INTERNATIONAL CONFERENCE ON DATA ASSIMILATION

Kalman Filter Design by Tuning its Statistics or Gains?

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KALMAN FILTER TOPICS TO BE DISCUSSED

1. **PRINCIPLES AND ITS APPLICATIONS**
2. **BRIEF FORMULATION**
3. **IMPORTANCE OF TUNING**
4. **TUNING BASED ON THE FILTER STATISTICS Po, Q, AND R**
5. **TUNING BASED ON THE CONSTANT GAIN APPROACH**

KALMAN FILTER

- **The Kalman filter in the present form is one of the finest innovations of the twentieth century had its early success in the moon mission. It contains the triplet of**
- **(i) mathematical prediction,**

  (ii) experimental observations, and

  (iii) an intuitive subjective criterion used objectively to obtain the improved estimates.

The development of Kalman Filter is similar to that of any physical theory.

CONCEPT: CHANGE, CAPTURE, AND CORRECT

- **Such a feature is shown to exist even in ancient Indian astronomy. More on this later.**

DEFINITION OF KALMAN FILTER

The Kalman filter assimilates the measurement information with uncertain system and measurement models based on probabilistic weighted linear addition of the predicted state and the measurement data to adapt both the state and measurement models in a statistically consistent way.

Another interpretation is the Kalman filter (KF) is an expanding knowledge front by assimilating newer information in a meaningful way.

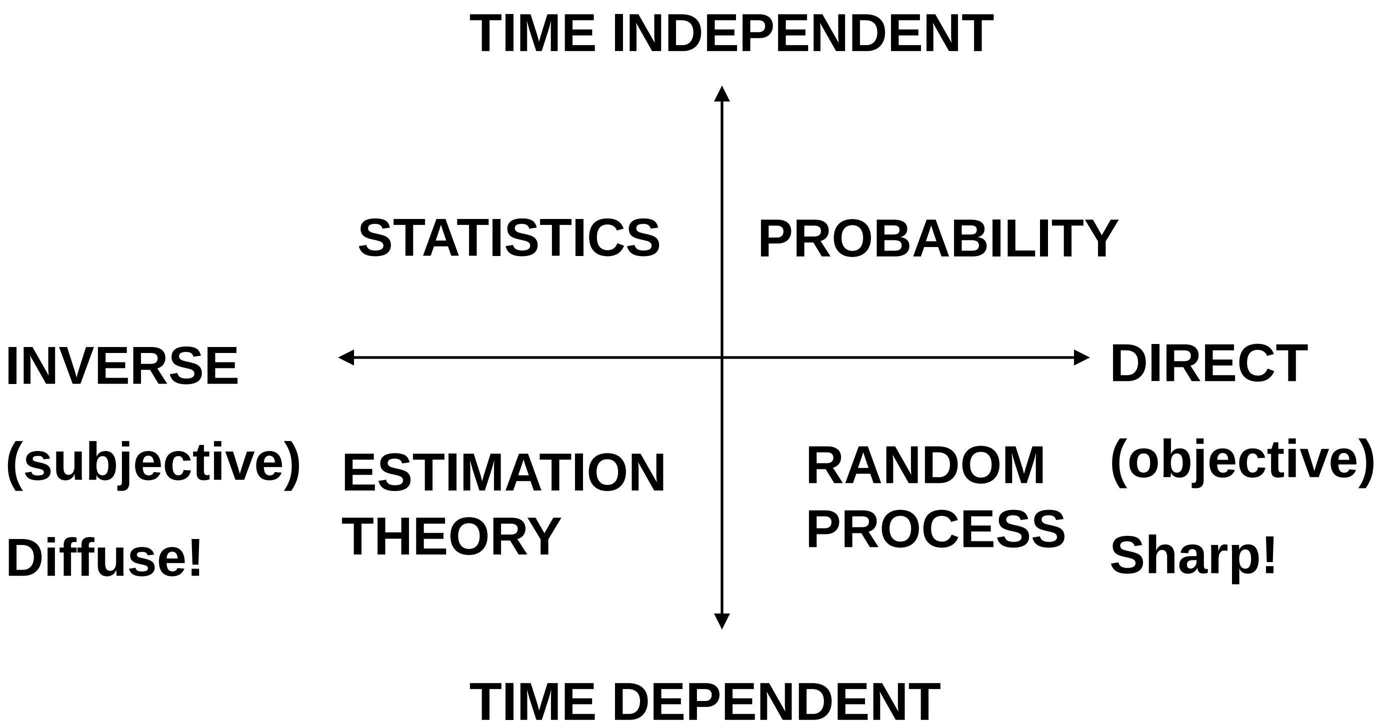
SOME PROBLEMS HANDLED BY KALMAN FILTER

- ONE OF THE EARLIEST USE OF KALMAN FILTER IS

  FOR THE APOLLO MISSION TO THE MOON
- DEPRESSION FOLLOWED INITIAL EUPHORIA!
- FILTER PARAMETERS HAVE TO BE TUNED
- BASICALLY AN OPTIMISATION PROBLEM
- PARAMETER ESTIMATION OF SYSTEMS
- TARGET TRACKING OF AEROSPACE VEHICLES
- THE EXPANSION OF THE SCENARIO

  (Evolution of the Space Debris, Airplane Accident Analysis)
- FUSION OF GPS AND INS DATA
- ATMOSPHERIC DATA ASSIMILATION

RELATIONSHIP AMONG PROBABILITY, STATISTICS, RANDOM PROCESS, AND ESTIMATION THEORY

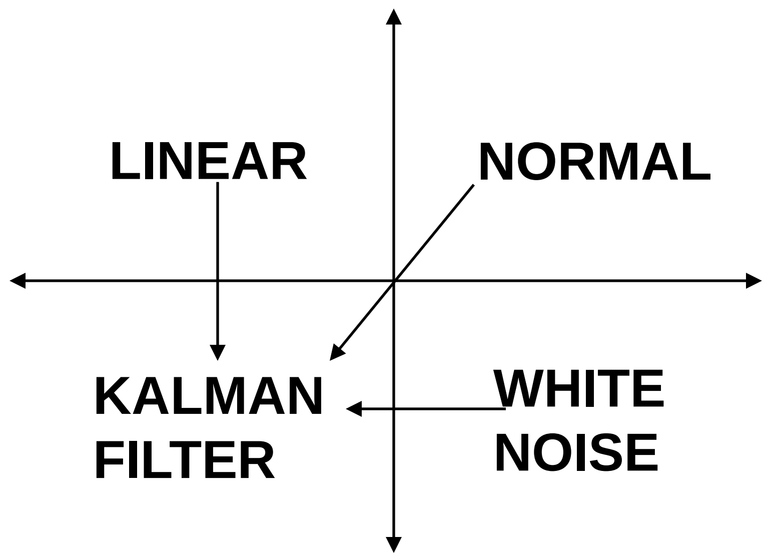


WHAT IS COMMON IN ALL QUADRANTS IS RANDOMNESS!

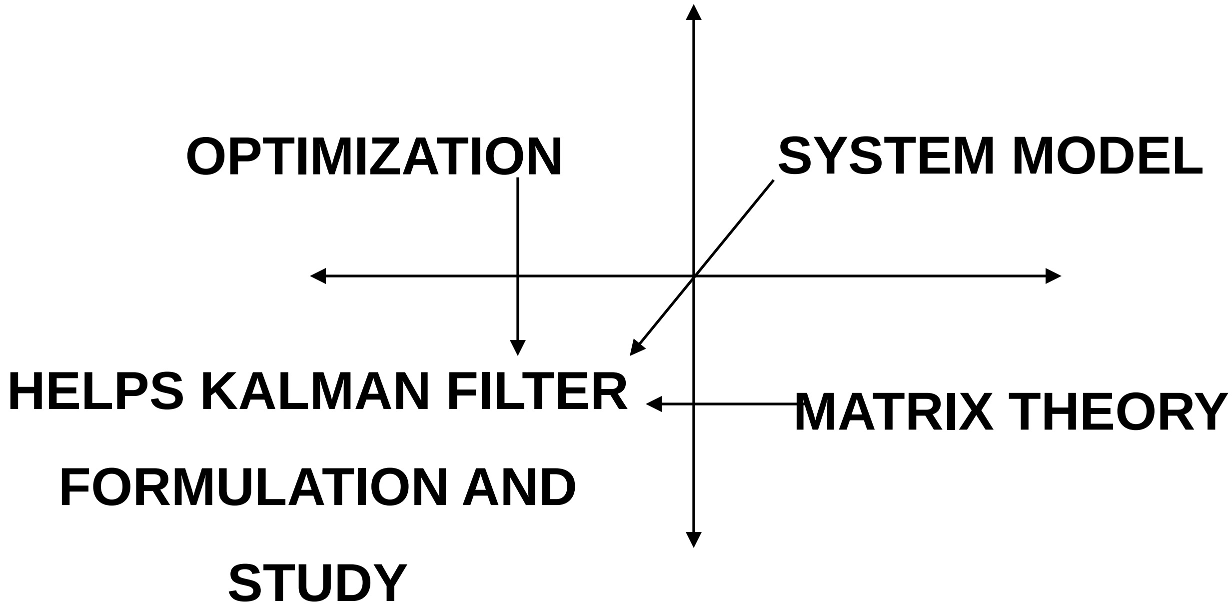
CHANCE AND CREATIVITY

- From Aristotle to mid 19th century chance indicated ignorance making predictions impossible. But chance is inherent in all phenomena and the only way is to study chance and its rules. Chance may be an obstructer and an irritant but it can also create. We have learnt to make chance to work for the mankind.
- All knowledge in the final analysis is history.
- All sciences in the abstract is mathematics.
- All methods of acquiring knowledge are essentially statistics.
- All time dependent statistics is data analysis with time.
- All sequential random time series data analysis off line, online or in real time is Kalman Filtering!

INITIAL APPROACH WITH THE LEAST AMOUNT OF INFORMATION(!) AND GOOD MATHEMATICAL TRACTABILITY!



MODERN MATHEMATICAL TOOLS THAT HELP SOPHISTICATED KALMAN FILTER THEORY



SUBJECTIVITY IN ESTIMATION THEORY

In the previous diagram in the direct problems of probability and random processes namely on the right hand quadrants given a set of rules right or wrong in the objective consequences can all be worked out. In the left hand side in statistics and estimation theory due to the uncertainties prevailing in the data using the subjectivity of the analyst only the best possible probabilistic answers can be provided. The root cause for compulsive subjectivity in the analysis is the uncertainty in the data.

A GENERAL CASE OF SYSTEM AND MEASUREMENT

To keep the discussion simple, for linear systems one has

$$dX/dt = A\,X(t) + B\,U(t) + w(t);\ X(0) = X$$

$$Z = C\,X(t_k) + D\,U(t_k) + v(t_k);\ k = 1,2,\ldots N$$

where the A, B, C, D are appropriate matrices with all the unknown parameters in them denoted by Θ. The process and measurement noise have zero mean with covariance Q and R respectively. Due to the process noise, the estimate and covariance propagation of the state X are given in terms of the transition matrices ϕ and ψ by

$$X_k^- = \phi_{k-1}\,X_{k-1} + \psi_{k-1}\,U_{k-1};\ X(0) = X_0$$

$$P_k^- = \phi_{k-1}\,P_{k-1}\,\phi_{k-1}^T + Q_k;\quad P(0) = P_0$$

and the measurement equation is

$$Z_k = H_k X_k + v_k$$

A GENERAL CASE OF SYSTEM AND MEASUREMENT

The above superscript (-) and the latter (+) indicate the estimates before and after using the measurement information. We have at time t_k an estimate

$$X_k^- \text{ with covariance } P_k^-$$

and another from the measurement namely

$$Z_k \text{ with covariance } R$$

when combined appropriately leads to

$$X_k^+ = X_k^- + K_k[\, Z_k - H_k\, X_k^-\,] = X_k^- + K\, \nu_k$$

$$P_k^+ = [\, I - K_k H_k\,]\, P_k^-\, [\, I - K_k H_k\,]^T + KR_k\, K^T$$

where ν_k is the innovation and the optimal value for the Kalman Gain

$$K_k = P_k^-\, H_k^T\, [\, H_k\, P_k^-\, H_k^T + R_k\,]^{-1}$$

A GENERAL CASE OF SYSTEM AND MEASUREMENT

The innovation follows a Gaussian distribution whose probability when maximised leads to the cost function J defined as

$$J = \sum_{i=1}^{N} \nu_k \left[H_k P_k^- H_k^T + R_k \right] \nu_k^T$$

$$= J(X_0, P_0, Q, R, \Theta)$$

which is a batch processing nonlinear optimization problem and the numerical effort cannot be swept under the rug! When there is no process noise $w_k = 0$, $Q = 0$, and the above reduces to the MMLE with measurement noise only also known as the output error method. If Θ is considered as an additional state then generally it needs linearization at every point known as the Extended Kalman Filter (EKF).

PERFORMANCE INDEX

The performance of the algorithm is dependant on the cost function used which needs to be as discriminating as possible. Cost functions must be designed carefully: one might be found to be more effective with a certain type of optimization problem, while another might be better in some other cases.

TABLE. SOME SALIENT DIFFERENCES BETWEEN USUAL OPTIMIZATION AND KALMAN FILTERING.

- **TYPE OF SYSTEM/COST FUNCTION: STATIC VERSUS DYNAMIC.**
- **NUMBER OF DATA POINTS: FIXED VERSUS INCREASING.**
- **MODEL IN THE COST FUNCTION: FIXED VERSUS VARYING.**
- **UNKNOWNS IN THE COST FUNCTION: DETERMINISTIC VERSUS PROBABLISTIC.**
- **UPDATE OF UNKNOWNS: CAN GENERALLY BE DONE IN ANY ORDER BUT IN KALMAN FILTER IT MAY NOT BE POSSIBLE TO UPDATE IN ANY ORDER.**

PERFORMANCE INDEX

- **The first cost function ISE considers each error contribution with a different weight a_i . For example, an error of 1m in position might not be as important as a 1m/s deviation in velocity.**
- $J = (1/T) \int (1/N) \Sigma\, a_i (x_i - x_m)^2\, dt$
- **Here N is the number of states in the estimator, and the integral is taken over the evaluation interval T of the Kalman filter.**
- **The second cost function ITAE (Integral of Time multiplied by Absolute Error) considers the time as a scaling factor. This perfectly makes sense since it is important in this case to ensure a zero error after the filter has converged. The second performance index is represented by the following equation:**
- $J = (1/T) \int (1/N) \Sigma\, a_i |x_i - x_m|\, dt$

A GENERAL CASE OF SYSTEM AND MEASUREMENT

Another cost function that is useful to study the effects of inadequate modelling in state estimation problems which is very common in Kalman filter studies. This is defined as

$$J^* = (1/N) \sum_{i=1}^{i=N} (x(i) - x_{ref}(i))^T (P)^{-1} (x(i) - x_{ref}(i))$$

where the suffix 'ref' refers to a desired reference trajectory to be followed, and the argument in X(.) denotes the time step or point. The P is the covariance matrix obtained with nominal values for the unknown disturbances. If the variations or deficiencies in the modeling is beyond the statistical fluctuations as denoted by the above covariance then cost function changes substantially and indicates a degradation of the performance of the filter.

GLOBAL FORMULATION

A previous global formulation based on the innovation is operationally equivalent to minimizing the cost function

$$J = (1/N)\, \Sigma\, \nu(k)\, [H(k)\, P^{-}(k)\, H^{T}(k) + R\,]^{-1}\, \nu^{T}(k)$$

$$= (1/N)\ \Sigma\, \nu(k)\, [\, \Re\,]^{-1}\, \nu^{T}(k) = J\, (X_0, P_0, Q, R, \Theta)$$

with Σ denoting the summation over all the N measurements. A Kalman filter analyst has to choose or tune the statistics P_0, Q, R, and estimate Θ (all of which may be uncertain or unknown) for a proper filter design and operation.

KALMAN FILTER TUNING

TUNING Choose X_O, P_O, Q and R to minimise the cost function such that Kalman gain is adjusted to extract the information from the data fully.

