -\frac{2}{\varepsilon}\zeta_t^\varepsilon Z_t^\varepsilon,$$
$$\dot{\zeta}_t^\varepsilon = -\frac{1}{\varepsilon^2}\zeta_t^\varepsilon + \frac{1}{\varepsilon}X_t^\varepsilon Z_t^\varepsilon + \frac{1}{\varepsilon}\dot{W}_t^1,$$
$$\dot{Z}_t^\varepsilon = -\frac{1}{\varepsilon^2}Z_t^\varepsilon + \frac{1}{\varepsilon}X_t^\varepsilon \zeta_t^\varepsilon + \frac{1}{\varepsilon}\dot{W}_t^2.$$

Here X is slow and (ζ, Z) is a two dimensional fast process, and we fix $\varepsilon = 10^{-3}$.
– the effect of $\frac{1}{\varepsilon}XZ$ term on ζ is small compared to the other terms
– but, the invariant measure of $\dot{\zeta}_t^\varepsilon = -\frac{1}{\varepsilon^2}\zeta_t^\varepsilon + \frac{1}{\varepsilon}\dot{W}_t^1$ is Gaussian with mean zero.
Similar is the case with the process Z.
For the observation process we consider

$$\dot{Y}_t = h(X_t^\varepsilon, \zeta_t^\varepsilon, Z_t^\varepsilon) + \sigma_y \dot{B}_t \tag{36}$$

with $\sigma_y = 0.1$ and

$$h(x, \zeta, z) = \frac{1}{4}x^2 + x(\zeta^2 + (z+1)^2).$$

the law of X as $\varepsilon \to 0$ is same as that of an SDE whose drift and diffusion are obtained as

$$\begin{cases} \bar{b}(x) & = \int (-2\zeta_1 z_2 - 2z_1\zeta_2)\, \mu_x(d\zeta_1, dz_1, d\zeta_2, dz_2), \\ \bar{\sigma}^2(x) & = 2 \int \mu(d\zeta_1, dz_1)\,(2\zeta_1 z_1) \int_0^\infty \mathbb{E}[2\zeta^1_{\varepsilon^2 s} Z^1_{\varepsilon^2 s}] ds, \end{cases} \tag{37}$$

where $\mu_x(d\zeta_1, dz_1, d\zeta_2, dz_2)$ is the invariant measure of the fast processes $(\zeta_1, \zeta_2, z_1, z_2)$:

$$\begin{cases} \dot{\zeta}^1_t & = -\frac{1}{\varepsilon^2}\zeta^1_t + \frac{1}{\varepsilon}\dot{W}^1_t, \\ \dot{\zeta}^2_t & = -\frac{1}{\varepsilon^2}\zeta^2_t + \frac{1}{\varepsilon^2} x Z^1_t, \\ \dot{Z}^1_t & = -\frac{2}{\varepsilon^2} Z^1_t + \frac{1}{\varepsilon}\dot{W}^2_t, \\ \dot{Z}^2_t & = -\frac{2}{\varepsilon^2} Z^2_t + \frac{1}{\varepsilon^2} x \zeta^1_t. \end{cases} \tag{38}$$

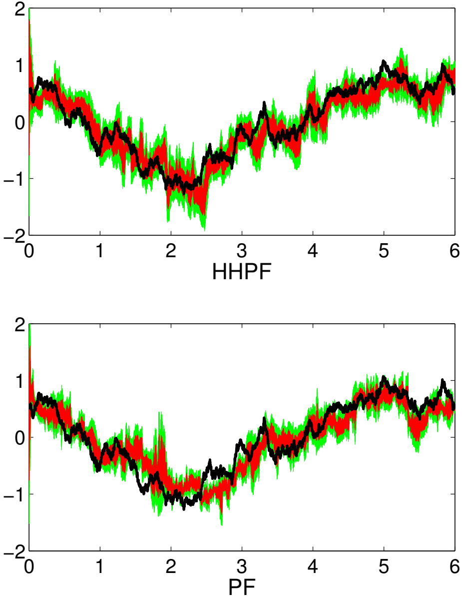


Figure: Comparison of HHPF and PF

black line – true signal, red shade – spread of estimated $var(X)$ and green shade – spread of $2var(X)$ about the estimated mean.

