Multiscale Dynamics and Information: Dimensional Reduction and Data Assimilation

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Joint work with:

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January 8-11, 2014

Workshop on Nonlinear filtering and data assimilation

TIFR Centre for Applicable Mathematics, Bangalore, India.

Research Supported by: NSF EFRI -1024772, CMMI -1030144, and AFOSR FA9550-12-1-0390

Signal Process:

$$\varepsilon \dot{Z}_t^\varepsilon = g(X_t^\varepsilon, Z_t^\varepsilon, \xi_t^\varepsilon), \; Z_0^\varepsilon = z \in \mathbb{R}^m, \text{ core atmosphere model}$$
$$\dot{X}_t^\varepsilon = f(X_t^\varepsilon, Z_t^\varepsilon, \xi_t^\varepsilon), \; X_0^\varepsilon = x \in \mathbb{R}^n, \text{ core ocean model}$$

where, z represent the general circulation of the atmosphere; x are the ocean components such as the density gradients, angular momentum, etc.. ξ^ε represents the unmodeled dynamics of the system or an additive noise.

Observation process:

For example, Meridional Overturning Circulation and Heatflux Array (MOCHA): Current Meters, CTDs (salinity and temperature every hour or so for a period of up to two years), buoys, acoustic releases. Deployed along 26.5° N in the Atlantic.

The observation process is a function of the signal process corrupted by noise

$$Y_t^\varepsilon = \int_0^t h^\varepsilon(X_s^\varepsilon, Z_s^\varepsilon) ds + V_t,$$

where $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is called the sensor function.

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Mathematical methodology for multiscale estimation

Multiscale process

$$
\begin{aligned}
&(\text{slow})\ \dot{X}_t^\varepsilon = B(X_t^\varepsilon, Z_t^\varepsilon, \theta, \xi_t), \quad X_0^\varepsilon = x \in \mathbb{R}^m \\
&(\text{fast})\ \dot{Z}_t^\varepsilon = F(\varepsilon, X_t^\varepsilon, Z_t^\varepsilon, \theta, \zeta_t), \quad Z_0^\varepsilon = z \in \mathbb{R}^n \\
&(\text{obs})\ Y_t^\varepsilon = H(X_t^\varepsilon, Z_t^\varepsilon, \chi_t), \quad Y_0^\varepsilon = 0 \in \mathbb{R}^d
\end{aligned}
$$

Nonlinear filtering
Compute conditional expectation of signal process, given observations:
$\pi_t^\varepsilon(\varphi) \overset{\text{def}}{=} \mathbb{E}_{\mathbb{P}^\varepsilon}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon) \middle| Y_{0:t}^\varepsilon\right]$

Maximum likelihood estimation
Maximize likelihood of signal, given parameter value:
$\theta^* \overset{\text{def}}{=} \arg\max_\theta \log \mathbb{E}_{\mathbb{P}_\alpha^\varepsilon}\left[\frac{d\mathbb{P}_\theta^\varepsilon}{d\mathbb{P}_\alpha^\varepsilon} \middle| Y_{0:T}^\varepsilon\right]$

Difficulty: $\mathbb{R}^{m+n}$ is high dimensional state-space; numerical computation issues

Question: for each $t \geq 0$, find the *conditional law* of X_t (only the ocean state) given $\mathscr{Y}_t$ – the information available at time $t > 0$. $\Longrightarrow$ only interested in estimating X^ε dynamics

We develope methods that combine techniques of model reduction (from RDS) and nonlinear filtering (from SPDE), that enable more effective data assimilation and prediction of complex systems with multiple scales.