EXACT Very few due to difficult optimization problem

HEURISTIC Offers best chance since based on physical insight with good guesses

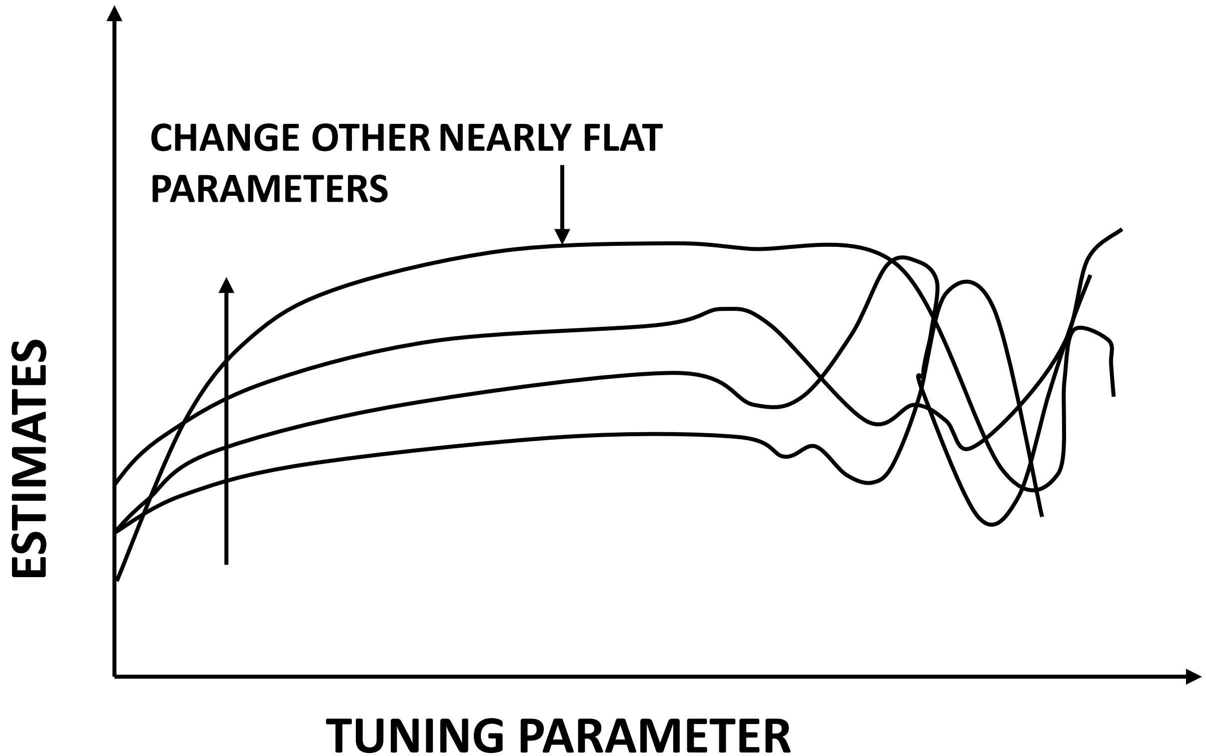
AD HOC Plenty. Manual. Results could be grossly in error

KALMAN FILTER TUNING

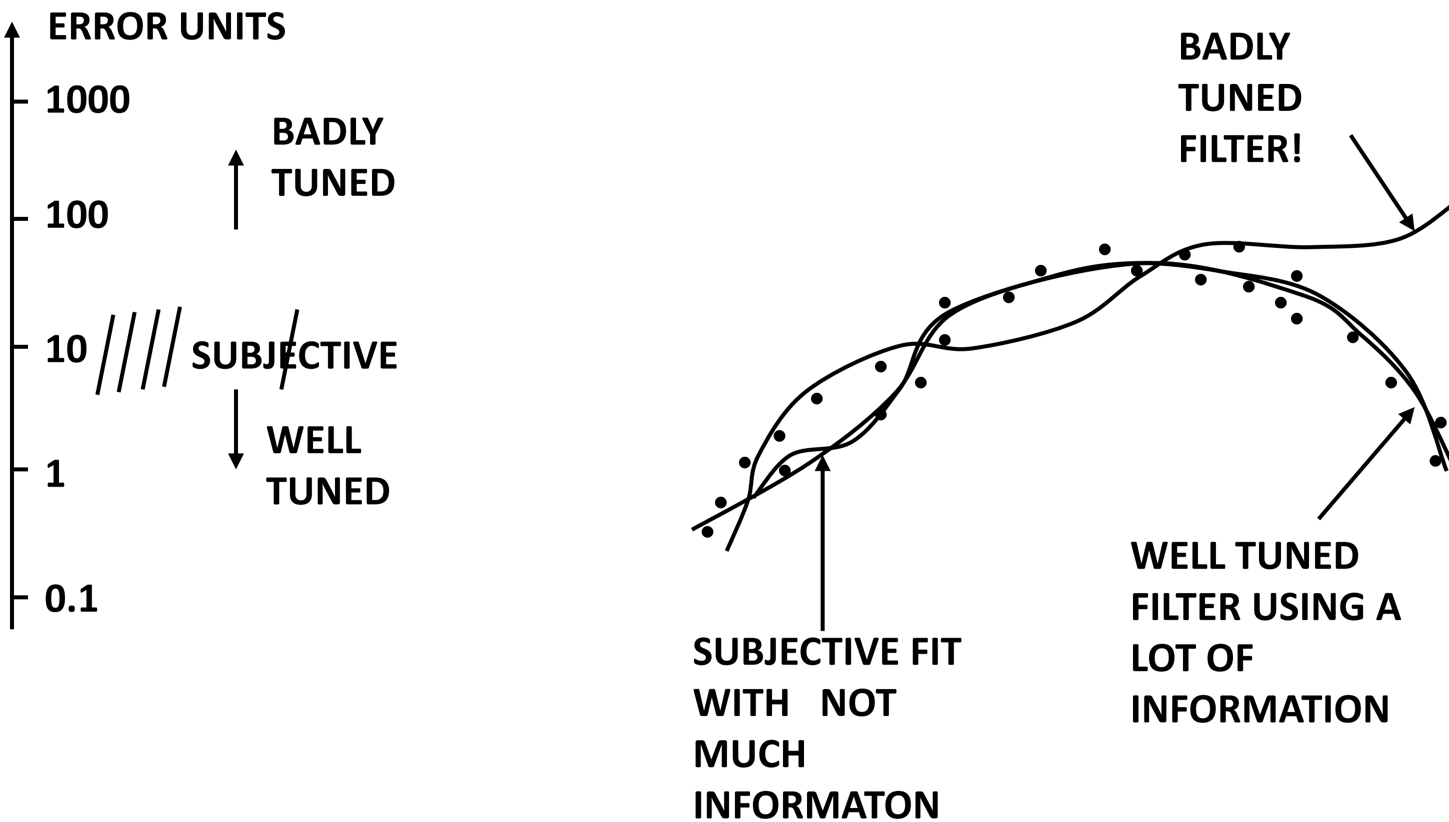
ATTITUDE OF THE USERS IN PRESENTING THEIR EXPERIENCE IN APPLYING THE KALMAN FILTER

Plenty	**Utilise**
Few	**Analyse**
Very few	**Report!**

Most reports and publications write out detailed standard filter equations but the tuning procedures is not known, reported or kept as a secret!



CHANGE OTHER PARAMETERS
NEARLY FLAT
ESTIMATES
TUNING PARAMETER


ERROR UNITS
1000
100
10
1
0.1
BADLY TUNED
SUBJECTIVE
WELL TUNED
BADLY TUNED FILTER!
SUBJECTIVE FIT WITH NOT MUCH INFORMATON
WELL TUNED FILTER USING A LOT OF INFORMATION

Tuning of Kalman Filter Statistics

This is carried out by minimizing the normalized covariance of the innovation as defined over a time span of the measurement data. Such a complex optimization problem requires enormous numerical effort that cannot be swept aside. The optimum solutions being not easy generally suboptimal solutions have to be sought.

Tuning of Kalman Filter Statistics

It is not well known that after Kalman proposed his filter there was an euphoria only to be followed soon after by depression since the solution was only formal. After the 1960 paper of Kalman by the time of Gelb's book in 1974 most filter approaches, applications and numerical procedures were in place. But the tuning of the filter statistics had not matured to a level for routine use. One has to tune the statistics P_0, Q and R for a satisfactory filter operation and even now this is generally done manually!

TUNING OF R and Po

- **The measurement noise covariance R is fairly objective and can be determined from the data by ‘scaling’ the measurement noise.**
- **The initial state covariance P_0 is tricky and is partially objective as well as subjective since the quantities can be reasonably guessed and generally the off diagonal elements are set to zero and the diagonal elements are set to large values but however their relative ‘sizing’ values are crucial for an optimum filter operation. The P_0 controls the handover from the initial transient to Q for steady state filter behavior and also has to be chosen carefully.**
- **The Po is important like for a child who has deficient or overdose components of nutrition when young will have some shortfall even with appropriate nutritious food he/she may have in later life.**

TUNING OF Q

- **The Q is notoriously difficult and is highly subjective but it helps to inject uncertainty into the state equations to assist the filter to learn from the measurements and it also controls the steady state filter response. Even when there is no state noise but only the measurement noise in the data, starting with some initial estimate for Θ to assist the filter to learn from the measurements a non zero Q has to be injected into the state equations. A too large a value of Q will lead to a short transient with large steady state uncertainty of the estimates and a small Q vice versa. In spite of being labeled as notorious it is only by 'shaping' the Q that an analyst can handle the unmodelled or unmodellable errors and even account for numerical errors! The Q is also helpful to track systems whose dynamical equations are not known.**

EXAMPLES FOR USE OF Q (PROCESS NOISE)

- Though Q is considered notorious ***it is the life line*** of the Kalman filter doing good work all the time. Grossly misunderstood just like good people thought to be bad for a long time! Some classic examples for such systems are the GPS receiver clocks, satellite, trajectory of aircraft, missiles, and reentry vehicles. These are handled by using the kinematic relations between the position, velocity, and acceleration all assumed to be driven by white Gaussian noise Q of suitable magnitude to enable the filter to track these systems. Generally the filter parameters are tuned off line using simulated data and subsequently used for on line and real time applications perhaps with efficient numerical procedures for processing the data.

CHANGES IN THE COST FUNCTION FORMULATION FROM STATISTICS TO GAINS

$$J = (1/N) \Sigma\, \nu(k) \, [H(k)\, P^{-}(k)\, H^{T}(k) + R\,]^{-1}\, \nu^{T}(k)$$

$$= (1/N) \;\Sigma\, \nu(k) \, [\,\Re\,]^{-1}\, \nu^{T}(k)$$

$$= J\,(X_0, P_0, Q, R, \Theta)$$

= J (X_0, Θ, K (traded for P_0,Q,R)) and all the covariance equations have disappeared!

DIFFERENCE IN THE KALMAN FILTER FORMULATION FROM STATISTICS TO GAINS

STEPS IN KALMAN FILTER USING

	STATISTICS	GAINS
1. STATE PROPAGATION.	YES	YES
2. COVARIANCE PROPAGATION	YES	NO
3. GAIN CALCULATION	YES	YES
4. STATE UPDATE	YES	NO
5. COVARIANCE UPDATE	YES	YES

NOTE: 1, and 4 refer to sample while 2, 3 and 5 refer to ensemble characterization.

ADVANTAGE OF CONSTANT GAIN KALMAN FILTER

- **Most importantly one need not propagate the covariance equations of the usual Kalman filter thus enormously speeding up the computations.**
- **The gains have to be chosen using a suitable cost function based on the normalised innovation over a period of time.**
- **The gain K or KH as appropriate lie between 0 and 1.**
- **The Kalman Gains are more robust parameters in the filter formulation than the statistics Po, Q and R.**

THE DESIGN OF THE KALMAN FILTER

Similar to the derivation of the filter equations there could be different levels of implementing the same for any practical situation.

- **(i) Time varying full matrices Q and R**
- **(ii) Time varying diagonal matrices Q and R**
- **(iii) Time invariant diagonal matrices Q and R**
- **(iv) Work with constant Kalman gain matrix**
- **(v) Work with important Kalman gain elements**

The first and second are unwieldy, but the third is generally the most preferable. The fourth and fifth ones provide acceptable and more easily implementable solutions though not necessarily optimal.

BEYOND KALMAN FILTER

- **Linearised Kalman Filter**
- **Extended Kalman Filter**
- **Information Filter**
- **Schmidt-Kalman Filter**
- **Unscented Kalman Filter**
- **Interactive Multiple Model Filter**
- **Particle Filter**
- **Ensemble Kalman Filter**
- **Computational variations exist among these but all of them are chased by the ghost of filter tuning!**

TUNING THE KALMAN FILTER BASED ON THE FILTER STATISTICS Po, Q, AND R THE INITIAL STATE, PROCESS, AND MEASURMENT NOISE COVARIANCES

A Heuristic Adaptive Kalman Filtering Approach

Exact solutions are very hard, approximate choice can lead to inappropriate results but heuristic approaches provide the middle path in designing the Kalman filter like the one presented here. This has been successful in a variety of problems. The general methodology of any filter estimation algorithm is to update the X_0, P_0, Θ, R and Q at a point, over a window, after a pass or after multiple passes by applying some corrections to them based on changes, iterations or sample statistics such that the numerical solution does not diverge but converges to the best possible estimates based on the information contained in the data.

A Heuristic Adaptive Kalman Filtering Approach

A heuristic approach was proposed by Myers and Tapley to estimate R and Q. Later Gemson and Ananthasayanam showed the importance of P_0 for parameter estimation and gave a recipe to estimate all the above uncertain or unknown initial condition, statistics and parameters. The optimization of J estimates the deterministic Θ and the statistics P_0, Q and R. Unlike in classical problems, convergence being not possible in general with simultaneous update of all the statistics is worth studying further.

A Heuristic Adaptive Kalman Filtering Approach

- **The flow chart in the next slide shows the steps to adaptively estimate the Kalman filter statistics. These are (i) Choose an initial suitable diagonal matrix for R and set $Q \equiv 0$, (ii) Make a scouting pass through the data with diagonal $P_0 = P_g$, a guess matrix or I an identity matrix. After such a pass obtain P_0 from the inverse of the information matrix (IIM), (iii) With the above P_0 = IIM iterate for Q using MT algorithm till the cost J converges, (iv) Using the above converged Q obtain an updated estimate of R using MT algorithm, and (v) Use the above updated R and after resetting $Q \equiv 0$ repeat from Step (ii) until R converges.**

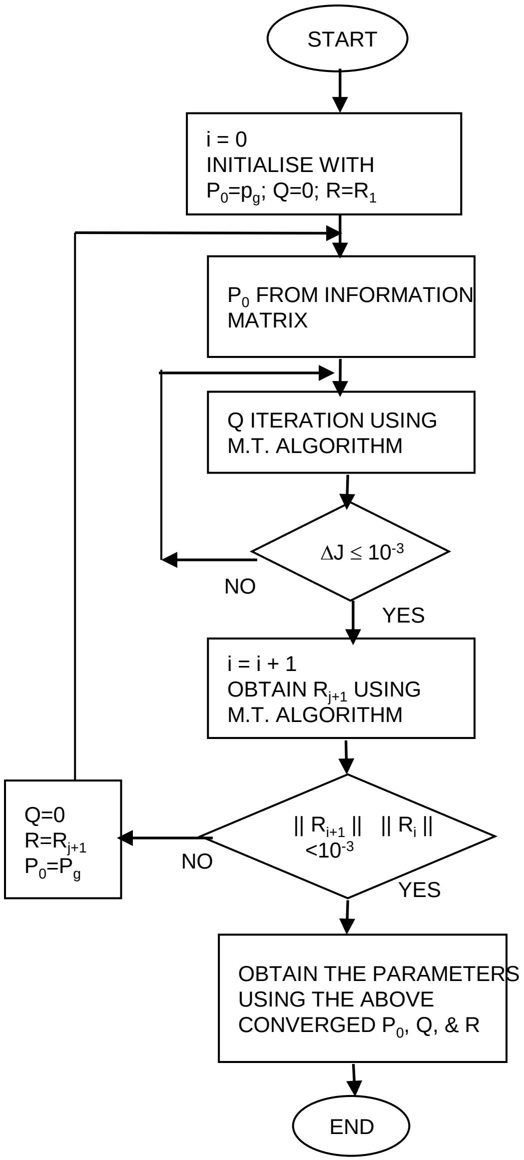


Figure. Flow Chart of an Adaptive Tuning Method For P_o , Q, and R for the EKF Formulation.