"Close, but no cigar" when it comes to data assimilation in chaotic systems with sparse data – observations taken over discrete times close to "error doubling time".

Difficult Issues:

- High-dimensional complex systems with positive Lyapunov exponents – small errors in the estimate of the current state can grow to have a major impact on the subsequent forecast.
- HHPF faces the well known problem of particle collapse – very few of the particles end up close to the actual location, and hence receive large fraction of the weight.

Solutions:

- Need to superimpose a control on the particle dynamics which drives the particles to locations most representative of the observations.
- Need to make sure not to over do the control - otherwise the sample diversity is lost.

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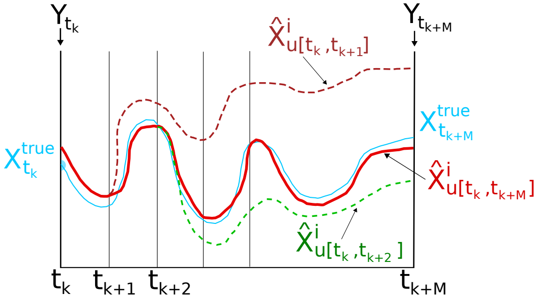
Optimal Nudging in Particle Filtering

Nudging consists of adding forcing terms to the "prognostic" equations, that steer the particles toward the observations – at the same time, not to over control so that the sample diversity is lost.



In directions of large signal noise amplitude, allow for more correction and in directions where the quality of the observation is poor, we allow our particle to be further away from the observation.

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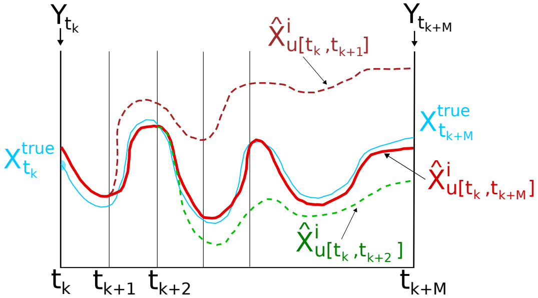

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Control the particles to construct a new proposal measure

Find the optimal control u which minimizes the cost:

$$J(t_k, x_k; u) = \hat{\mathbb{E}}_{t_k,x_k}\left[\frac{1}{2}\int_{t_k}^{t_{k+1}} u(s)^T Q^{-1} u(s)\, ds + g(\hat{X}(t_{k+1}))\right], \qquad (39)$$

where $g(x) = \frac{1}{2}(Y_{k+1} - h(x))^T R^{-1}(Y_{k+1} - h(x))$ and $\hat{\mathbb{E}}_{t_k,x_k}$ is the probability measure generated by the controlled process $\hat{X}$:

$$d\hat{X}(s) = \bar{b}(\hat{X}(s))ds + u(s)ds + \sigma dW, \quad t_k \le t \le t_{k+1}, \quad \hat{X}(t_k) = x_k.$$

- less penalty on the size of the control in the directions with large noise amplitude.
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Let $V(t,x)$ be the value function defined by $V(t,x) := \inf_u J(t,x;u)$ for $t \in [t_k, t_{k+1}]$. Then, $V(t,x)$ is the solution of HJB equation.

$$-\frac{\partial V}{\partial t} + H(t,x,D_x V, D_x^2 V) = 0 \quad \text{with} \quad V(t_{k+1},x) = g(Y_{t_{k+1}},x), \tag{40}$$

where the Hamiltonian of the associated control problem is

$$\begin{aligned} H(t,x,p,B) &\stackrel{\text{def}}{=} \sup_u \left[-\left(\bar{b}(t,x)+u\right)^T p - \frac{1}{2}u^T Q^{-1} u - \frac{1}{2}\text{tr}(QB)\right] \\ &= \left[-\bar{b}(t,x)^T p + \frac{1}{2} p^T Q p - \frac{1}{2}\text{tr}(QB)\right]. \end{aligned}$$

(Note that the supremum in the above equation is achieved with $u = -Qp$.)