If only interested in estimating X^ε dynamics $\Longrightarrow$ make use of process X^0

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Lecture 2: Multiscale filtering and estimation

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Multiscale process

$$(\text{slow})\ dX_t^\varepsilon = b(X_t^\varepsilon, Z_t^\varepsilon, \theta)dt + \sigma(X_t^\varepsilon, Z_t^\varepsilon, \theta)dW_t, \quad X_0^\varepsilon = x \in \mathbb{R}^m$$

$$(\text{fast})\ dZ_t^\varepsilon = \varepsilon^{-1} f(X_t^\varepsilon, Z_t^\varepsilon, \theta)dt + \varepsilon^{-1/2} g(X_t^\varepsilon, Z_t^\varepsilon, \theta)dV_t, \quad Z_0^\varepsilon = z \in \mathbb{R}^n$$

$$(\text{obs})\ Y_t^\varepsilon = \int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)ds + B_t, \quad Y^\varepsilon \in \mathbb{R}^d$$

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Outline of nonlinear filtering

Normalized filter

Filter $\pi_t^\varepsilon(\varphi) \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{Q}}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon) \middle| Y_{0:t}^\varepsilon\right]$

Introduce $\mathbb{P}^\varepsilon$ by Girsanov's Theorem:

$$\left.\frac{d\mathbb{P}^\varepsilon}{d\mathbb{Q}}\right|_{\mathcal{F}_t} =: D_t^\varepsilon = \exp\left(-\int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)^* dY_s^\varepsilon + \frac{1}{2}\int_0^t |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds\right)$$

Distribution of $(X^\varepsilon, Z^\varepsilon)$ is the same under $\mathbb{P}^\varepsilon$ as under $\mathbb{Q}$ but Y^ε becomes a Brownian motion *independent* of W, V.

$$\pi_t^\varepsilon(\varphi) = \frac{\mathbb{E}_{\mathbb{P}^\varepsilon}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon)(D_t^\varepsilon)^{-1} \middle| \mathcal{Y}_t^\varepsilon\right]}{\mathbb{E}_{\mathbb{P}^\varepsilon}\left[\mathbb{1}_{(X_t^\varepsilon, Z_t^\varepsilon)}(D_t^\varepsilon)^{-1} \middle| \mathcal{Y}_t^\varepsilon\right]} =: \frac{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} u_t^\varepsilon(x,z)\varphi(x,z)dzdx}{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} u_t^\varepsilon(x,z)dzdx},$$

where u_t^ε satisfies the Zakai equation

Zakai PF

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▸ Zakai ▸ PF

Outline of nonlinear filtering (cont'd)

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Define $\rho_t^\varepsilon(\varphi) := \mathbb{E}_{\mathbb{P}^\varepsilon}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon)(D_t^\varepsilon)^{-1}\middle|\, \mathcal{Y}_t^\varepsilon\right] = \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} u_t^\varepsilon(x,z)\varphi(x,z)dzdx$

Then, $\pi_t^\varepsilon(\varphi) = \frac{\rho_t^\varepsilon(\varphi)}{\rho_t^\varepsilon(\mathbb{1})}$

We have

$$du_t^\varepsilon(x,z) = (\mathcal{L}^\varepsilon)^* u_t^\varepsilon(x,z)dt + u_t^\varepsilon(x,z)h^*(x,z,t)dY_t^\varepsilon, \quad u_0^\varepsilon(x,z) = p_0(x,z)$$

where $\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_F + \mathcal{L}_S$:

$$\mathcal{L}_F \stackrel{\text{def}}{=} \sum_{i=1}^n f_i(x,z)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i,j=1}^n (gg^*)_{ij}(x,z)\frac{\partial^2}{\partial z_i \partial z_j},$$

$$\mathcal{L}_S \stackrel{\text{def}}{=} \sum_{i=1}^m b_i(x,z)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^m (\sigma\sigma^*)_{ij}(x,z)\frac{\partial^2}{\partial x_i \partial x_j}$$