An Algorithm for Estimating P_0, Q and R

- The methodology to estimate the filter statistics is as follows. The filter has many sample quantities such as $X^+(k)$, $X^-(k)$, $Z(k)$, $h(X^-(k))$, $h(X^+(k))$ and also their ensemble statistical characteristics like $P^+(k)$, $P^-(k)$, $H(k)P^-(k)H^T(k)$, $H(k)P^+(k)H^T(k)$. Generally by using heuristic arguments and mathematics (!) one can estimate the covariances Q and R. Some possible intuitive sample sequences for measurement and state noise are respectively $r(k) = [Z(k) - h(X^-(k))]$ and $q(k) = [X^+(k) - X^-(k)] = [X^+(k) - \varphi(k-1)X^+(k-1)]$ from which unbiased estimates for the measurement and state noise covariance has been inferred to be

An Algorithm for Estimating $P_{0,}$ Q and R

- R_{est} $= (1/N)\ \boldsymbol{\Sigma}\ [\{r(k) - r_m)(r(k) - r_m)^T\} - H(k)P^{-}(k)H^T(k)]$
- Q_{est} $= (1/N)\ \boldsymbol{\Sigma}\ [\{(q(k) - q_m)(q(k) - q_m)^T\}$

  $- \{\varphi(k\text{-}1)P^{+}(k\text{-}\ 1)\varphi^T(k\text{-}1) - P^{+}(k)\}]$

  where the subscript 'm' denotes the mean and $\boldsymbol{\Sigma}$ the summation over a window of N time points. A variety of similar intuitive choices of sequences are possible. Also at some stage of estimation if the positive definiteness is not maintained then 'ad hoc' corrections can be made to continue the numerical procedure without stumbling. For P_0 the proposed choice with $\boldsymbol{\Sigma}$ as above is the inverse of the information matrix.
- P_0 $= [\boldsymbol{\Sigma}(1/N)\ (\phi(k)H(k)R(k)H^T(k)\phi^T(k))]^{-1}$

MODEL FOR SIMULATION STUDIES

Model used for simulated studies

This is a model provided by Candy[6]. The state variable X(t) is continuous and Y(k) are discrete measurements at time instants t_k.

$$\dot{X}(t) = a1X + a2X^2 + w(t)$$
$$Y(k) = X^2(t_k) + X^3(t_k) + v_k \qquad \text{A (1,2)}$$

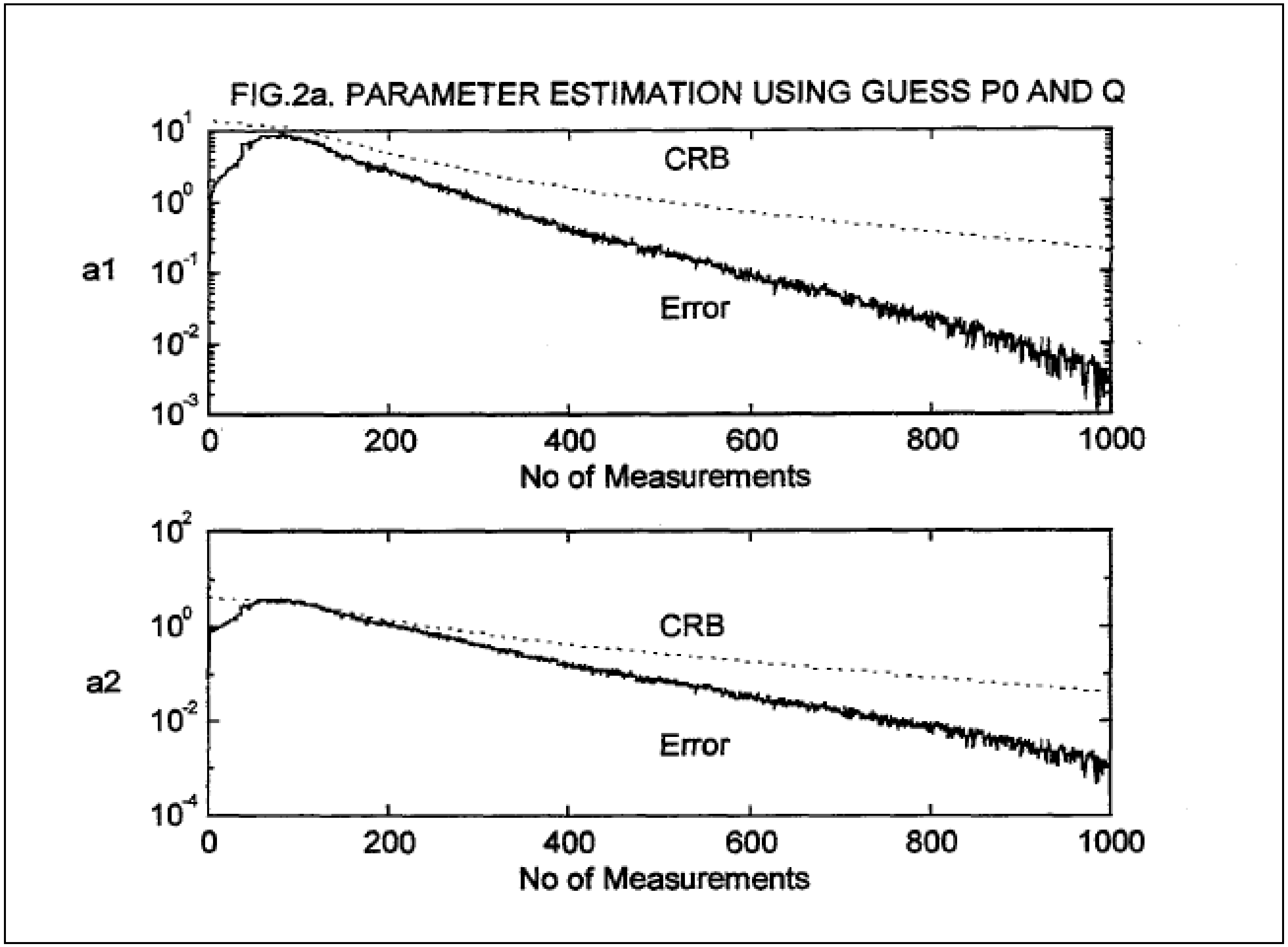
The true values of the parameters are

a1 = –0.05; a2 = 0.04.

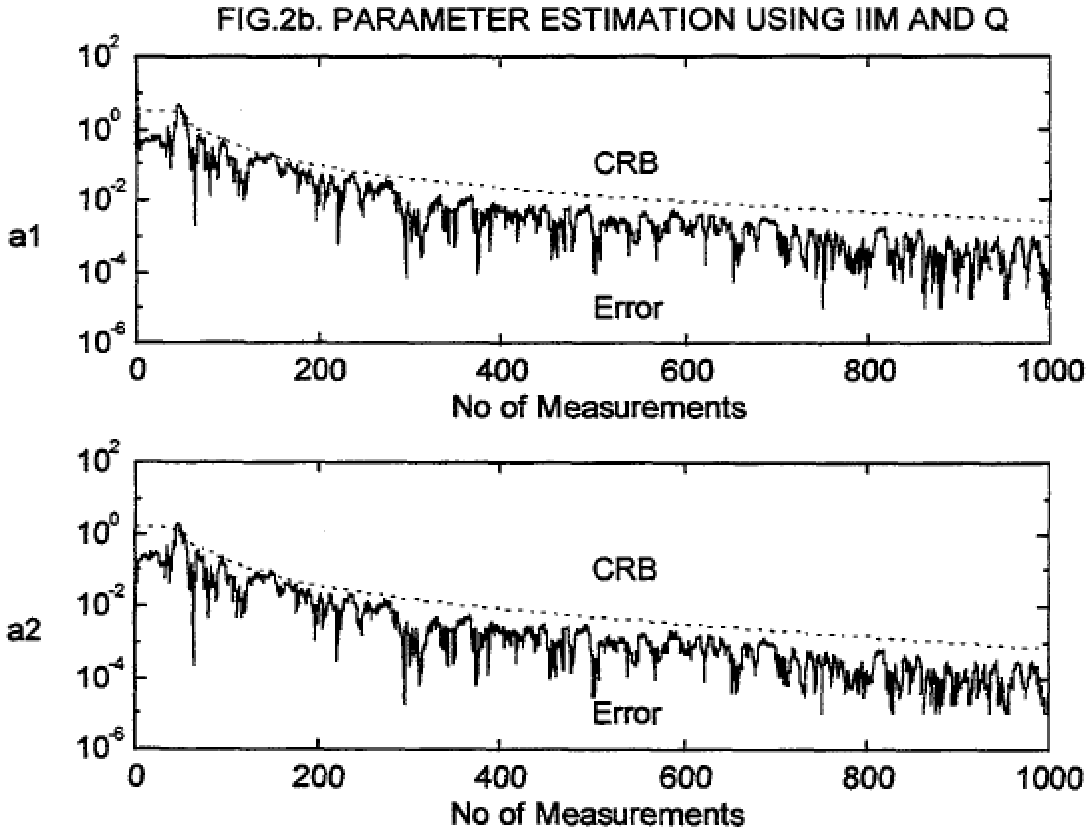
Noise characteristics :

Process noise	$w(t) \sim N[0.0, 10^{-6}]$
Measurement noise	$v_k \sim N[0.0, 0.09]$

MODEL FOR SIMULATION STUDIES



MODEL FOR SIMULATION STUDIES



AN AIRPLANE

STATE AND MEASUREMENT MODELS USED FOR REAL FLIGHT TEST DATA STUDIES: DYNAMICAL EQUATIONS FOR THE AIRCRAFT

$$\dot{\alpha} = q - \left(\frac{\bar{q}S}{mV}\right)(C_L + C_{L0}) + \left(\frac{g}{V}\right)(\cos\varphi_m \cos\theta os\theta c_c + \sin\theta in\theta s_c)$$
$$-\beta_c(p_m \cos\alpha_c + \gamma_m \sin\alpha_c)$$

$$\dot{q} = \left(\frac{\bar{q}S\bar{c}}{I_{yy}}\right)\left\{C_{m_\alpha}.\alpha + C_{m_q}\left(\frac{q\bar{c}}{2V}\right) + C_{m_{\delta_e}}\delta_e + C_{m_0}\right\} + \gamma_m p_m \left(\frac{I_{zz} - I_{xx}}{I_{yy}}\right)$$

$$\dot{\theta} = \dot{\theta}_0 + q\cos\varphi_m - \gamma_m \sin\varphi_m$$

The quantities α, q, θ, φ, β are the angle of attack, pitch rate and pitch angle bank angle and side slip angle respectively. The moments of inertia about the x, y and z axes are I_{xx}, I_{yy} and I_{zz} respectively. The quantities with suffix m denote the measurements and those with suffix c represent the measurements corrected for sensor location effects.

THE MEASUREMENT EQUATIONS

$$\alpha_m = K_\alpha . \alpha - K_\alpha . X_\alpha \left(\frac{q}{V}\right)$$

$$q_m = q$$

$$\theta_m = \theta$$

$$a_{nm} = \left(\frac{\bar{q}s}{mg}\right) C_N + \frac{X_{an}}{g} \dot{q}$$

$$a_{xm} = - \left(\frac{\bar{q}s}{mg}\right) C_A + \frac{Z_{ax}}{g} \dot{q}$$

where $C_L = C_{N \cos\alpha} - C_{A \sin\alpha}$

with $C_N = C_{NO} + C_{N_\alpha} . \alpha + C_{A_{\delta_e}} . \delta_e$

$$C_A = C_{A_0} + C_{A_\alpha} . \alpha + C_{A_{\alpha 2}} . \alpha^2 + C_{A\delta_e} . \delta_e$$

and K_α = Flow amplification factor

X_α = x location parameter of the α sensor

X_{an} = x location parameter of the normal accelerometer

A_x = z location parameter of the longitudinal accelerometer

The unknown parameter set

$$\{\Theta\}^T = \left\{C_{N\alpha}, C_{N\delta\delta}, C_{Lo}, C_{M\alpha}, C_{Mq}, C_{mo}, \theta_o, C_{No}, C_{A\alpha}, C_{A\alpha 2}, C_{A\delta\delta}, C_{Ao}\right\}$$

Further the constants used in the analysis are

$$\bar{C} = 5.58; S = 184.0; m = 172.667$$

$$I_{xx} = 4142.9;\ I_{yy} = 3922.4;\ I_{zz} = 7642.5;$$

$$g = 32.2;\ V_{\delta} = 403.1;\ \bar{q} = 83.08;\ K_{\alpha} = 1.0;$$

$$X_{\alpha} = -0.0279;\ X_{an} = 0.101;\ Z_{ax} = -1.17$$

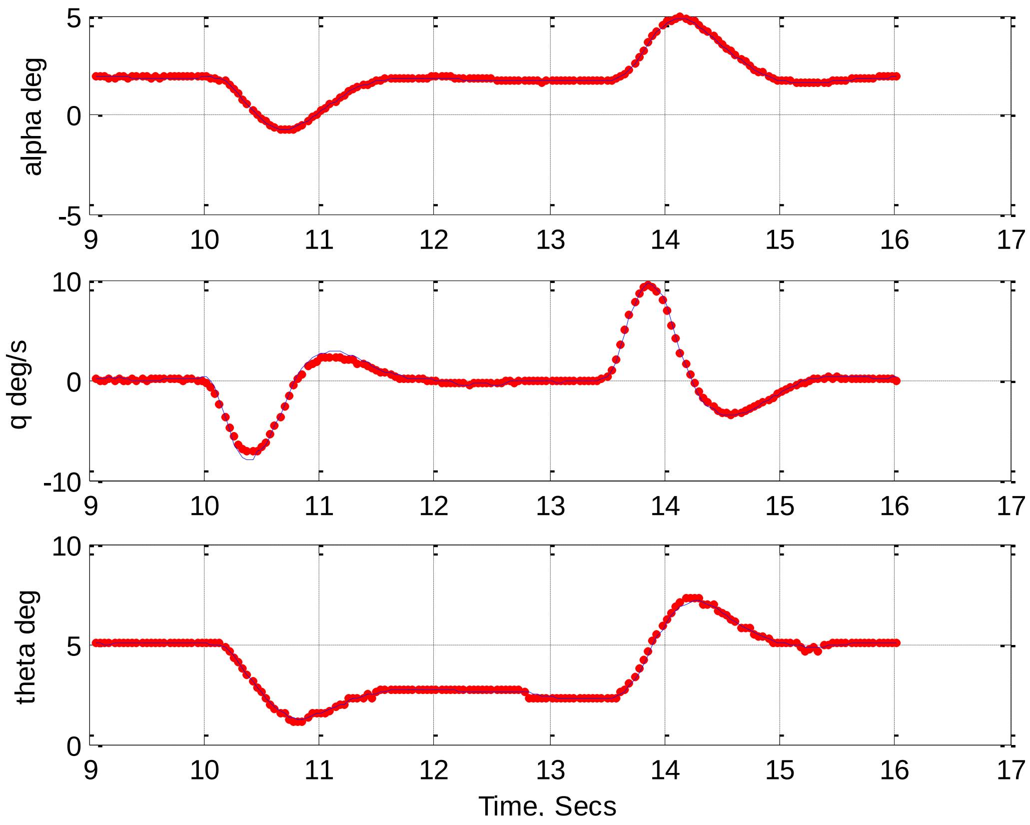
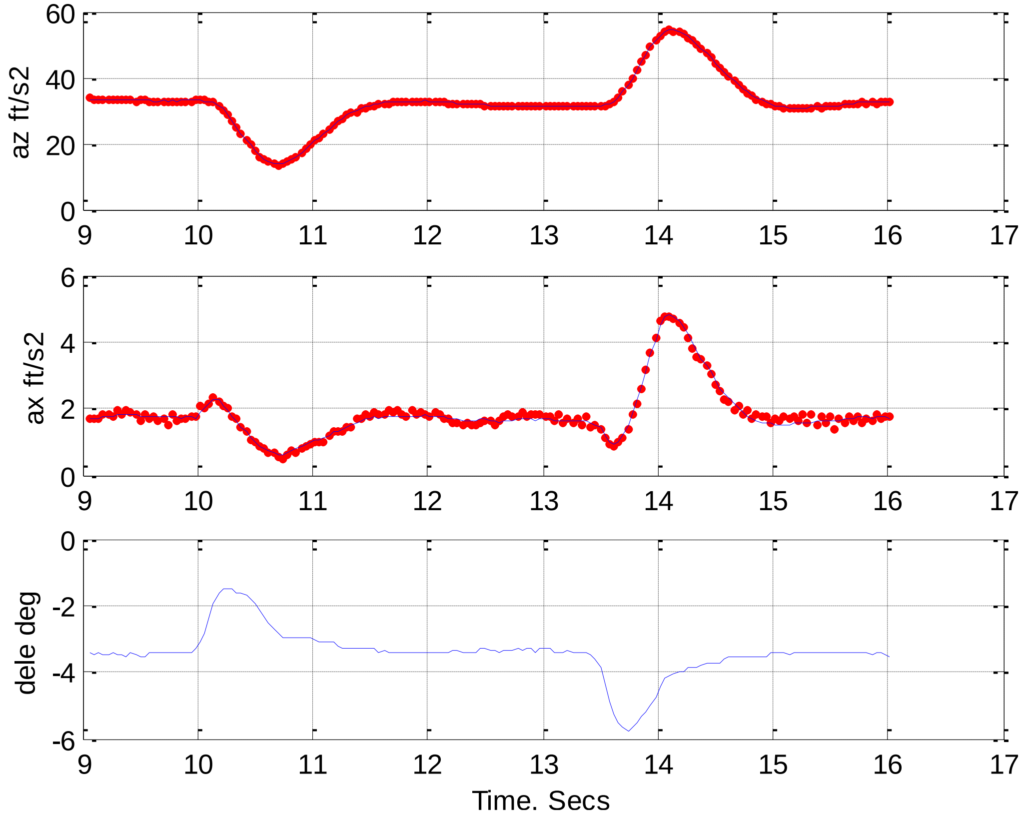


Figure. Comparison of Model Output with the Measurements.