Solution to the HJB equation using the Feynman-Kac formula

The quadratic nonlinearity in the HJB equation can be removed using

$$V(t,x) = -\log \Phi(t,x)$$

yields the the optimal control as

$$u(t) = -Q\, D_x V(t, \hat{X}(t)) = \frac{1}{\Phi(t, \hat{X}(t))} Q D_x \Phi(t, \hat{X}(t)),$$

and Φ satisfies

$$\frac{\partial \Phi}{\partial t} + \bar{b}^T(t,x) D_x \Phi + \frac{1}{2} \operatorname{tr}\left(Q D_x^2 \Phi\right) = 0 \quad \text{with} \quad \Phi(t_{k+1}, x) = \exp\left(-g(x)\right).$$

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Feynman-Kac formula yields

$$\Phi(t,x) = \mathbb{E}_{t,x}\left[\exp\left(-g(\tilde{X}(t_{k+1}))\right)\right], \tag{41}$$

where $\mathbb{E}_{t,x}$ is the probability measure generated by a process $\tilde{X}$ evolving according to

$$d\tilde{X}(t) = \bar{b}(\tilde{X}(t))dt + \sigma dW \quad \text{with} \quad \tilde{X}(t) = x. \tag{42}$$

Optimal Nudging Embedded on HHPF

The explicit solution for the optimal control is $u(t) = -\sigma\sigma^T D_x V(t, \hat{X}(t))$, and taking $u(t) = \sigma v(t)$ for notational simplicity, we construct the proposal measure from the controlled process

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Updating the weights

We evolve the particles according to

$$d\hat{X}(t) = \bar{b}(\hat{X}(t))dt + \sigma dW + \sigma v(t)dt, \; t_k \leq t \leq t_{k+1},$$
$$\hat{X}(t_k) = x_k,$$

then, using the principle of importance sampling, the weights should be updated according to

$$w_i^{k+1} \propto \exp\left(-g(Y_{k+1}, \hat{X}_i(t_{k+1}))\right) \frac{d\mu^i_{\tilde{X}}}{d\mu^i_{\hat{X}}} w_i^k, \qquad (44)$$

where $\mu^i_{\tilde{X}}$ and $\mu^i_{\hat{X}}$ are the measure generated by the original and controlled processes, respectively, evolving for $t_k \leq t \leq t_{k+1}$ with starting point x at t_k. The weights are updated according to (44) with (use Girsanov)

$$\frac{d\mu^i_{\tilde{X}}}{d\mu^i_{\hat{X}}} = \exp\left(-\int_{t_k}^{t_{k+1}} v^T(s)\, dW(s) - \frac{1}{2}\int_{t_k}^{t_{k+1}} v(s)^T v(s)\, ds\right)$$

Consider the linear signal and observation dynamics:

$$dX_t = AX_t dt + \sigma dW_t \tag{45}$$

$$Y_{t_k} = HX_{t_k} + V_{t_k}$$

Letting $Q = \sigma\sigma^*$ and $R = cov(V)$, for linear systems, the control $u(t,x)$ is obtained as follows:

$$u(t,x) = Q(e^{A(t_{k+1}-t)})^*[I + H^*R^{-1}H\Sigma]^{-1}H^*R^{-1}[(Y_{t_{k+1}} - H\mu)] \tag{46}$$

where $\mu := e^{A(t_{k+1}-t)}x$, and $\Sigma := \int_t^{t_{k+1}} e^{A(t_{k+1}-s)}Q(e^{A(t_{k+1}-s)})^* ds$ are the mean and variance of the system (45) at time t_{k+1} when it starts at time t at x.

Lorenz '63 model - A simple toy model of atmospheric convection

$$\begin{aligned}
\dot{X}_t &= -\sigma X_t + \sigma Y_t && + \xi_x(t) \\
\dot{Y}_t &= \rho X_t - Y_t - X_t Z_t && + \xi_y(t) \\
\dot{Z}_t &= -\beta Z_t - X_t Y_t && + \xi_z(t)
\end{aligned}$$

- Typical values for parameters are $\sigma = 10$ (Prandtl number), $\rho = 8/3$ and β can vary; model exhibits chaotic behavior at $\beta = 28$
- "True" signal generated and observations taken every 50 timesteps (sparse data); signal noise added is vector of Gaussian random numbers premultiplied by correlation matrix $\begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$; observation is taken as signal plus Gaussian noise with correlation matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- Particle filters implemented with 20 particles, resample when effective number of particles falls below 5