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Then,

$$\begin{aligned} d\left\langle u_t^\varepsilon, \varphi\right\rangle &= \left\langle (\mathcal{L}^\varepsilon)^* u_t^\varepsilon, \varphi\right\rangle dt + \left\langle u_t^\varepsilon h^*, \varphi\right\rangle dY_t^\varepsilon \\ &= \left\langle u_t^\varepsilon, \mathcal{L}^\varepsilon\varphi\right\rangle dt + \left\langle u_t^\varepsilon, h\varphi\right\rangle dY_t^\varepsilon \end{aligned}$$

so

$$d\rho_t^\varepsilon(\varphi) = \rho_t^\varepsilon\left(\mathcal{L}^\varepsilon\varphi\right)dt + \rho_t^\varepsilon\left(h\varphi\right)dY_t^\varepsilon, \quad \rho_0^\varepsilon(\varphi) = \mathbb{E}\left[\varphi\left(X_0^\varepsilon, Z_0^\varepsilon\right)\right].$$

Nonlinear filtering in high dimensions

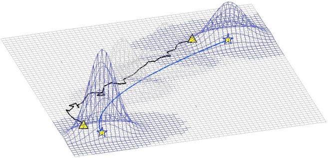


Figure: Evolution of density

Nonlinear filtering theory

Filter: $\pi_t^\varepsilon(\varphi) \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{Q}}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon) \middle| Y_{0:t}^\varepsilon\right]$

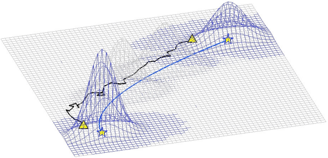
Unnormalized filter: $\rho_t^\varepsilon(\varphi) \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{P}^\varepsilon}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon)(D_t^\varepsilon)^{-1} \middle| Y_{0:t}^\varepsilon\right]$, where $\pi_t^\varepsilon(\varphi) = \frac{\rho_t^\varepsilon(\varphi)}{\rho_t^\varepsilon(\mathbb{1})}$

DMZ equation[a]: $\underbrace{du_t^\varepsilon(x,z) = (\mathcal{L}^\varepsilon)^* u_t^\varepsilon(x,z)dt}_{\text{Fokker-Planck equation}} + u_t^\varepsilon(x,z)h^*(x,z,t)dY_t^\varepsilon$

[a] $D_t^\varepsilon = \exp\left(-\int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)^* dY_s^\varepsilon + \frac{1}{2}\int_0^t |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds\right)$

▸ D^ε

Nonlinear filtering in high dimensions



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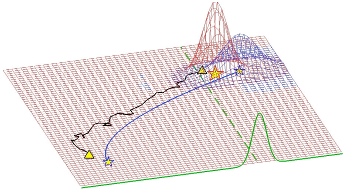

Figure: Evolution of conditional density

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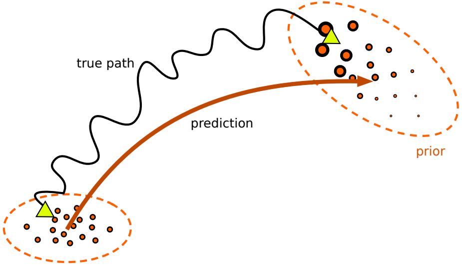
D^ε

Nonlinear filtering in high dimensions



Particle method

- (weighted) particles to represent (conditional) density [a]
- "curse of dimensionality" [b]

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[a] N. J. Gordon, D. J. Salmond, A. F. M. Smith, Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEEE Proceedings F*, 140(2), 1993

[b] T. Bentsson, P. Bickel, B. Li, Curse-of-dimensionality revisited: Collapse of the particle filter in very large scale systems. *IMS Collections: Probability and Statistics, Vol. 2: Essays in Honor of David A. Freedman*, 2008

Our interest is to estimate the slowly varying signal (coarse-grained signal) X_t^ε at time t on the basis of the sigma-algebra $\sigma\{Y_s^\varepsilon : 0 \le s \le t\}$.