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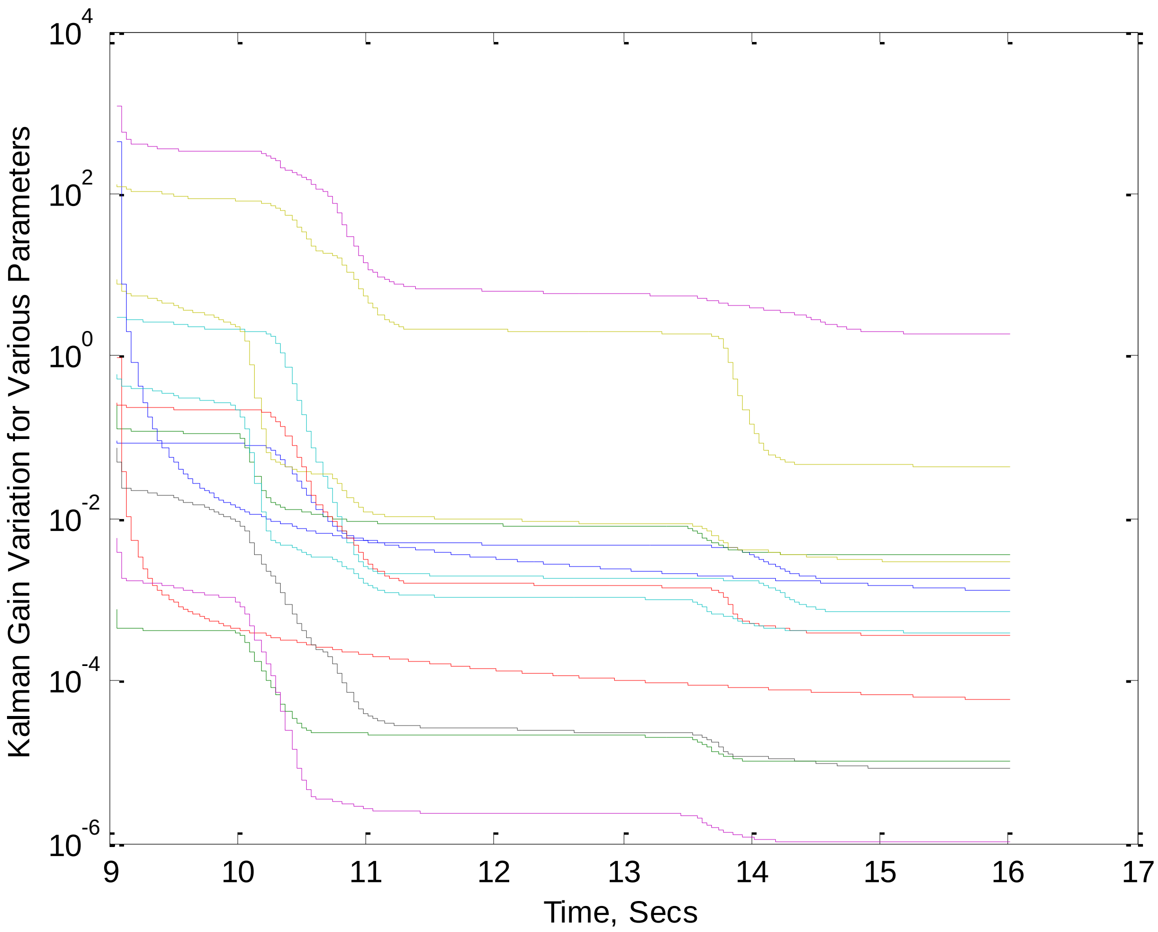


Figure. Variation of the Kalman Gain for the Unknown Parameters in the Longitudinal Airplane Dynamics

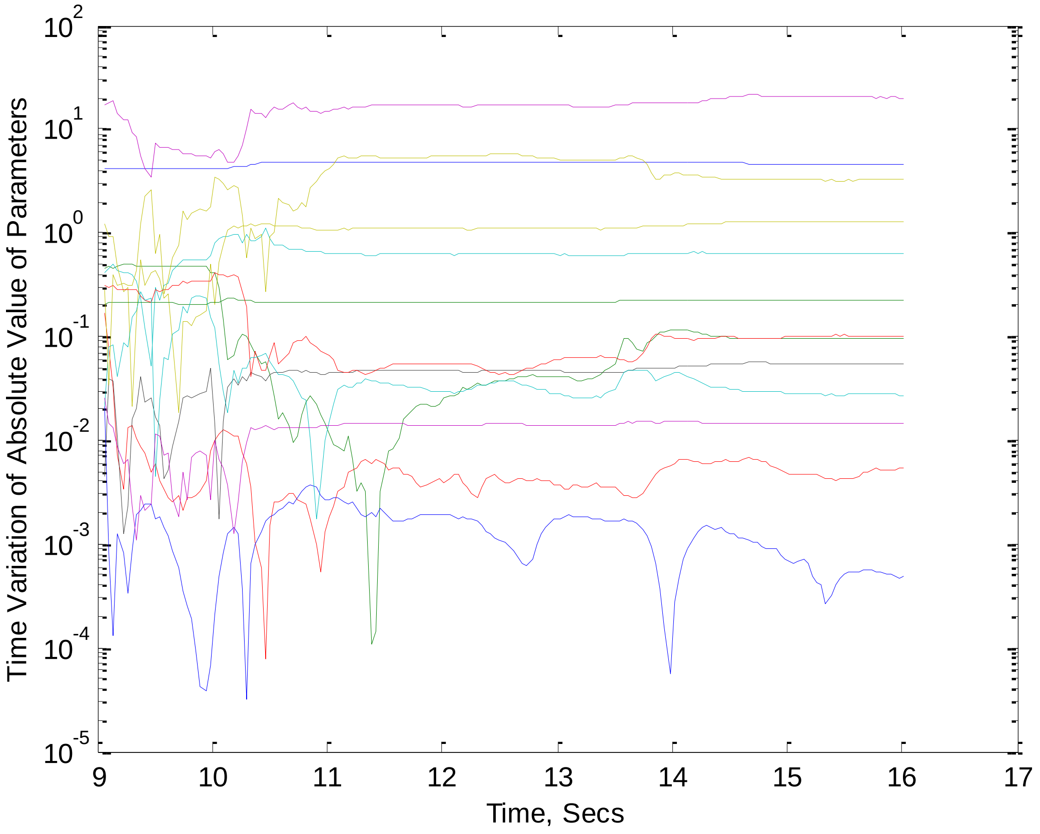
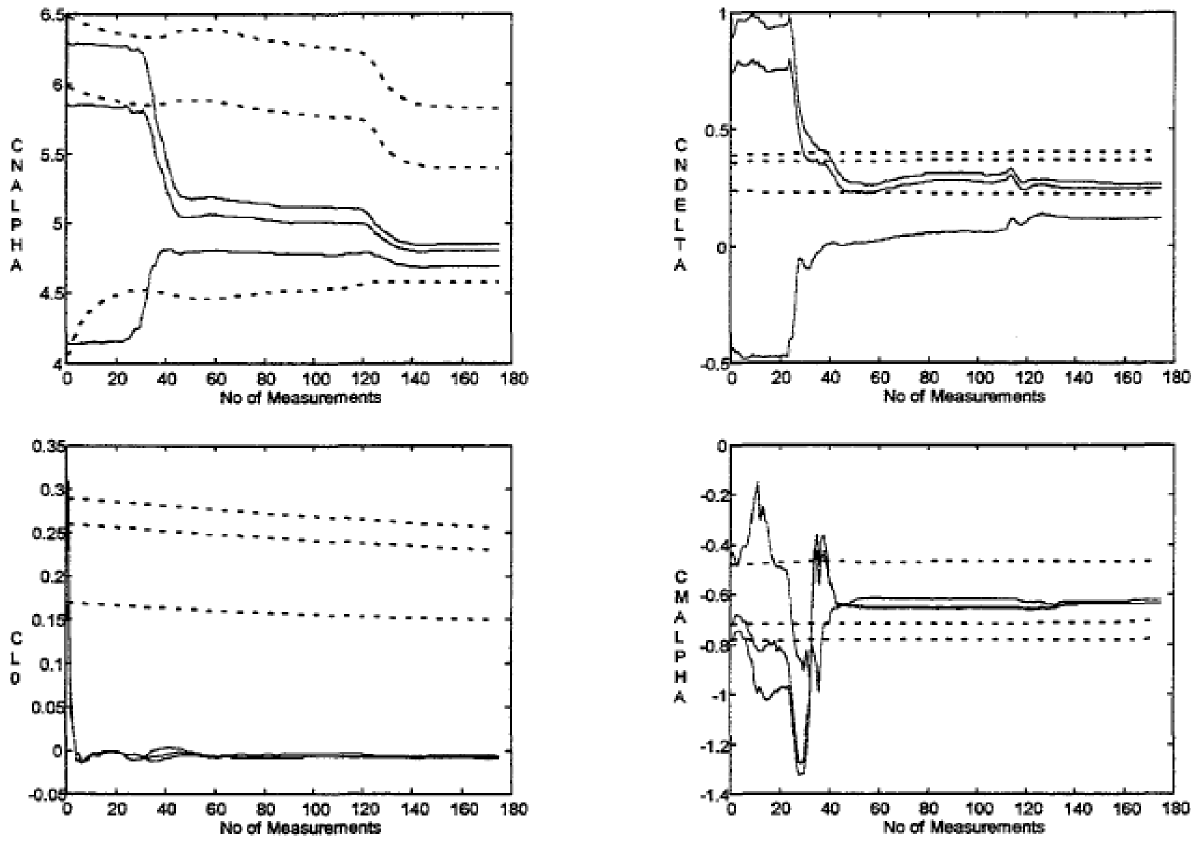


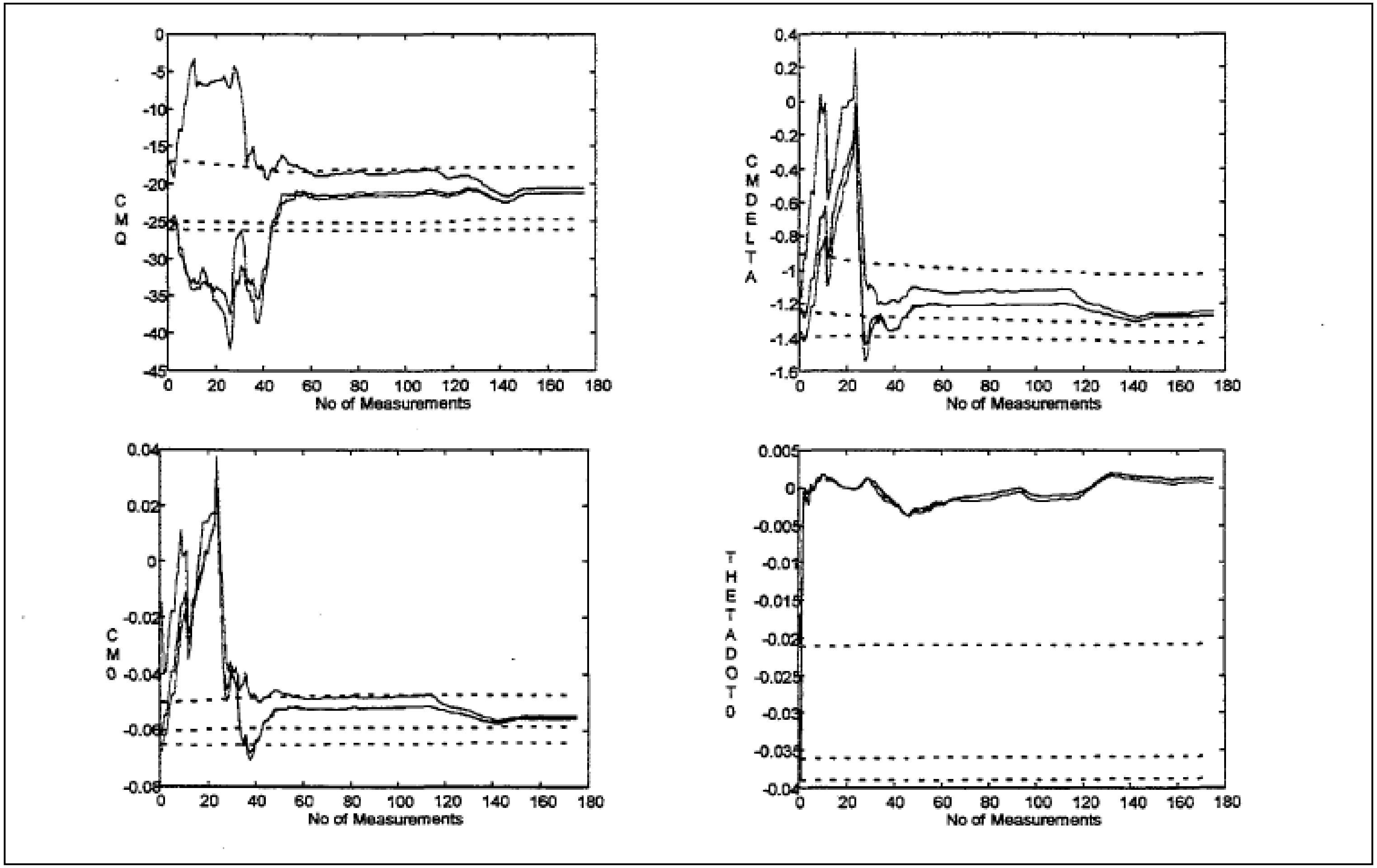
Figure. Variation of the 13 Parameters Through the Final Filter Pass.

AIRCRAFT LONGITUDINAL PARAMETER ESTIMATION (AIAA-98-4153)

FIG. 4a LONGITUDINAL PARAMETER HISTORY (----- P_0 ; _____ IIM)

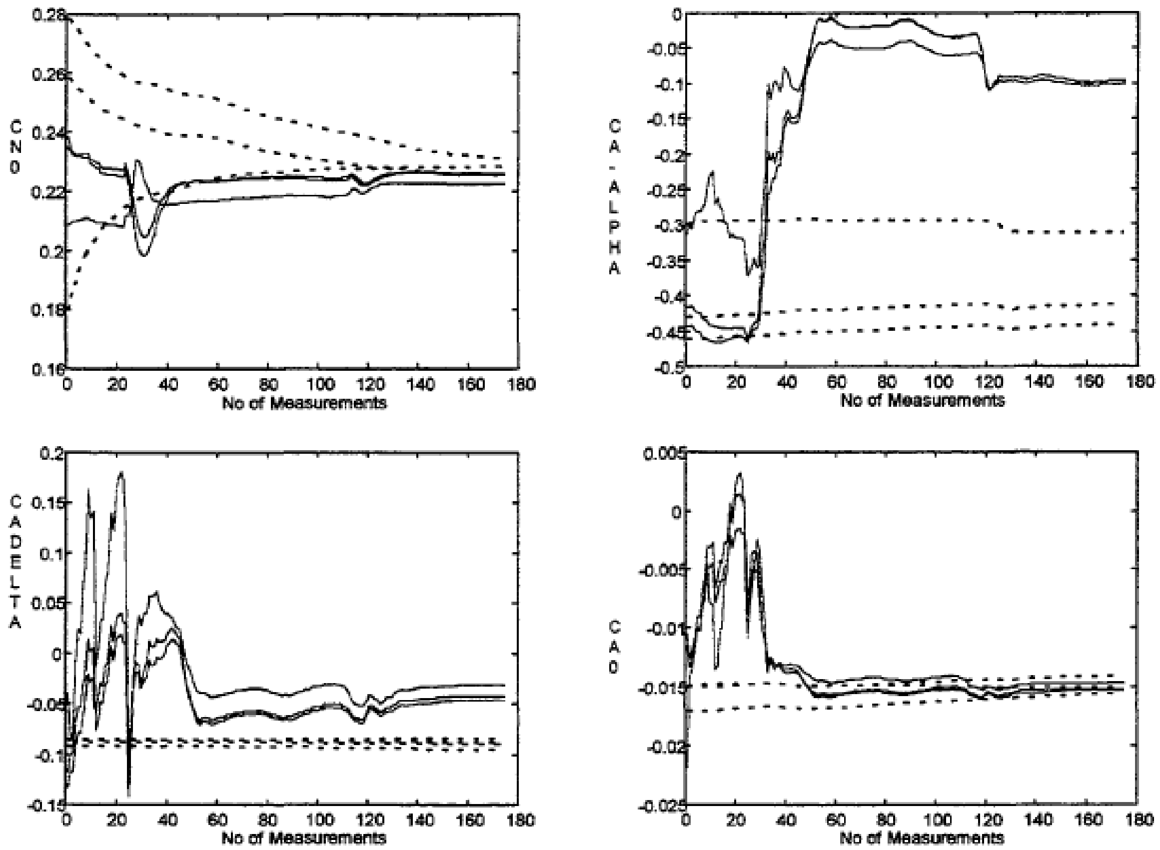


AIRCRAFT LONGITUDINAL PARAMETER ESTIMATION (AIAA-98-4153)



AIRCRAFT LONGITUDINAL PARAMETER ESTIMATION (AIAA-98-4153)

FIG. 4a LONGITUDINAL PARAMETER HISTORY (Contd.)