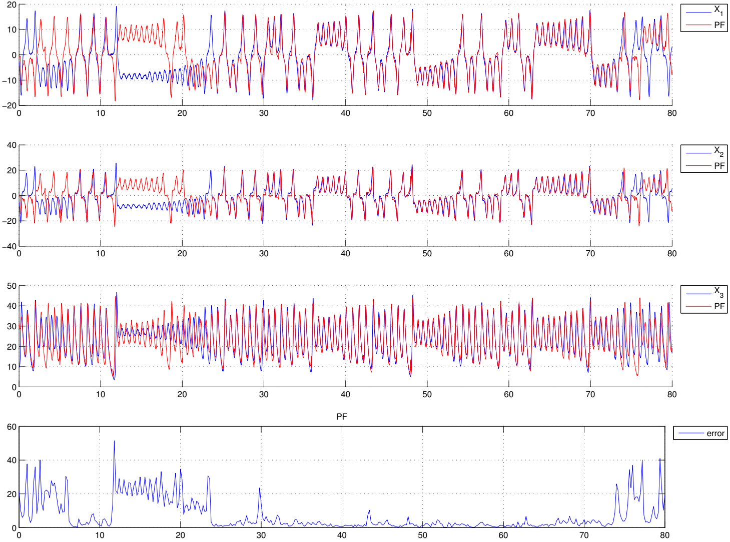


Figure: Particle filter without optimal control; $N_s = 3$

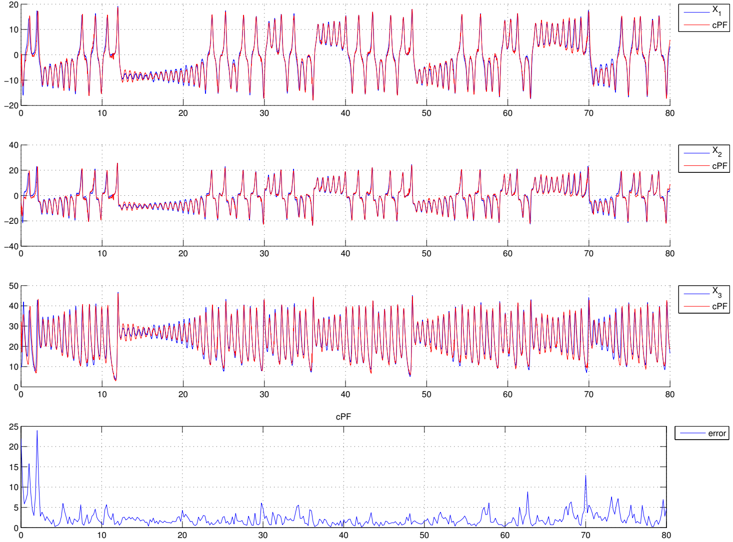


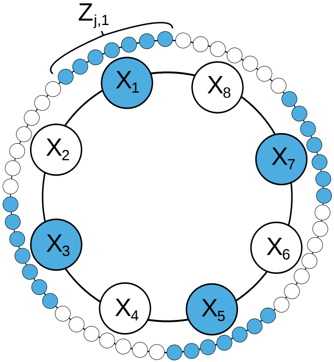
Figure: Particle filter with optimal control; $N_s = 3$

Lorenz-96[8]: Heuristic midlatitude atmospheric model

the large scale, low frequency X variables and the small scale, high frequency Z variables

$$\dot{X}_k = \overbrace{-X_{k-1}(X_{k-2} - X_{k+1})}^{\text{advection}} - \overbrace{X_k}^{\text{dissipation}} + \underbrace{F_x}_{\text{forcing}} + \frac{h_x}{J}\sum_{j=1}^{J} Z_{j,k}$$

$$\dot{Z}_{j,k} = \frac{1}{\varepsilon}(-Z_{j+1,k}(Z_{j+2,k} - Z_{j-1,k}) - Z_{j,k} + h_z X_k) + \frac{1}{\sqrt{\varepsilon}}\zeta(t)$$



- $J = 36,\ K = 10,\ \varepsilon = 1/128$

- sensitive to initial conditions
- HHPF as defined requires modifications

[8] E. N. Lorenz, *Proc. Seminar on Predictability*, 1(1), 1996

Lorenz-96[8]: Heuristic midlatitude atmospheric model

the large scale, low frequency X variables and the small scale, high frequency Z variables

$$\dot{X}_k = \overbrace{-X_{k-1}(X_{k-2} - X_{k+1})}^{\text{advection}} - \overbrace{X_k}^{\text{dissipation}} + \underbrace{F_x}_{\text{forcing}} + \frac{h_x}{J}\sum_{j=1}^{J} Z_{j,k}$$