- More precisely for each $t \ge 0$, we want to find the conditional law of theslowly varying signal (coarse-grained signal)

$$\pi_t^\varepsilon(A) \stackrel{\text{def}}{=} \mathbb{P}\{X_t^\varepsilon \in A | Y_s^\varepsilon : 0 \le s \le t\}, \quad \text{for all } A \in \mathscr{B}(\mathbb{R}^n).$$

Primary objective is to show that

- if the multi-scale signal process $\{X_t^\varepsilon\}$ converges to coarse-grained process $\{X_t^0\}$ in a weak sense,

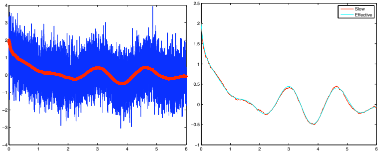

Figure: Original Signal processes and Averaged process

- then the conditional law $\{\pi_t^\varepsilon\}$ for the coarse - grained dynamics converge to a process $\{\pi_t^0\}$ that is governed by a lower dimensional recursive linear SPDE.

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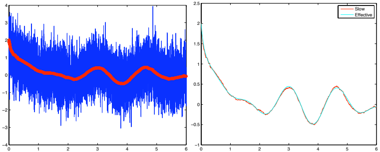

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Mathematical Challenges of Scaling in Information

While specific calculations are sometimes clear, the combination of scaling and information presents some new challenges.

– The twist here is that the convergence of $(X^{\varepsilon}, Y^{\varepsilon})$ to (X^0, Y^0) itself does not guarantee the convergence of filters.

Moral is that one must be careful when "averaging" out high-frequency information.

Simplest Example

To see explicitly some of these issues in action, let us consider the *simplest* problem we can think of. We consider

$$\boxed{\begin{aligned} d\theta_t^\varepsilon &= \frac{1}{\varepsilon} dB_t \qquad \text{quickly-diffusing angle} \\ dX_t^\varepsilon &= b(\theta_t^\varepsilon) dW_t \qquad \text{order-one-diffusing axial coordinate} \end{aligned}} \quad \text{plant/signal}$$

$$dY_t^\varepsilon = h(\theta_t^\varepsilon, X_t^\varepsilon)dt + dV_t \qquad \text{observations}$$

Here b and h are one-periodic in θ and B, W, and V are all independent standard Brownian motions.

We want to understand $\mathbb{P}\{X_t^\varepsilon \in dx | \vec{Y}_t^\varepsilon\}$ (where $\vec{Y}_t^\varepsilon \overset{\text{def}}{=} \{Y_s^\varepsilon \mid 0 \le s \le t\}$); i.e., we want to understand the conditional law of the slow part of the plant conditioned on observations.

- The main challenge in multiscale modeling is to recognize their simplicity. Take advantage of the multi-scales in the problem to see how information interacts with these complex structures and scales.

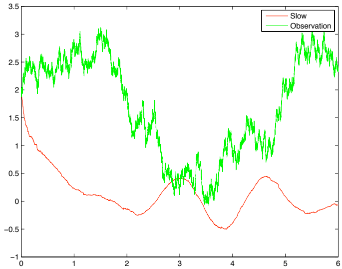

Figure: Fast and Slow Signal Processes and Observation Process

We expect that in the limit, we should have the problem

$$dX_t^0 = \bar{b} dW_t \qquad \text{plant: effective axial coordinate}$$
$$dY_t^0 = \bar{h}(X_t^0)dt + dV_t \qquad \text{effective observations}$$

where

$$\bar{b} \overset{\text{def}}{=} \sqrt{\int_0^1 b^2(\theta)d\theta} \quad \text{and} \quad \bar{h}(x) \overset{\text{def}}{=} \int_0^1 h(\theta, x)d\theta.$$

Our approach is to use limit theorems to "guess" what the filter should be.

We should have that $\mathbb{P}\{X_t^\varepsilon \in dx | \vec{Y}_t^\varepsilon\} \approx \mathbb{P}\{X_t^0 \in dx | \vec{Y}_t^0\}$. ??????

(our interest here is to showcase some parts of the calculations—these calculations can be generalized).