TUNING OF THE KALMAN FILTER BASED ON THE CONSTANT GAIN APPROACH

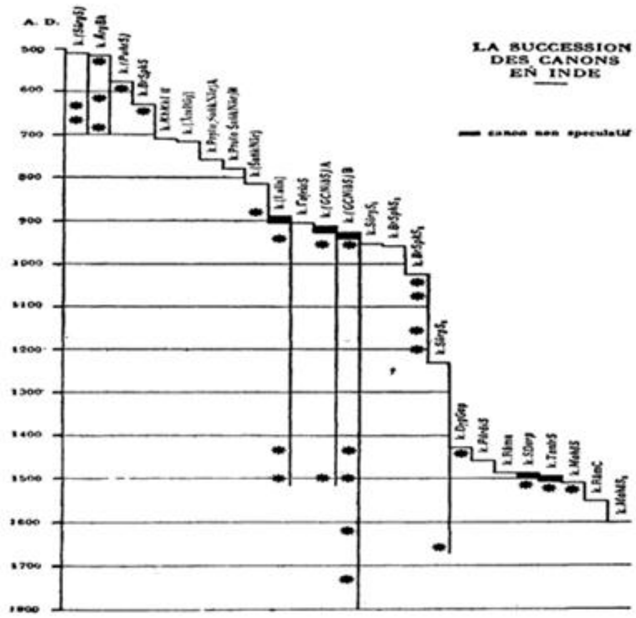
KALMAN GAIN APPROACH IN ANCIENT INDIAN ASTRONOMY

1. **How did the ancients achieve this? The sensitive Kalman filter statistics P_O, Q, and R can be traded to the more robust Kalman gain K, a fact utilized to handle many highly nonlinear problems of the present day. The K is a measure of the relative importance to be given to the predicted model values to the observations.**

2. **Similar to the filter statistics the gain K is also subjective but as shown in such problems it can be systematically chosen based on a cost function.**

3. **As much as the filter statistics are chosen even today manually in a subjective manner by trial and error the ancient Indian astronomers must have chosen the gain K in a very similar subjective manner to update the earlier *yuga* parameters with the newer observations.**

The Various Canons of Indian Astronomy (Billard 1977)



CANONS FROM AROUND 500 AD TO 1600 AD

CONCEPT OF CHANGE, CAPTURE AND CORRECT BY KERALA ALMANACISTS!

APPROACH OF ANCIENT INDIAN ASTRONOMERS

- **THE UPDATED PLANETARY PARAMETER =**

  THE EARLIER PLANETARY PARAMETER +

  (SOME QUANTITY) x

  (MEASURED POSITION OF THE PLANET –

  PREDICTED POSITION OF THE PLANET)

- **SOME QUANTITY = KALMAN GAIN!**
- **NOTE THAT THE MEASURED QUANTITY LONGITUDE IS DIFFERENT FROM THE UPDATED QUANTITY WHICH IS THE NUMBER OF REVOLUTIONS IN A YUGA.**
- **THIS IS AS IN MANY PRESENT DAY KALMAN FILTER APPLICATIONS!**

NILAKANTA'S ON USES OF ECLIPSES TO REFINE THE COMPUTATIONAL METHODS

Nilakantha had stated "the eclipses cited in Siddhantadipika can be computed and the details verified. Similarly, other eclipses traditionally known as well as those currently observable are to be studied. In the light of such experience future ones can be computed and predicted. Or, eclipses occurring at other places can be studied taking into account the latitude and longitude of the places and on this basis the method for the true Sun, Moon, ….can be perfected. Based on these, the past and future eclipses of one's own place can be studied and verified with appropriate refinement of the technique". DATA FUSION AND SMOOTHING!

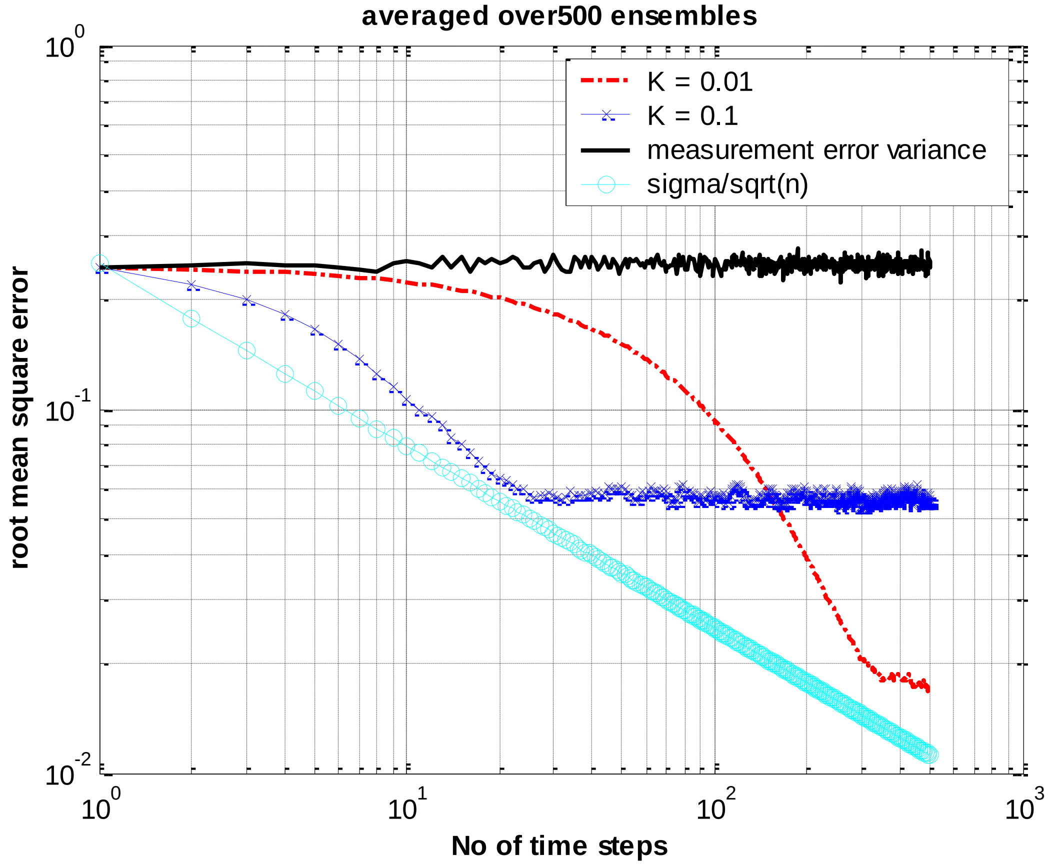


Figure. Variation of the Error for Different Constant Kalman Gains.

TRACKING A RANDOMLY MOVING OBJECT WHOSE DYNAMICS IS UNKNOWN!

- **REAL WORLD STATE EQUATION**

  A randomly moving target with time along the vertical axis.

- **FILTER STATE EQUATION**

  Xminus(t) = Xplus(t-1) ; no randomness! The initial Xminus(0) has to be chosen.

- **MEASUREMENT EQUATION**

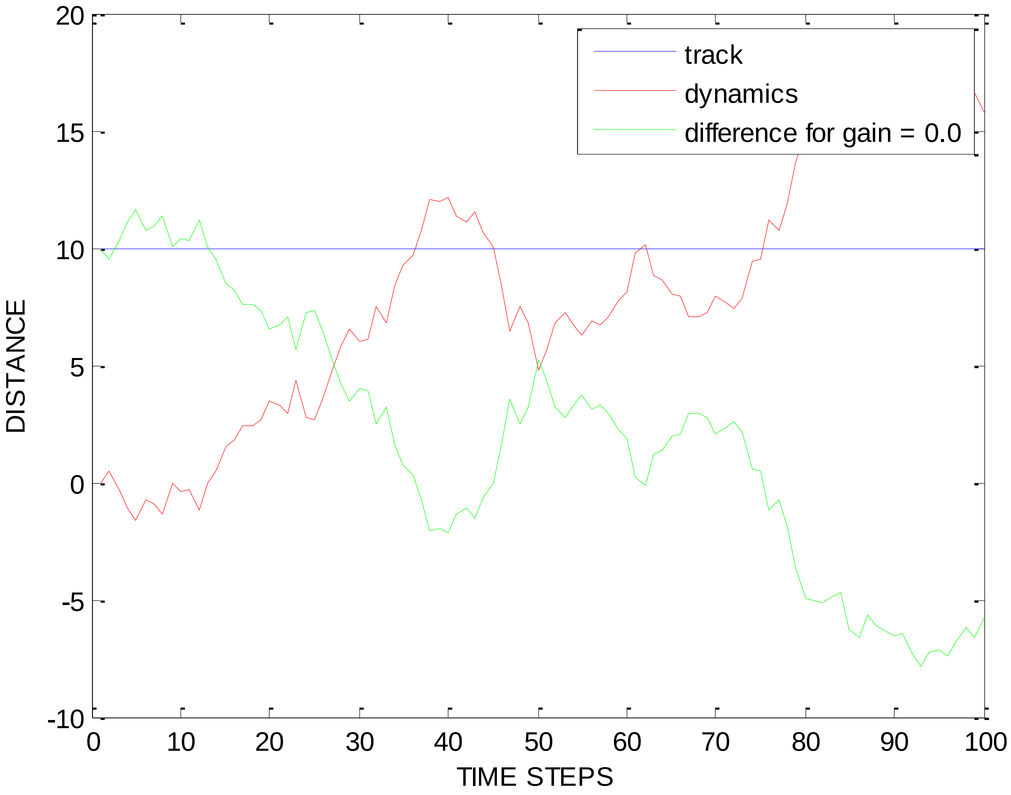
  Z(t) = Xminus(t) + randn(1);

- **UPDATE EQUATION**

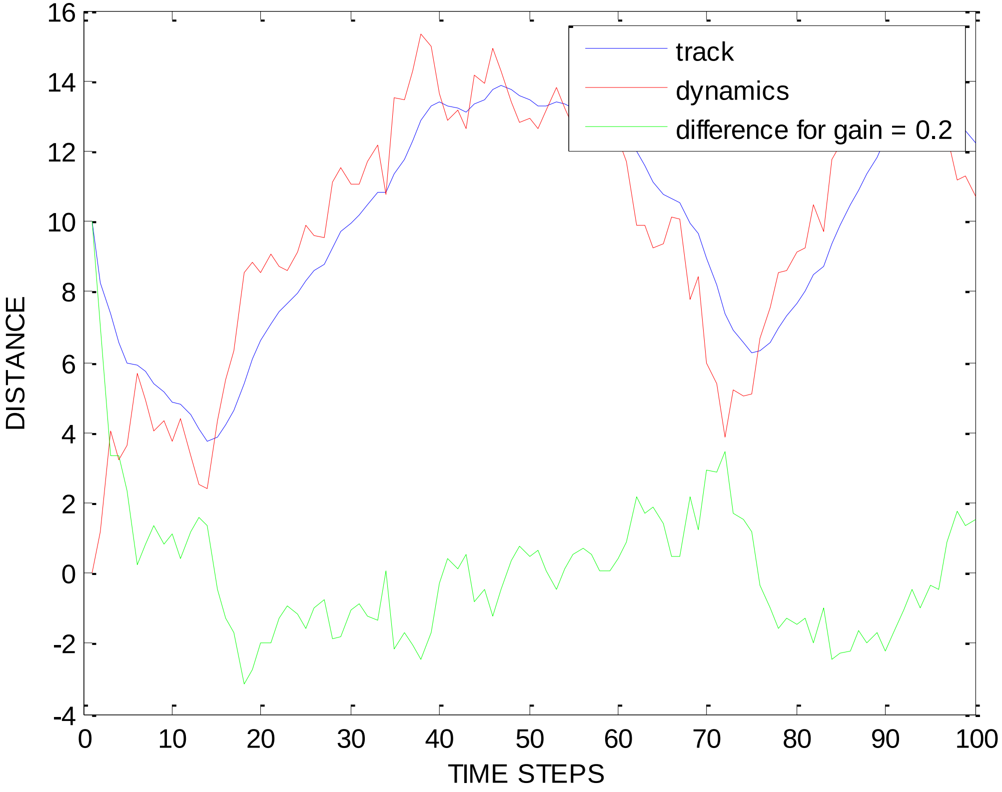
  Xplus(t) = Xminus(t) + K*[Z(t) - Xminus(t)]

  where K is the constant Kalman gain. The experiments are performed for varying constant gains.