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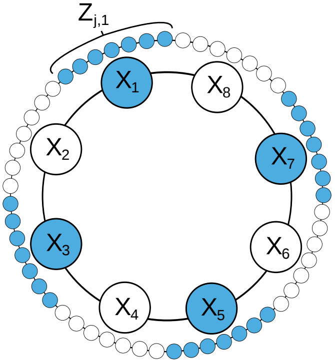


- $J = 36,\ K = 10,\ \varepsilon = 1/128$

- sensitive to initial conditions
- HHPF as defined requires modifications

[8]E. N. Lorenz, *Proc. Seminar on Predictability*, 1(1), 1996

Model parameters

- # of slow, large amplitude modes $K = 36$;
- # of fast, low amplitude modes $J = 10$ implies 360 total fast modes;
- constant external forcing $F_x = 10$;
- coupling coefficients $h_x = -0.8$, $h_z = 1$;
- $\varepsilon = 1/128$

Numerical integration

- $T = 20$ (time) units $= 10$ days, micro timestep $\delta t = 2^{-11} = 35 sec$;
- macro timestep $\Delta t = 2^{-4} = 45$ min;
- $N_s = 100$
- Processor: Intel Xeon DP Hexa-core X5675s (dual, 3.07GHz)
- Integration scheme: Euler-Maruyama (stochastic), Runge-Kutta (deterministic) in MATLAB (R2010b)

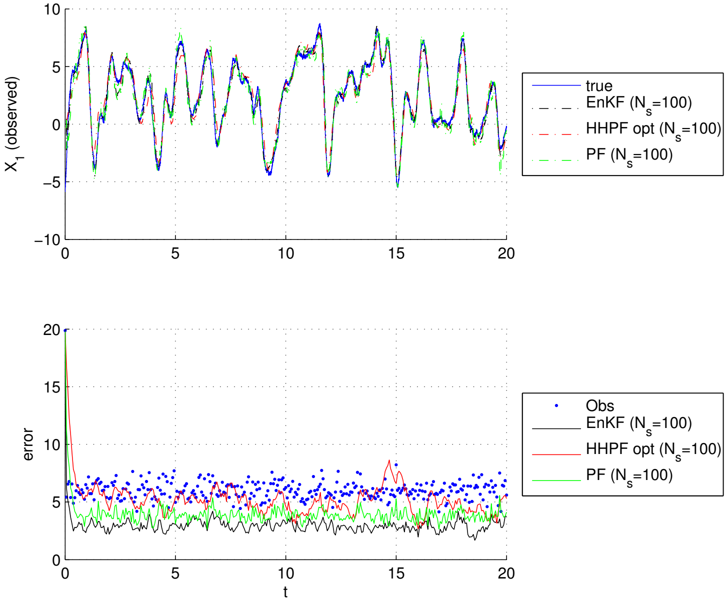


Figure: Comparison of HHPF optimum with other filters: Observation every 45 min

Computation times for different nonlinear filters

N_s	Opt. HHPF	Direct HHPF	PF	EnKF
2	5	5	1727	N/A
10	16	15	1716	N/A
20	30	20	1757	540
50	67	46	1936	651
100	134	86	2169	920
200	177	176	3002	1415
400	401	539	3591	3152

Table: Typical computation times (in sec.) for different sample sizes for the different nonlinear filters, in the case of non-sparse observations.