To this end, we write the filter for the coarse-grained dynamics for each $A \in \mathscr{B}(\mathbb{R})$ as,

$$\boxed{\mathbb{P}\{X_t^\varepsilon \in A | \vec{Y}_t^\varepsilon\} = \frac{\int_{x \in A} \int_{0 \le \theta < 1} u^\varepsilon(t, x, \theta) d\theta\, dx}{\int_{x \in \mathbb{R}} \int_{0 \le \theta < 1} u^\varepsilon(t, x, \theta) d\theta\, dx},}$$

then $u^\varepsilon(t, x, \theta)$, the *un-normalised density* of the original problem, solves the Zakai equation

$$\begin{aligned} du^\varepsilon(t, x, \theta) = (\mathscr{L}_\varepsilon u^\varepsilon)(t, x, \theta)dt + h(x, \theta)u^\varepsilon(t, x, \theta)dB_t \\ + h(x, \theta)h(\mathbf{X}_t^\varepsilon, \boldsymbol{\Theta}_t^\varepsilon)u^\varepsilon(t, x, \theta)dt, \end{aligned} \tag{31}$$

with $u^\varepsilon(0, x, \theta) = p(x, \theta)$.

For each $\varepsilon \in (0,1)$, let

$$u^\varepsilon(t,x,\theta) \stackrel{\text{def}}{=} u^0(t,x) + \Phi^\varepsilon(t,x,\theta) + R^\varepsilon(t,x,\theta),$$

where Φ^ε and R^ε are to be chosen; here, Φ^ε is the corrector and R^ε is the error, such that, $u^0(t,x)$ solves the averaged Zakai equation

$$\boxed{du^0(t,x) = (\bar{\mathscr{L}}_S u^0)(t,x)dt + \bar{h}(x)u^0(t,x)dB_t + \bar{h}(x)\bar{h}(\mathbf{X}_t^0)u^0(t,x)dt} \qquad (32)$$

This separates our problem into two qualitatively distinct parts – First show that the corrector Φ^ε is small and finally show that the remainder R^ε is small.

We can show that

$$u^\varepsilon - u^0 \to 0 \quad (\text{as}\, \varepsilon \to 0).$$

Naturally, we show that

$$\mathbb{P}\{X_t^\varepsilon \in dx | \vec{Y}_t^\varepsilon\} \to \mathbb{P}\{X_t^0 \in dx | \vec{Y}_t^0\} \quad (\text{as}\, \varepsilon \to 0).$$

For each $\varepsilon \in (0,1)$, let

$$u^\varepsilon(t,x,\theta) \stackrel{\text{def}}{=} u^0(t,x) + \Phi^\varepsilon(t,x,\theta) + R^\varepsilon(t,x,\theta),$$

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But the observations we are *given* are Y^ε, even though we might prefer to work with Y^0.

Rumsfeld dilemma:

To paraphrase, Rumsfeld, "you filter with the observations you have, not the observations you might want or wish to have." When combining scales and filtering, one has to remember that the observations you want (which in our case represent a limiting dynamics Y^0) are not the observations you have (which are the prelimit ones Y^ε).

To come out of this conundrum, we recall that the filter is a "map from observations to distributions," and then show that this filter is sufficiently continuous that it works well enough when the limiting observations are replaced by the prelimit ones.

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Resolution: In fact, both

$$\mathbb{P}\{X_t^\varepsilon \in A | \vec{Y}_t^\varepsilon\} = \pi_t^\varepsilon(A, \vec{Y}_t^\varepsilon) \qquad \text{and} \qquad \mathbb{P}\{X_t^0 \in A | \vec{Y}_t^0\} = \pi_t^0(A, \vec{Y}_t^0)$$

are measurable maps on $C[0,t]$ (the space of observations); where $\varphi \mapsto \pi_t^\varepsilon(\cdot, \varphi)$ and $\varphi \mapsto \pi_t^0(\cdot, \varphi)$ are both maps from $C[0,t]$ to $\mathscr{P}(\mathbb{R})$ (the collection of probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$).

We want to show that the maps $\varphi \mapsto \pi_t^\varepsilon(\cdot, \varphi)$ and $\varphi \mapsto \pi_t^0(\cdot, \varphi))$ are close when we apply them (π_t^0) to the *true* observations ($\varphi = \vec{Y}_t^\varepsilon$), that is,

$$\pi_t^0(\cdot, \vec{\mathbf{Y}}_{[0,t]}^\varepsilon) \quad \text{is close to} \quad \pi_t^\varepsilon(\cdot, \vec{\mathbf{Y}}_{[0,t]}^\varepsilon)$$

.