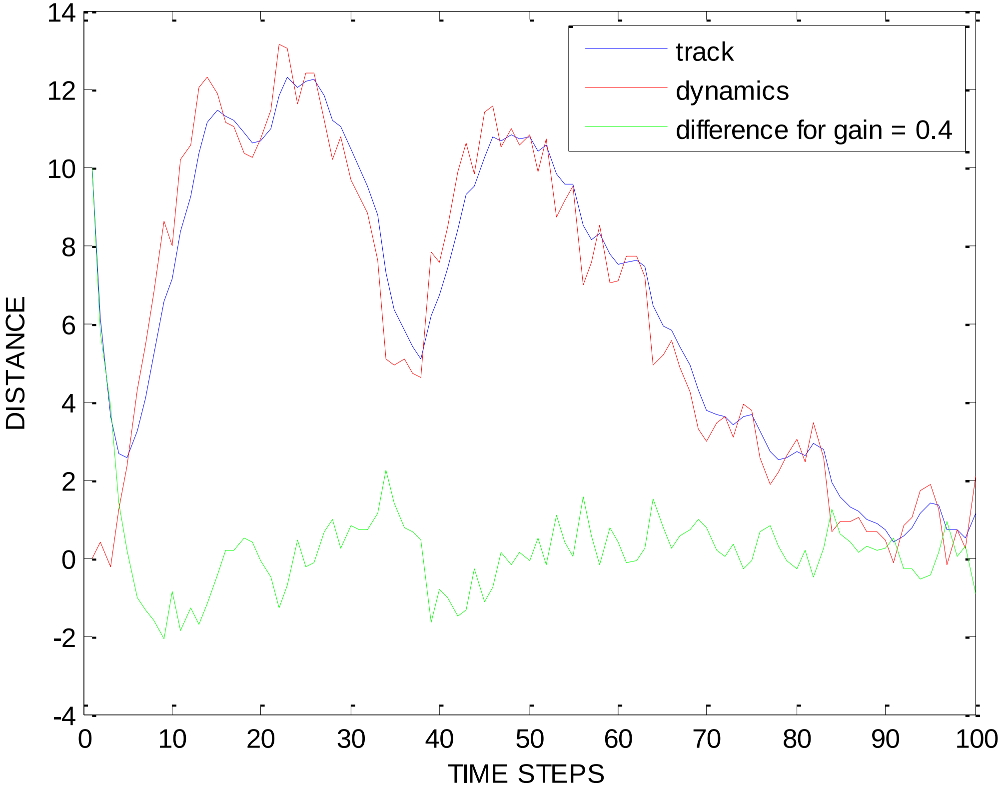
TRACKING A RANDOMLY MOVING OBJECT



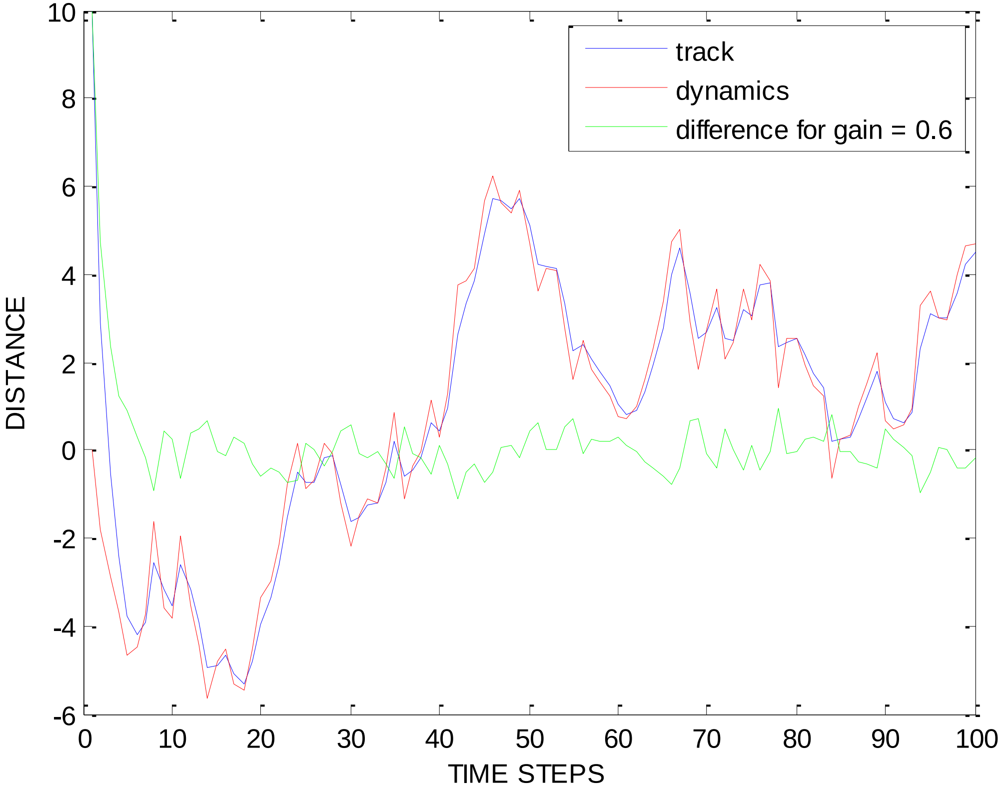
TRACKING A RANDOMLY MOVING OBJECT



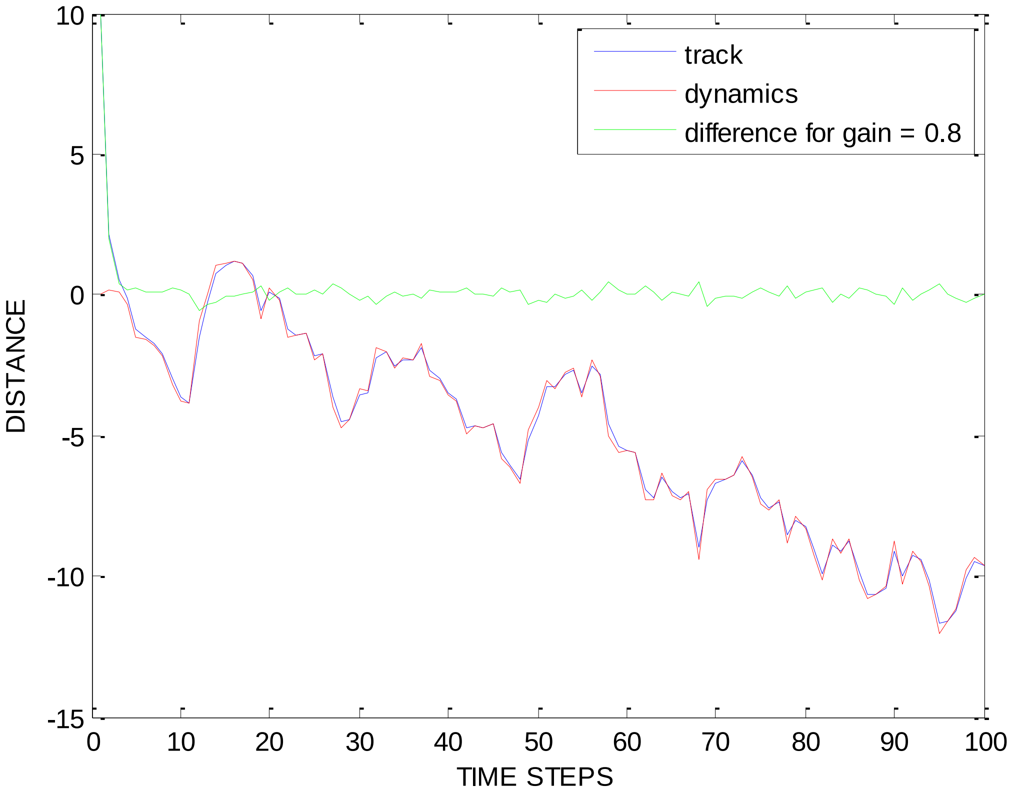
TRACKING A RANDOMLY MOVING OBJECT



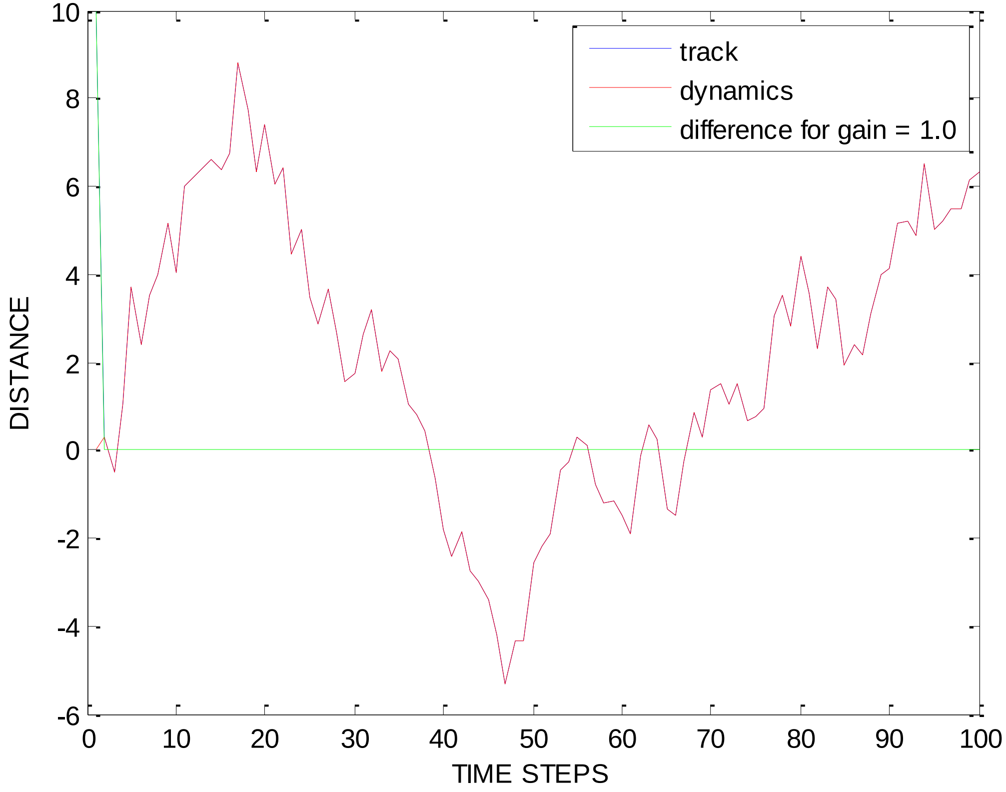
TRACKING A RANDOMLY MOVING OBJECT



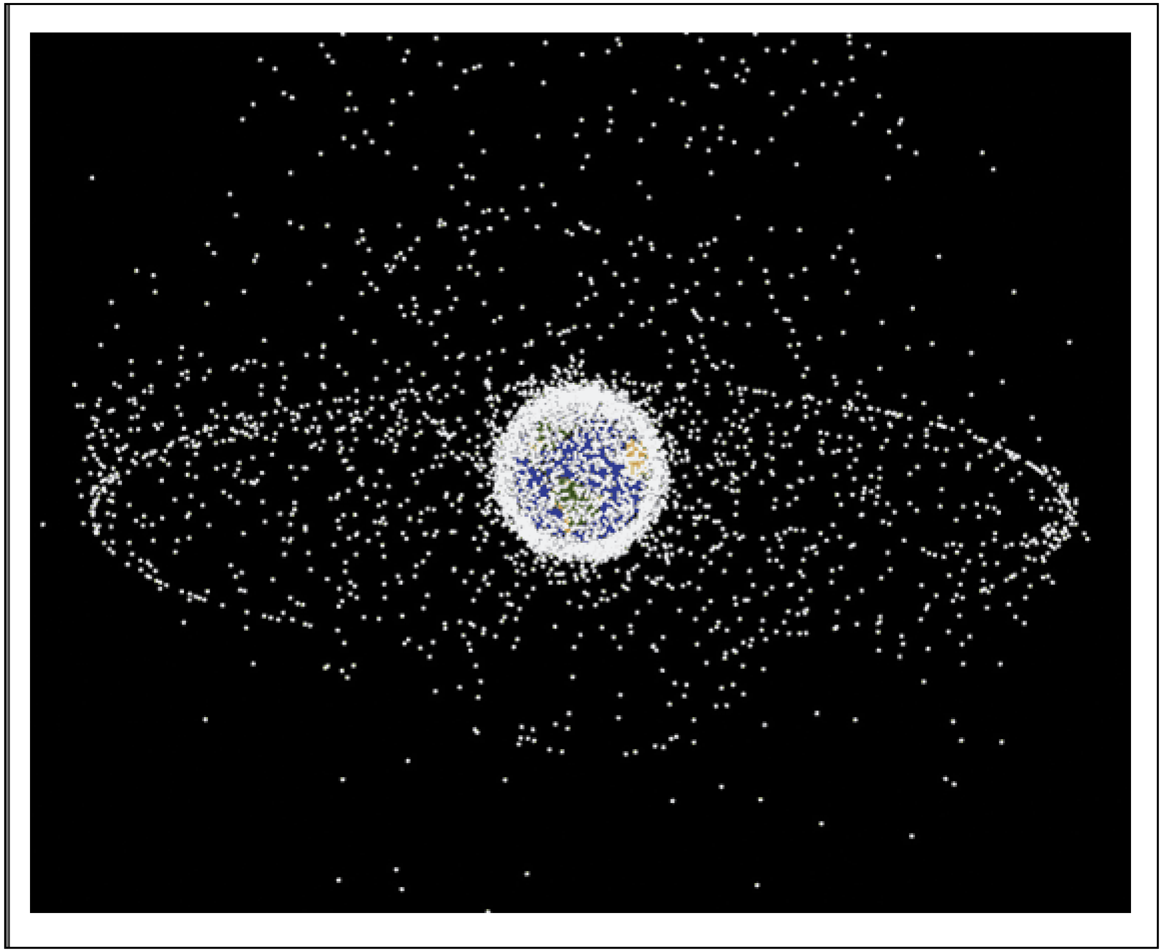
TRACKING A RANDOMLY MOVING OBJECT



TRACKING A RANDOMLY MOVING OBJECT



SPACE DEBRIS



SPACE DEBRIS AND GAS MOLECULES

Table 1 Differences between orbital space debris and gas molecules[a]

No.	Feature	Gas molecules	Orbital space debris
1	History	Studied for centuries	Studied for about a few decades
2	Flow regimes	Free molecular to Continuum	Free molecular
3	Constituents	Equal size, shape and mass	Unequal size, shape and mass
4	Evolution	Binary collision among themselves and any body in space	Mainly convection and at times collision
5	Environment	Similar molecules and body	Randomly varying atmosphere, Earth's gravitational harmonics, Solar Radiation Pressure, Luni Solar Perturbations.
6	Distribution function	$f(r, v, t)$	$f(r, v, m, S, t)$
7	Source terms	Generally not present	Source due to explosions and collisions and sink terms due to decay are present
8	Periodic measurements	Not essential (theory is adequate)	Essential
9	End objective	Forces and heat transfer experienced by the body.	Risk analysis, orbit selection among many other uses
10	Present status of subject	Well developed theory with experimental supplement	Not fully developed. Difficult for theory and experiment

[a] r = location in space; v = velocity; m = mass; S = cross sectional area; t = time.

REAL WORLD SCENARIO

Table 2 Real-world scenario without binning

Quantity	Description
The state variables	(a, e, B) of each of the individual fragments.
The state input	Suitable model for the complex environment.
The state noise	Random variations in real situation from the above model environment.
The initial state	Fragment characteristics based on ASSEMBLE or any other suitable model.
The measured variables	The number of fragments in the various bins.
The measurement noise	Measurement errors in the orbital characteristics assigned to the individual debris fragment due to errors in tracking and data processing.

FILTER WORLD SCENARIO

Table 3 Filter world scenario with binning (in the present simulation studies)

Quantity	Description
The state variables	(a, e, B) of the EQF in each of the bins propagated and redistributed and also accounting for fragments from additional breakups.
The state input	Suitable model for the complex environment. Presently only the air drag effect is considered.
The state noise	The unmodelable inaccuracies in assigning the values for (a, e, B) for the EQF and its further propagation, redistribution, and also the environment.
The initial state	Fragment characteristics based on ASSEMBLE or any other suitable model to obtain the EQF.
The measured variables	The (a, e, B) for each of the individual fragments as in the real world but propagated with atmospheric drag alone and later converted to the number of objects in each of the bins.
The measurement noise	In the present simulation there is no measurement noise because all the fragments are propagated and based on the changed values of the orbital parameters (a, e) they are assigned to the appropriate bins.

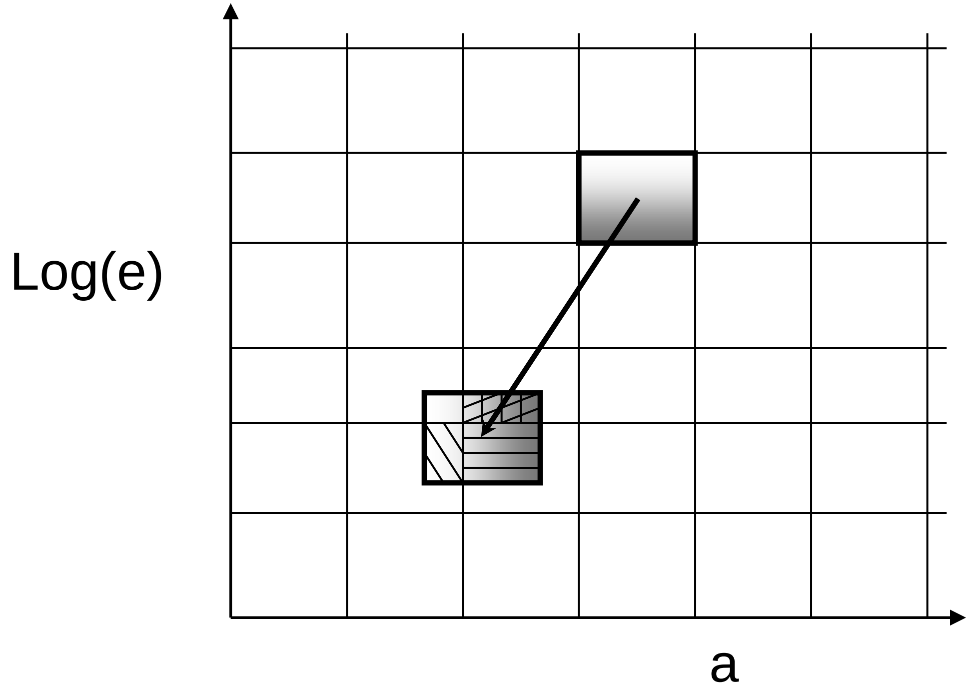


Figure. 1. Graphical Representation of the Orbital Propagation Algorithm in the Eccentricity ‘e’ vs. Semi major axis ‘a’ Space for the Debris in the above Bins.

UPDATE OF EQF CHARACTERISTICS

It may be noted that the present approach considers

1) a,

2) log(e), and

3) log(B)

The third parameter B has been presently used because the orbital parameters are sensitive to the air drag and thus change with time. A question that would occur is that in general there could be errors due to discretization and approximation in specifying the mean values for a and e in the various bins and further in the propagation due to unaccounted or even unmodelable forces. Such errors can be accounted for by process noise in the state equations describing the propagation of the EQF. An important feature of the present approach is that because the individual representative object of each bin is propagated the computing time is almost independent of the debris population size.

Filter World State Equations

Here the various EQF, which are specified by the parameters:

1) The number of fragments in the particular (a, e) bin,

2) The equivalent semi major axis,

3) The equivalent eccentricity, and

4) The equivalent ballistic coefficient.

The first parameter is clear and the second is arithmetic, with the third and fourth being the geometric mean of the fragments in the bins. The (a, e) bins are not changing and the EQF moves in the (a, e) plane like any other single fragment and later gets redistributed based on a certain rule. Thus the states presently considered in each and every one of the (a, e) bins are the number of objects N and their equivalent ballistic coefficient B. The measurements are just the number of fragments in the above (a, e) bins. The state equations for the EQF in the various bins between measurements are

dN/dt = Σ [propagation across the (a, e) bins + redistribution + source terms] + state noise

dB/dt = 0 + state noise(=0)

Filter World Measurement Update Equations

The measured quantities are the true number density N_M in the various bins. These are obtained in simulation by propagating each and every one of the individual debris fragments. It is this that is used to update the predicted number density from the EQF propagation and redistribution to an improved value and also update the ballistic coefficient of the EQF in the various bins based on equations such as

$$N^+ = N^- + K_N (N_M - N^-) \quad \text{and} \quad B^+ = B^- + K_B (N_M - N^-)$$

with the superscript (+) and (-) denote respectively the post and pre updated values. The constant Kalman gains K_N and K_B refer respectively, to the number density and the equivalent ballistic coefficient. In the constant Kalman gain approach the number of such gains to be estimated is 200 namely the two K_N and K_B for each of the (a, e) bins.