Model parameters for sparse observations

- # of slow, large amplitude modes $K = 36$;
- # of fast, low amplitude modes $J = 10$ implies 360 total fast modes;
- constant external forcing $F_x = 10$;
- coupling coefficients $h_x = -0.8$, $h_z = 1$;
- $\varepsilon = 0.075$

Numerical integration

- $T = 20$ (time) units = 100 days, micro timestep $\delta t = 0.0005 = 3.6$ min;
- macro timestep $\Delta t = 0.05 = 6$ hrs;
- $N_s = 20$, Error doubling time = 36 hrs, observations at every 36 hrs
- Processor: Intel Xeon DP Hexa-core X5675s (dual, 3.07GHz)
- Integration scheme: Euler-Maruyama (stochastic), Runge-Kutta (deterministic) in MATLAB (R2010b)

HHPFs comparison

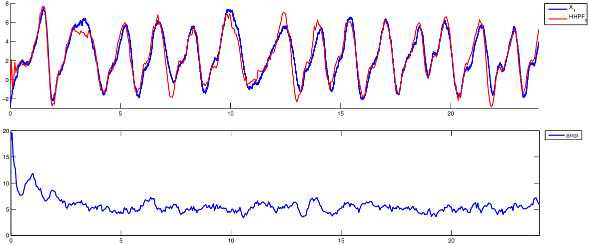


Figure: Optimized HHPF

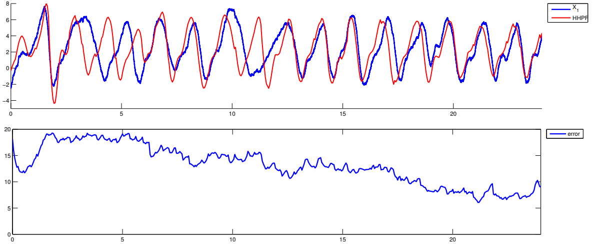


Figure: Direct HHPF

Upper figure: true solution in blue. Lower figure: absolute error vs time.

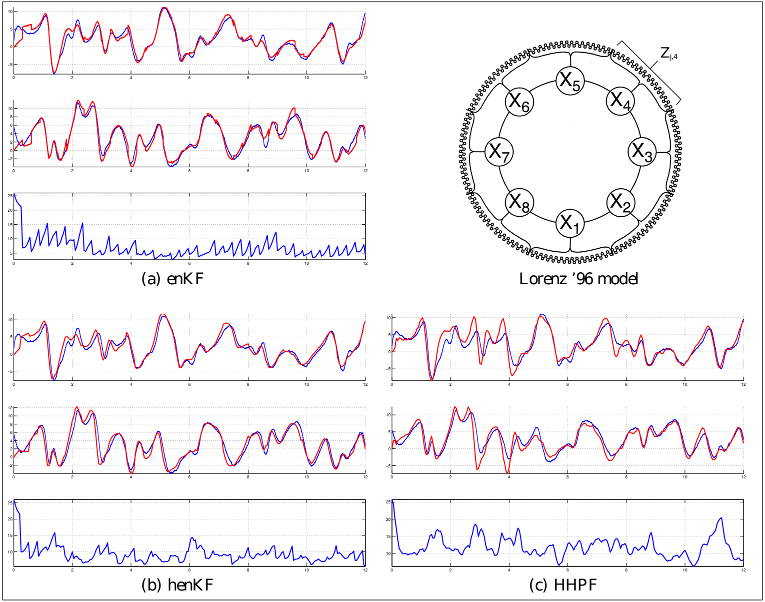


Figure: Estimation performance of the different filters; counterclockwise from the top left, the enKF, the henKF and the HHPF. The blue curve in the top two plots of each figure represents the true state trajectory while the red curve represents the estimated trajectory.

N_s	Opt. HHPF		Direct HHPF		Homog. EnKF		EnKF	
	time	*RMSE*	*time*	*RMSE*	*time*	*RMSE*	*time*	*RMSE*
10	82	26.04	26	69.27	26	32.53	785	25.96
20	97	24.52	27	58.09	27	26.92	587	20.05
50	241	24.78	68	42.71	70	24.53	703	17.32
100	493	24.92	139	36.77	134	23.75	861	16.21
200	1153	24.71	375	35.37	369	23.58	1539	16.46

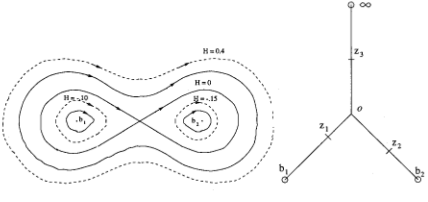
Table: Average computation times (in sec.) and RMSEs for different sample sizes. Observations recorded every 0.3 time units ($6\ \Delta t$), $\approx$ error doubling time of 36 hrs real time.