Homogenized Hybrid Nonlinear Filter Equations

Hence, the equation we are actually interested in is the reduced filtering equation driven by actual observation $\mathbf{Y}^{\varepsilon}$, which is given as

$$d\bar{v}^{\varepsilon}(t,x) = (\bar{\mathscr{L}}_S^* \bar{v}^{\varepsilon})(t,x)dt + \bar{h}(x)\bar{v}^{\varepsilon}(t,x)d\mathbf{Y}_t^{\varepsilon}. \tag{33}$$

Note that this is not a Zakai equation. We want to show that

$$u^{\varepsilon} - \bar{v}^{\varepsilon} \to 0.$$

We will naturally use u^0 of (32) in the proof.

Main Results

Lemma (Park, Namachchivaya & Sowers (2009, 2010))

We can show that for any $\delta > 0$ *and* $t > 0$,

$$\boxed{\lim_{\varepsilon \to 0} \mathbb{P}\left\{ \sup_{0 \le s \le t} d(\pi_t^\varepsilon(\cdot, \vec{Y}_t^\varepsilon), \pi_t^0(\cdot, \vec{Y}_t^\varepsilon)) \ge \delta \right\} = 0}$$

where d *is the Prohorov metric on* $\mathscr{P}(\mathbb{R})$.

1. Jun H. Park, N. Sri Namachchivaya and Richard B. Sowers, "A Problem in Homogenization of Nonlinear Filters", *Stochastics and Dynamics*, Vol. 8(3), 2009, pp. 543-560.
2. Jun H. Park, Richard B. Sowers and N. Sri Namachchivaya, "Dimensional Reduction in Nonlinear Filtering", *Nonlinearity*, Vol. 22, 2010, pp. 305-324.

Multi-dimensional case

$$dX_t^\varepsilon = b(X_t^\varepsilon, Z_t^\varepsilon)dt + \sigma(X_t^\varepsilon, Z_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x \in \mathbb{R}^m,$$
$$dZ_t^\varepsilon = \frac{1}{\varepsilon}f(X_t^\varepsilon, Z_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}g(X_t^\varepsilon, Z_t^\varepsilon)dV_t, \quad Z_0^\varepsilon = z \in \mathbb{R}^n,$$
$$Y_t^\varepsilon = \int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)ds + B_t, \quad Y_0^\varepsilon = 0 \in \mathbb{R}^d$$

Zakai equation: $du^\varepsilon(x,z,t) = (\mathcal{L}^\varepsilon)^* u^\varepsilon(x,z,t)dt + h^*(x,z,t)u^\varepsilon(x,z,t)dY_t^\varepsilon$

$$\boxed{\mathcal{L}^\varepsilon \neq (\mathcal{L}^\varepsilon)^*}$$

Can no longer utilize Levy's (Knight's) Theorem

Multi-dimensional case

Recall unnormalized filter:

$$\rho_t^\varepsilon(\varphi) = \mathbb{E}_{\mathbb{P}^\varepsilon}\left[\varphi(X_t^\varepsilon, Z_t^\varepsilon)(D_t^\varepsilon)^{-1}\Big|\,\mathcal{Y}_t^\varepsilon\right] = \int_{\mathbb{R}^{m+n}} \varphi(x', z') u_t^\varepsilon(x', z') dz' dx'$$

x-marginal:

$$\rho_t^{\varepsilon,x}(\varphi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{m+n}} \varphi(x') u_t^\varepsilon(x', z') dz' dx' = \int_{\mathbb{R}^{m+n}} \varphi(x') \mathrm{d}\mathbb{P}_{0,t}^\varepsilon(dx', dz')$$

Constructing the homogenized filter (multi-dimensional case)

Homogenized filter

Find ρ^0:

$$d\rho^0_t(\varphi) = \rho^0_t\left(\bar{\mathcal{L}}\varphi\right) dt + \rho^0_t\left(\bar{h}\varphi\right) dY^\varepsilon_t, \quad \rho^0_0(\varphi) = \mathbb{E}\left[\varphi\left(X^0_0\right)\right]$$

$\pi^0_t(\varphi) \stackrel{\text{def}}{=} \frac{\rho^0_t(\varphi)}{\rho^0_t(1)}$

Note: Actual observation Y^ε_t is used.