Filter World Measurement Update Equations

These can be obtained by minimizing the cost function

$$J = \Sigma\, \nu_K^{T}\, [\mathscr{R}]\, \nu_K$$

where the innovation $\nu_K = (N_M - N^-)$ measurement value model prediction, and < is the covariance matrix of the innovation. The Σ denotes the summation over all bins and times. If one uses P, Q, and $\mathscr{R}$ there could be some variation in the gain values during the transient and the steady state conditions. Assuming the $\mathscr{R}$ to be constant it can be estimated similar to the estimation of R in the method of maximum likelihood estimation with measurement noise alone or better known as output error method (OEM). Thus, a set of constant Kalman gains was derived based on the above cost function J by utilizing a genetic algorithm (GA).

Filter World Measurement Update Equations

In the GA each member of the population will thus have 200 gain values. The gains obtained by following the 10,000 objects was also used later without any change to follow the evolution with more break up and source terms. The simulation takes most of the time to follow the orbital characteristics of each of the 10,000 fragments. Later the 100 EQFs were propagated in each slice of B to form the cost function. Obviously this has to be repeated over the population size and generations. Each test case took 100hrs in a MATLAB environment (though not CPU) using a 1.7 MHz PC. The constant gains were obtained by following one single breakup alone and subsequently using the same gains to demonstrate the adequacy to follow the evolution with more breakups, and source terms. Such an approach shows the way in which a certain model of the space debris scenario can be followed for longer times.

EVOLUTION OF THE INDIVIDUAL FRAGMENTS AND THE EQF IN THE BINS

The state variables are initialized to the scenario at initial time. At present a large number of about 10,000 simulated debris fragments due to an explosion are considered and later the additional fragments due to further breakups are accounted for as source terms. The state propagation equations for both the fragments and the EQF are similar. The rate of change over one revolution in the elements "a" and "e" due to drag alone for the kth revolution is given by

$$a_k = a_{k-1} + \Delta a_{k-1}$$
$$e_k = e_{k-1} + \Delta e_{k-1}$$

where a_k and e_k stand for semi major axis and eccentricity, respectively, at time instant k with the change per revolution denoted by being functions of the density at the perigee height, density scale height, rotation of the atmosphere, mass, drag coefficient, and effectivereference area following King Hele. The EQF is propagated based on its assigned value of suitable a and e and could in general end up in just within another bin.

EVOLUTION OF THE INDIVIDUAL FRAGMENTS AND THE EQF IN THE BINS

However, the various fragments of which the above is made could end up in the surrounding bins as well. To mimic such a feature a proposed heuristic rule is to redistribute the fraction of the EQF among the bins. Such a rule takes the ratio between the area covered by the propagated rectangle and the area of the rectangle itself as shown in Figure. Later due to the launches and retrievals the changed debris number density, their orbital parameters, and the ballistic coefficient are also accounted.

Real World and Filter World Scenario

It is very interesting to see that in the state equations of the filter world scenario the coarser conditions brought about by the binning, forming the EQF, propagation, and redistribution lead to modeling error and thus need process noise. In the present simulation studies the propagation of each and every one of the individual debris fragments and counting their number in the various bins leads to no measurement noise. In a real world scenario, there would be measurement noise due to inaccuracies in the assigned orbital characteristics of the individual fragments. Under these conditions it is useful to invert the role of the above state and measurement equations. Thus, we are able to reach a situation as in the output error method (OEM) where the state equations have no noise but the measurement equations have noise. In the inverted roles of the state and measurement equations the measurement noise arises from binning, forming EQF, propagation, and redistribution as mentioned earlier.

Case Studies and the Results

To study the effect and the soundness of the above approach, a single explosion is simulated using an earlier model called ASSEMBLE. Considering an explosion, at a typical altitude of 800 km and eccentricity 0.00045, about 10,000 objects of varying ballistic coefficients were simulated. These objects were propagated by taking into account only the atmospheric drag effect for a period of 600 days and thus the basic measurement data of the number of objects in the three-dimensional (a, e, B) bins is generated. Three different approaches are possible to follow the number density of the fragments of a single break up with time. These are

- **Case A: updating number of objects alone,**
- **Case B: updating only the EQB, and**
- **Case C: updating both the number density and the EQB.**

Case B: Estimating the Number of Objects by Updating the EQB Alone

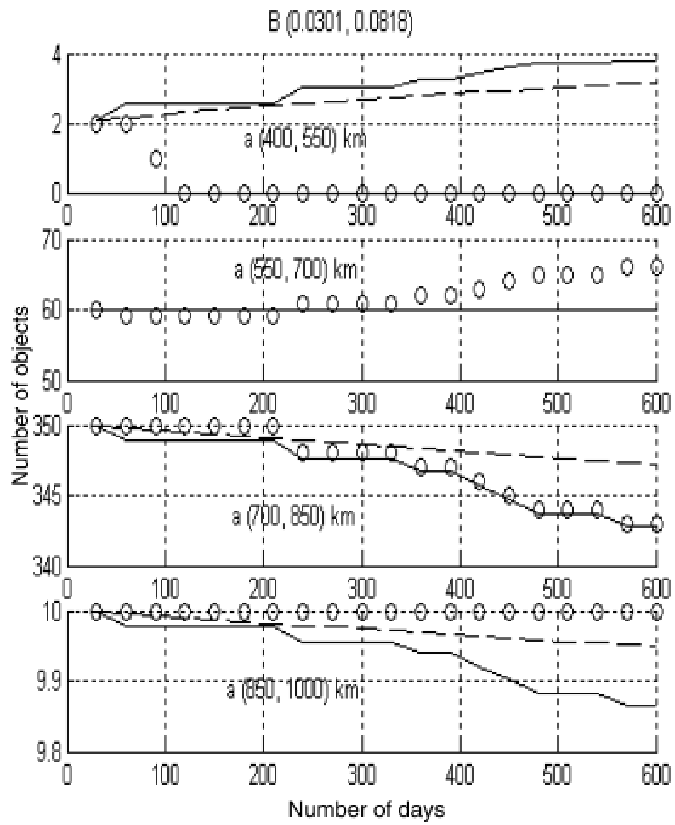


Fig. 2 Updating the ballistic coefficients only using constant gains for *B* bin (0.0301, 0.0818) (○) true; (solid line) filter; (dashed line) no update.

Case B: Estimating the Number of Objects by Updating the EQB Alone

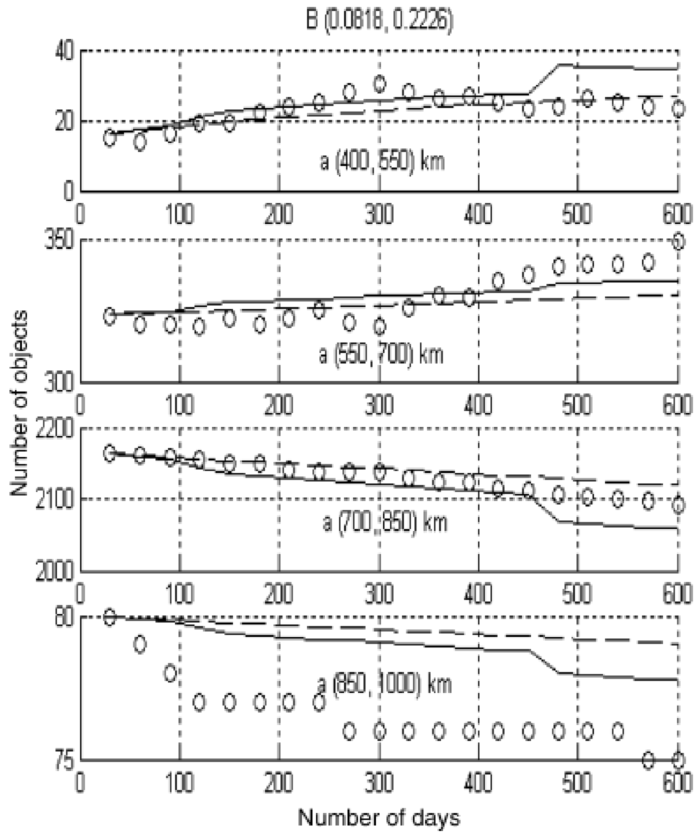


Fig. 3 Updating the ballistic coefficients only using constant gains for *B* bin (0.0818, 0.2226) (○) true; (solid line) filter; (dashed line) no update.

Case B: Estimating the Number of Objects by Updating the EQB Alone

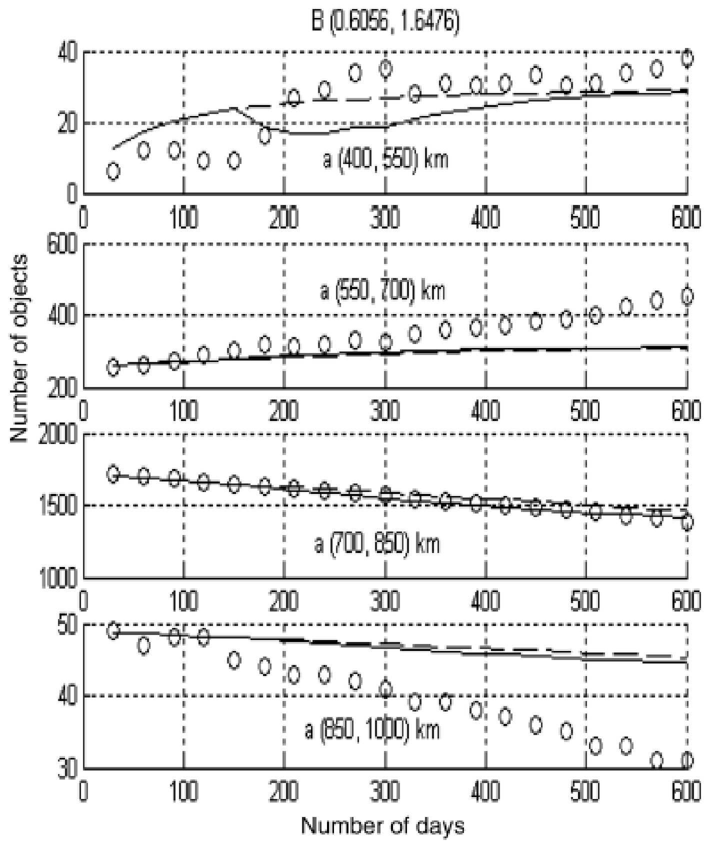


Fig. 4 Updating the ballistic coefficients only using constant gains for *B* bin (0.6056. 1.6476) (○) true; (solid line) filter; (dashed line) no update.

Case C: Estimating the Number of Objects by Updating both the Number Density and the EQB

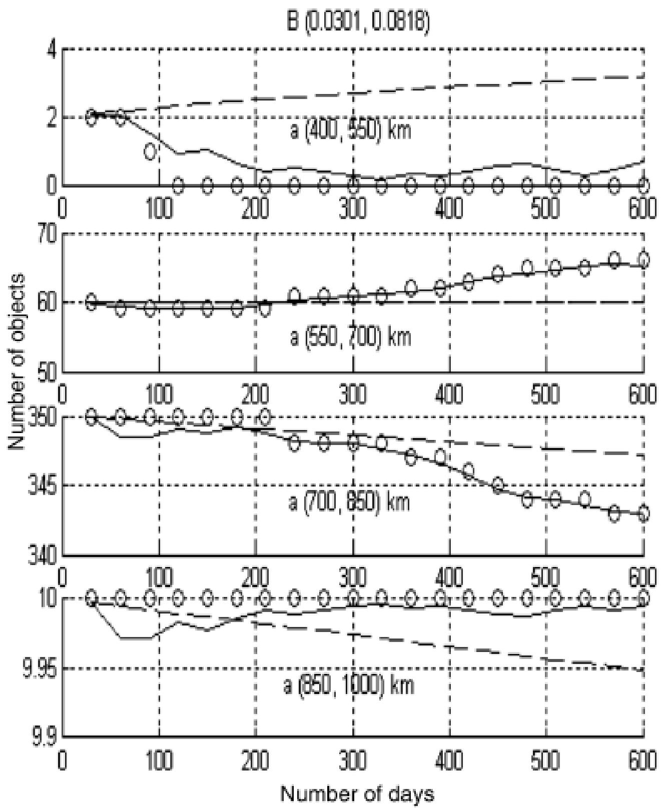


Fig. 5 Updating the ballistic coefficients and number of objects using constant gains for *B* bin (0.0301, 0.0818) (○) true; (solid line) filter; (dashed line) no update.

Case C: Estimating the Number of Objects by Updating both the Number Density and the EQB

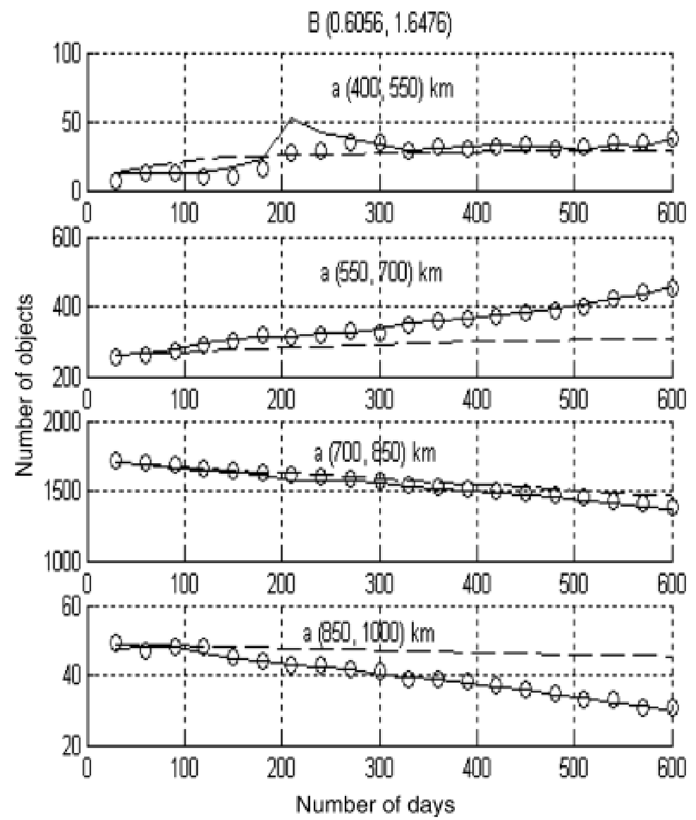


Fig. 6 Updating the ballistic coefficients and number of objects using constant gains for *B* bin (0.6056, 1.6476) (○) true; (solid line) filter; (dashed line) no update.

Variation of the Constant Kalman Gain for the Number of Fragments in Case C

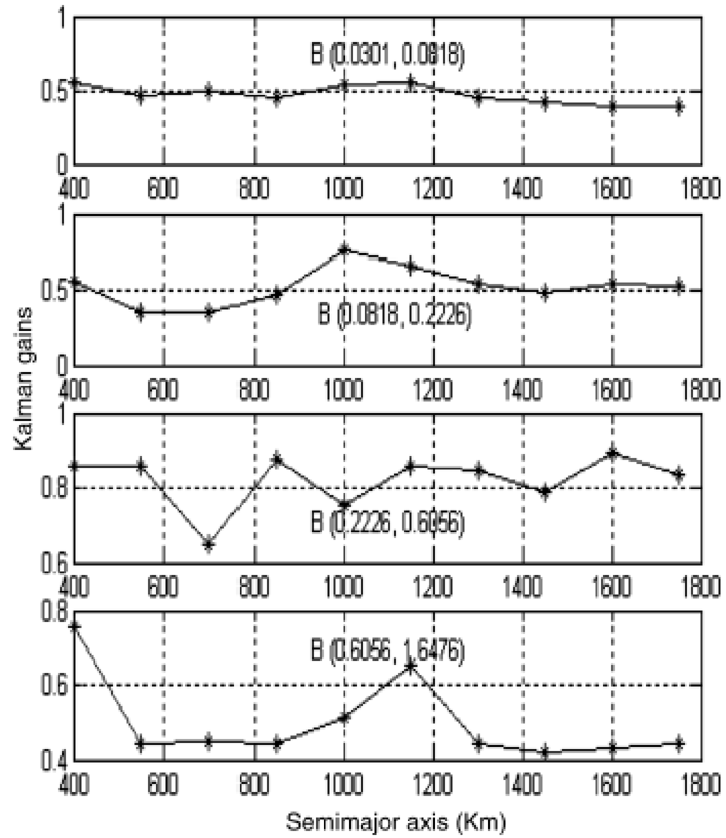


Fig. 7 Kalman gains for number of objects (K_N).