$$\ddot{u} = u - u^3 + \varepsilon \nu \dot{W}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} y \\ x - x^3 \end{pmatrix} dt + \nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW \qquad (1)$$

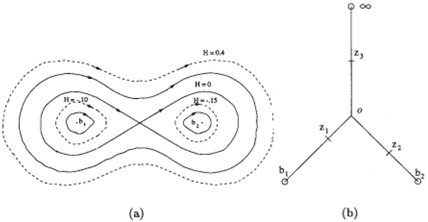
$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

$$dH^\varepsilon = \frac{1}{2}\nu^2 dt + \nu y dW$$



(a) (b)

$$H^\varepsilon \to \overline{H}, \qquad d\overline{H} = \frac{1}{2}\nu^2 dt + \sqrt{\langle \nu^2 y^2 \rangle_H} dW$$



$$H^\varepsilon \to \overline{H}, \qquad d\overline{H} = \frac{1}{2}\nu^2 dt + \sqrt{\langle \nu^2 y^2 \rangle (\overline{H})} dW$$

$\langle \nu^2 y^2 \rangle (\overline{H})$ is the average along H orbits—can be obtained analytically and involves elliptic integrals.

$$\mathbf{1}: \; H < 0, x < 0, \qquad \mathbf{2}: \; H < 0, x > 0, \qquad \mathbf{3}: \; H > 0$$

Gluing condition: $-f_1' - f_2' + 2f_3' = 0$.
Particle on reaching the homoclinic vertex, flips a 3-sided dice with probabilities $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ to decide on the leg.

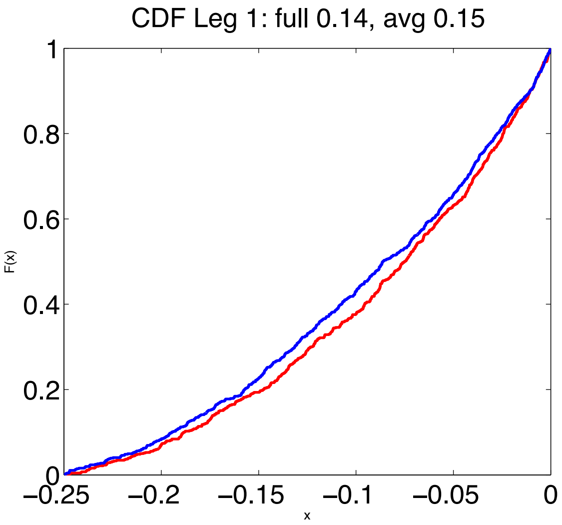
Euler scheme for $\overline{H}$:

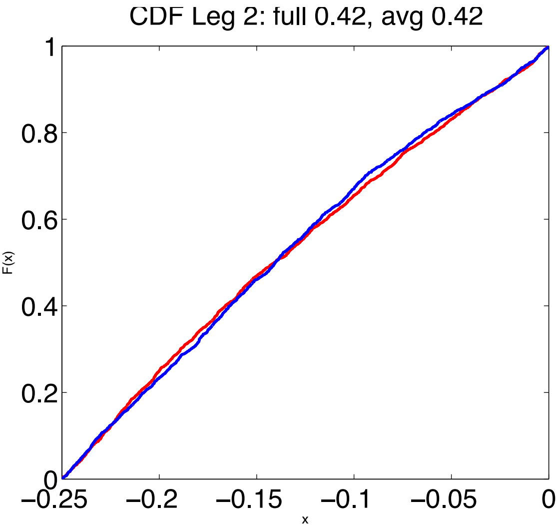
$$\overline{H}_{t+\Delta t} = \overline{H}_t + \frac{1}{2}\nu^2\Delta t + \sqrt{\langle \nu^2 y^2\rangle(\overline{H})}\sqrt{\Delta t}\mathcal{N}(0,1).$$

For particles in legs 1 or 2 at t: If $\overline{H}_{t+\Delta t} > 0$ they move to leg 3.

For particles in leg 3 at t: If $\overline{H}_{t+\Delta t} < 0$, they choose leg 1 or leg 2 with probability $\frac{1}{2}$.