Operator associated with homogenized process X^0

$$\bar{\mathcal{L}} = \sum_{i=1}^{m} \bar{b}_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m} \bar{a}_{ij}(x,z)\frac{\partial^2}{\partial x_i \partial x_j},$$

with $\bar{b}_i(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} b_i(x,z)\mu(x,dz)$, $\bar{a}_{ij}(x) = \int_{\mathbb{R}^n}(\sigma\sigma^*)_{ij}(x,z)\mu(x,dz)$

▸ ν^0 ▸ D^0

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$\pi_t^0(\varphi) \stackrel{\text{def}}{=} \frac{\rho_t^0(\varphi)}{\rho_t^0(1)}$

Note: Actual observation Y_t^ε is used.

Define the marginal filter $\rho_t^{\varepsilon,x}(\varphi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{m+n}} \varphi(x) u_t^\varepsilon(x,z)dzdx$

and $\pi_t^{\varepsilon,x}(\varphi) = \frac{\rho_t^{\varepsilon,x}(\varphi)}{\rho_t^{\varepsilon,x}(\mathbb{1})}$

▸ ν^0 ▸ D^0

Homogenization in multiscale filtering

Goal:

1. Find a suitable version of π^0 associated with (X^0, Y^ε) (homogenized process, actual observation)

2. Show that $\pi^{\varepsilon,x}$ is close to π^0 in L^p-sense as $\varepsilon \to 0$[a]

$$\boxed{\lim_{\varepsilon\to 0} \sup_{t\leq T} \mathbb{E}_{\mathbb{Q}}\left[d\left(\pi_t^{\varepsilon,x}, \pi_t^0\right)^p\right] = 0, \quad \forall T > 0}$$

[a] P. Imkeller, N. Sri Namachchivaya, N. Perkowski, H. Yeong, Dimensional reduction in nonlinear filtering: A homogenization approach. *Annals of Applied Probability*, Vol. 23, No. 6, 2290-2326, 2013 (extension of J. H. Park, R. B. Sowers, N. Sri Namachchivaya, Dimensional reduction in nonlinear filtering. *Nonlinearity*, 23, 2010)

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The actual proof in high dimension deals with the dual representation of the coarse-grained measure-valued process that solves a backward SPDEs.
Recall unnormalized filter: x-marginal:

$$\rho_t^{\varepsilon,x}(\varphi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \varphi(x') \int_{\mathbb{R}^m} u_t^{\varepsilon}(x', z') dz'\, dx' = \int_{\mathbb{R}^{m+n}} \varphi(x') \mathrm{d}\mathbb{P}_{0,t}^{\varepsilon}(dx', dz')$$

Markov property:

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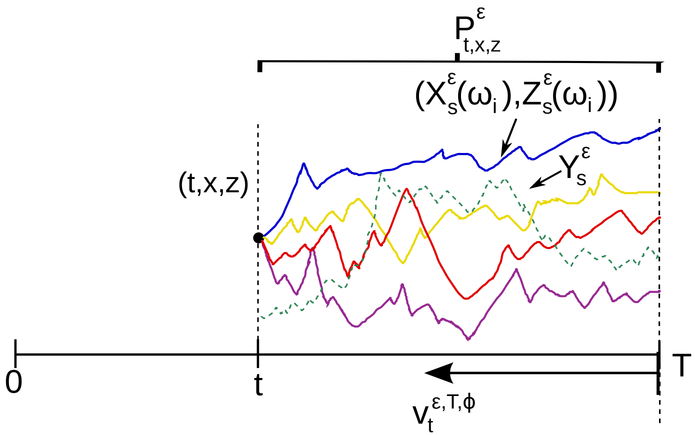