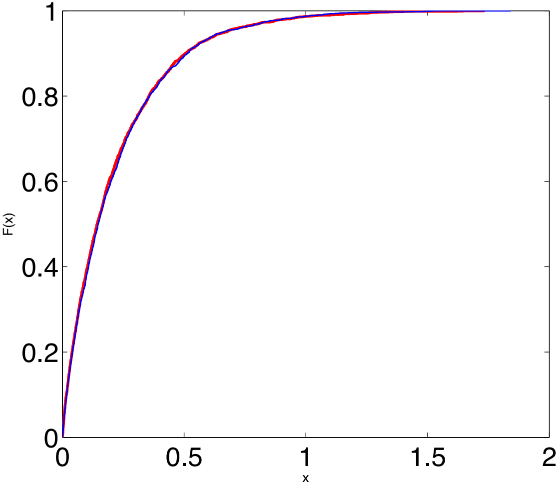
Test: $\nu = \frac{1}{2}$, initial position: start in leg 2 with initial h value -0.1.
Simulate for $T = 1$ with 5000 samples.
Red: Simulation of (x, y) system and recording H^ε with $\varepsilon = 0.05$.
Blue: Averaged system $\overline{H}$.


CDF Leg 1: full 0.14, avg 0.15
F(x)
x
1
0.8
0.6
0.4
0.2
0
−0.25
−0.2
−0.15
−0.1
−0.05
0



CDF Leg 2: full 0.42, avg 0.42
F(x)
x
1
0.8
0.6
0.4
0.2
0
−0.25
−0.2
−0.15
−0.1
−0.05
0

CDF Leg 3: full 0.44, avg 0.43



$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} y \\ x - x^3 \end{pmatrix} dt + \nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW \qquad (2)$$

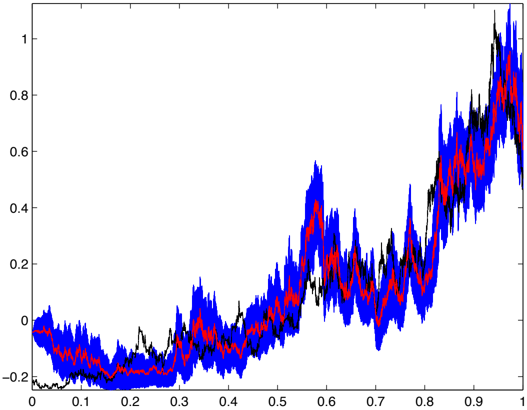
$$dY = \mathcal{H}(x(t), y(t))dt + dV.$$

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

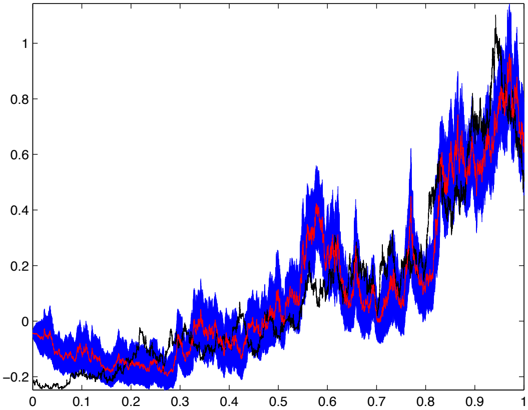
Interested in density of $H(t)$ given $Y_{[0,t]}$.

$$d\overline{H} = \frac{1}{2}\nu^2 dt + \sqrt{\langle \nu^2 y^2 \rangle_H} dW$$

Full filter — particles represent (x, y) location.
Black: true H; Red: estimated H; Blue: 1 std. dev. of sample.
Time: 816 sec.



Averaged filter — particles represent $\overline{H}$ location.
Black: true H; Red: estimated $\overline{H}$; Blue: 1 std. dev. of sample.
Time: 68 sec.



Conclusions: lower dimensional filters

We showed the efficient utilization of the low-dimensional models of the signal to develop a low-dimensional nonlinear filtering equations (Zakai-type equation) that determine the conditional law of the coarse-grained signal, X_t, of complex systems, in multi-scale environments.

Developed the *Homogenized Hybrid Particle Filter (HHPF)* which combined the results on reduced order nonlinear filtering with sequential Monte Carlo methods. HHPF reduces the effective number of variables in the evaluation of the conditional distribution needed in the Bayesian filter for data assimilation.

Numerical studies were presented to illustrate that in settings where the signal and observation dynamics are nonlinear a suitably chosen HHPF scheme can drastically outperform the regular particle filters.