▸ $D_{t,T}$

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Goal: Show $\lim_{\varepsilon\to 0} \mathbb{E}_{\mathbb{P}^{\varepsilon}}(|v_0^{\varepsilon}(x,z) - v_0^{0}(x)|^p) = 0$

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Filtering equations (asymptotic expansion)

v^ε is unique solution to backward SPDE [4]:

$$-dv_t^\varepsilon(x,z) = \mathcal{L}^\varepsilon v_t^\varepsilon(x,z)dt + h(x,z)^* v_t^\varepsilon(x,z)d\overleftarrow{Y}_t^\varepsilon$$
$$v_T^\varepsilon(x,z) = \varphi(x)$$

Expand v^ε:

$$v^\varepsilon(t,x,z) = \underbrace{u^0(t,x,z)}_{v^0(t,x)} + \underbrace{\varepsilon u^1\left(\frac{t}{\varepsilon},x,z\right)}_{\psi(t,x,z)} + \underbrace{\varepsilon^2 u^2\left(\frac{t}{\varepsilon},x,z\right)}_{R(t,x,z)}$$

Showing $\lim_{\varepsilon\to 0}\mathbb{E}_{\mathbb{P}^\varepsilon}(|v_0^\varepsilon(x,z) - v^0(x)|^p) = 0$
$\Leftrightarrow$ **Showing** ψ, $R \to 0$ **as** $\varepsilon \to 0$

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Corrector and remainder:

$$-d\psi(t,x,z) = \frac{1}{\varepsilon}\mathcal{L}_F \psi(t,x,z)dt + (\mathcal{L}_S - \bar{\mathcal{L}})v^0(t,x)dt$$
$$+ (h(x,z) - \bar{h}(x))^* v^0(t,x) d\overleftarrow{Y}^{\varepsilon}_t,$$
$$-dR(t,x,z) = \mathcal{L}^{\varepsilon} R(t,x,z)dt + \mathcal{L}_S \psi(t,x,z)$$
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with terminal conditions $\psi(T,x,z) = R(T,x,z) = 0$

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Results [6]

- Use BDSDE representations of v^0, $\psi(T,x,z)$, $R(T,x,z)$ ▸ BDSDE
- Utilize homogenization estimates from Pardoux and Veretennikov [a]:

As $\varepsilon \to 0$, ψ, R and their derivatives $\to 0$

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- Use BDSDE representations of v^0, $\psi(T,x,z)$, $R(T,x,z)$ ▸ BDSDE
- Utilize homogenization estimates from Pardoux and Veretennikov [a]:

As $\varepsilon \to 0$, ψ, R and their derivatives $\to 0$

[a] E. Pardoux, A. Y. Veretennikov, On Poisson equation and diffusion approximation 2.. *Ann. Probab.*, 31, 2003

$$\downarrow$$

$$\boxed{\lim_{\varepsilon\to 0} \mathbb{E}_{\mathbb{P}^\varepsilon}\left(|v_0^\varepsilon(x,z) - v_0^0(x)|^p\right) = 0}$$

$$\downarrow$$

$$\boxed{\lim_{\varepsilon\to 0} \mathbb{E}_{\mathbb{P}^\varepsilon}\left(|\rho_T^{\varepsilon,x}(x,z) - \rho_T^0(x)|^p\right) = 0}$$

$$\downarrow$$

$$\boxed{\lim_{\varepsilon\to 0} \sup_{t\le T} \mathbb{E}_{\mathbb{Q}}\left[d\left(\pi_t^{\varepsilon,x}, \pi_t^0\right)^p\right] = 0, \quad \forall T > 0}$$

▸ r

[6] P. Imkeller, N. Sri Namachchivaya, N. Perkowski, H. Yeong, Dimensional reduction in nonlinear filtering: A homogenization approach. *Accepted in Ann. Appl. Prob.*, 2012