Variation of the Constant Kalman Gain for the Number of Fragments in Case C

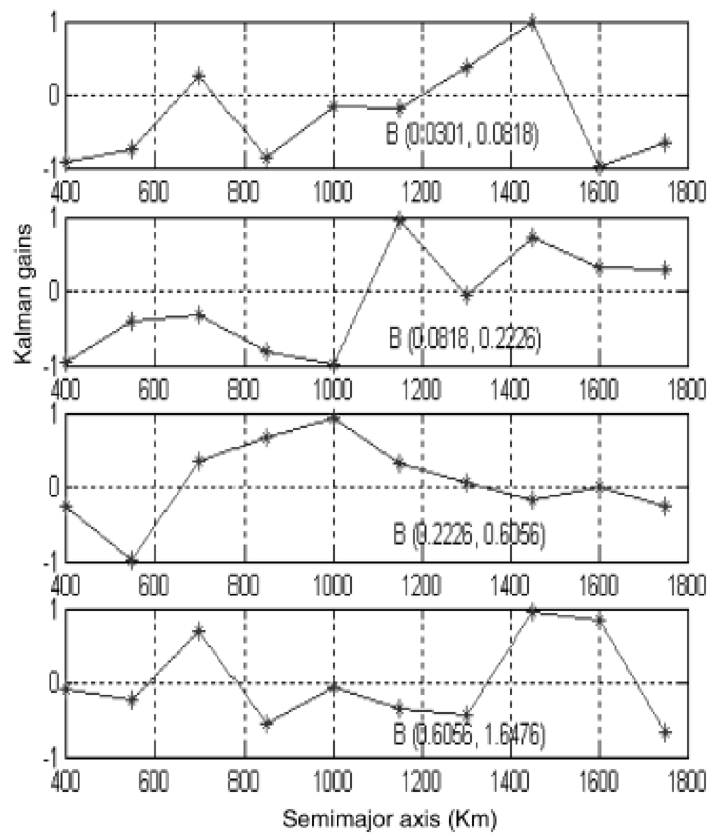


Fig. 8 Kalman gains for ballistic coefficients (K_B).

Variation of the Constant Kalman Gain for the Number of Fragments in Case C

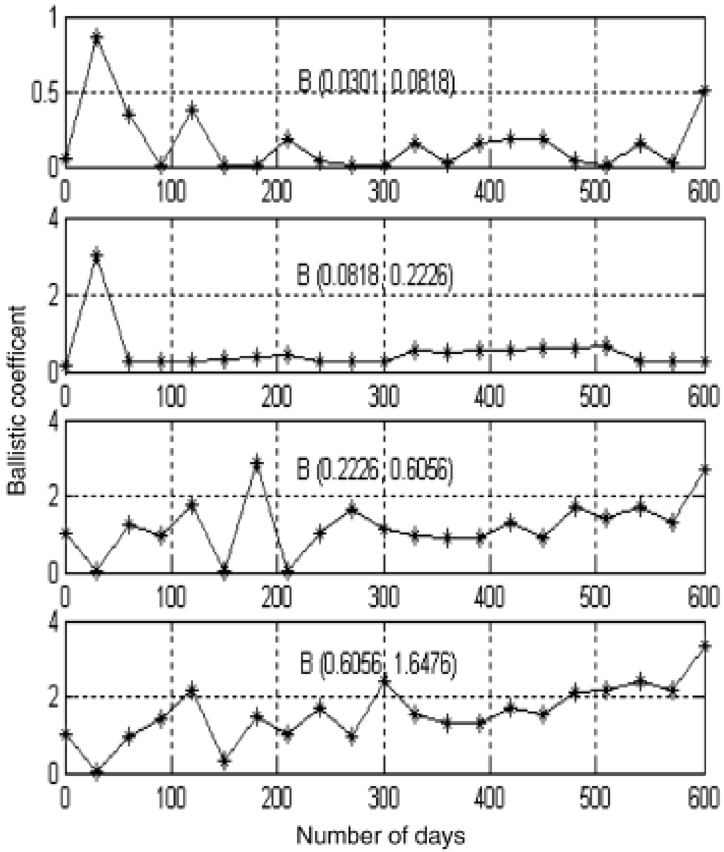


Fig. 9 The estimated ballistic coefficients.

Case C: Evolution of a Single Breakup Followed by Another Breakup

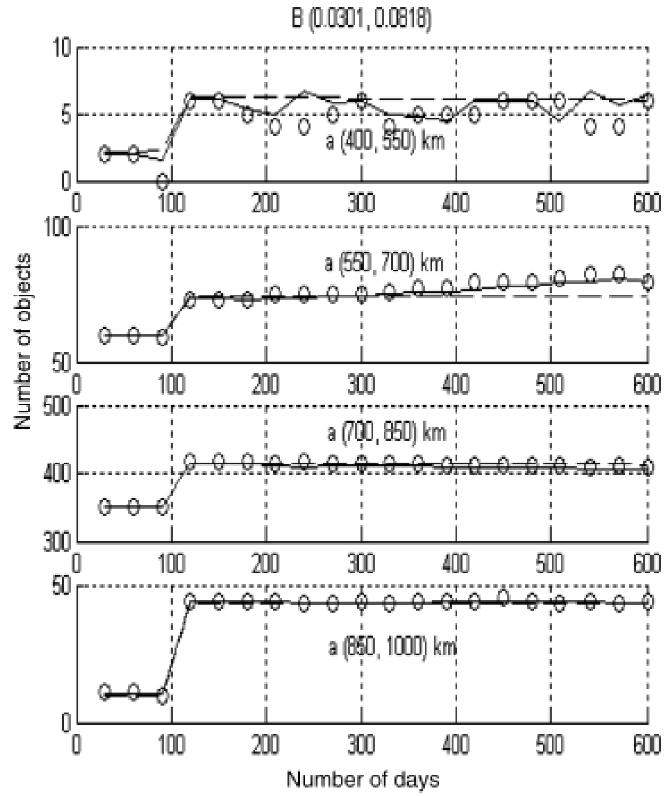


Fig. 10 One source of explosion included. Updating the ballistic coefficients and number of objects using constant gains for *B* bin (0.0301, 0.0818) (○) true; (solid line) filter; (dashed line) no update.

Case C: Evolution of a Single Breakup Followed by Another Breakup

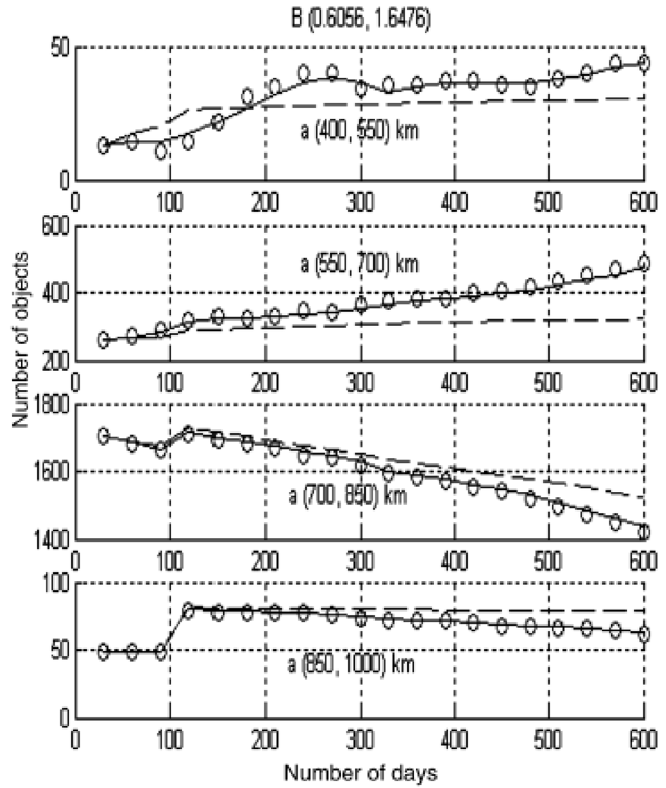


Fig. 11 One source of explosion included. Updating the ballistic coefficients and number of objects using constant gains for *B* bin (0.6056, 1.6476) (○) true; (solid line) filter; (dashed line) no update.

Case C: Evolution of a Single Breakup Followed by More than One Breakup

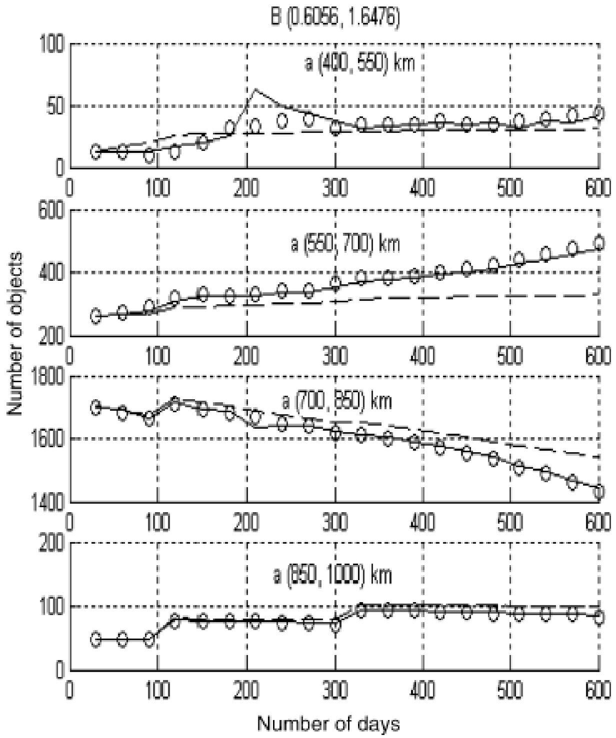


Fig. 12 Comparison between estimated and observed number of objects: two sources of explosions included. Updating the ballistic coefficients and number of objects using constant gains for *B* bin (0.6056, 1.6476) (○) true; (solid line) filter; (dashed line) no update.

Case C: Evolution of a Single Breakup Followed by More than One Breakup and Some Launch Activities

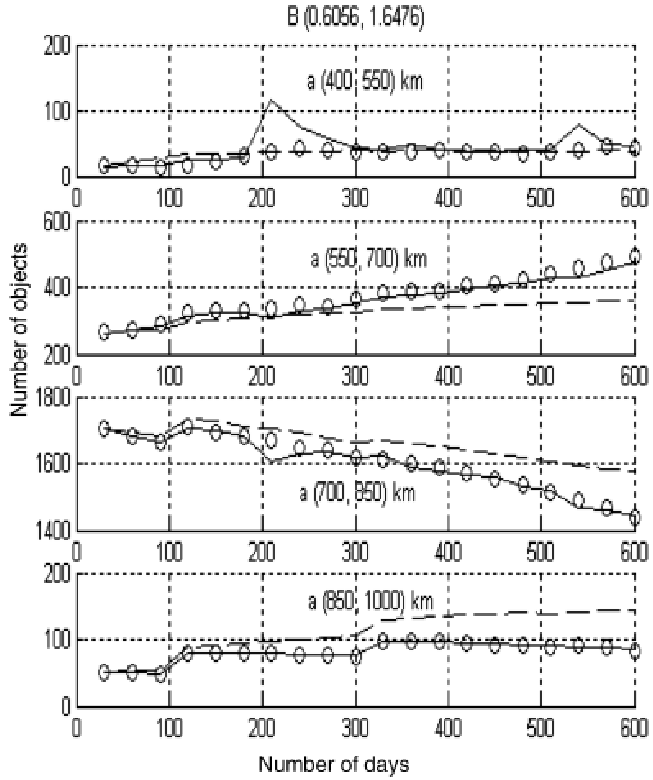


Fig. 13 Comparison between estimated and observed number of objects: two sources of explosions and some new launches included. Updating the ballistic coefficients and number of objects using constant gains for *B* bin (0.6056, 1.6476) (○) true; (solid line) filter; (dashed line) no update.

Analysis of a Typical Real World Scenario

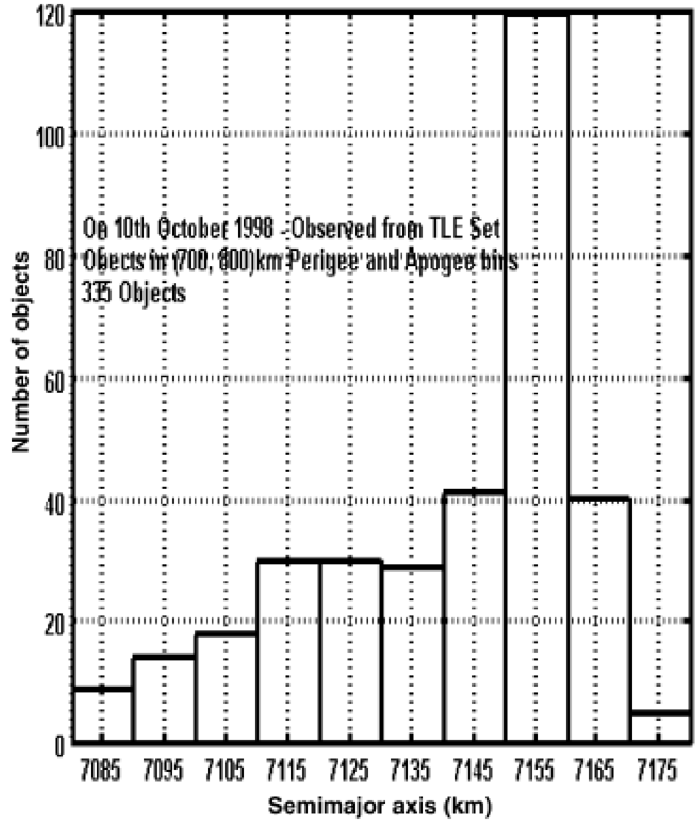


Fig. 14 Initial distribution of the debris number density with semimajor axis.

Analysis of a Typical Real World Scenario

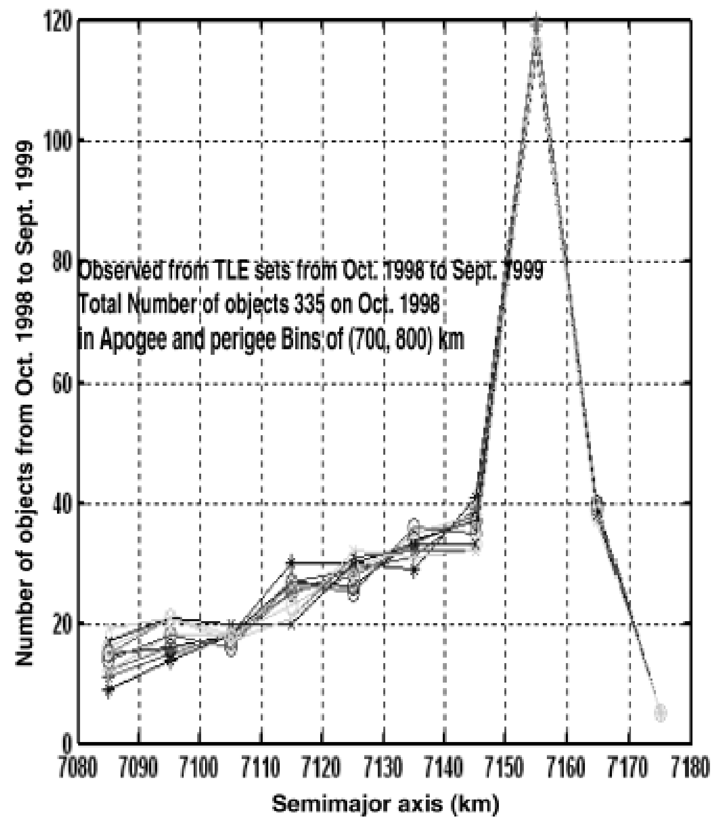


Fig. 15 Variation of the number density of the debris in the bins with time.

Analysis of a Typical Real World Scenario

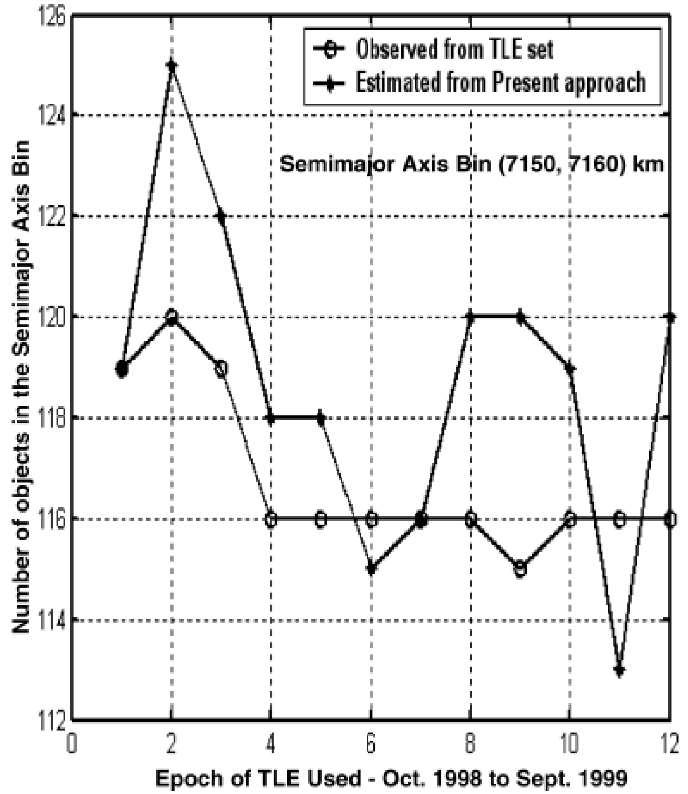


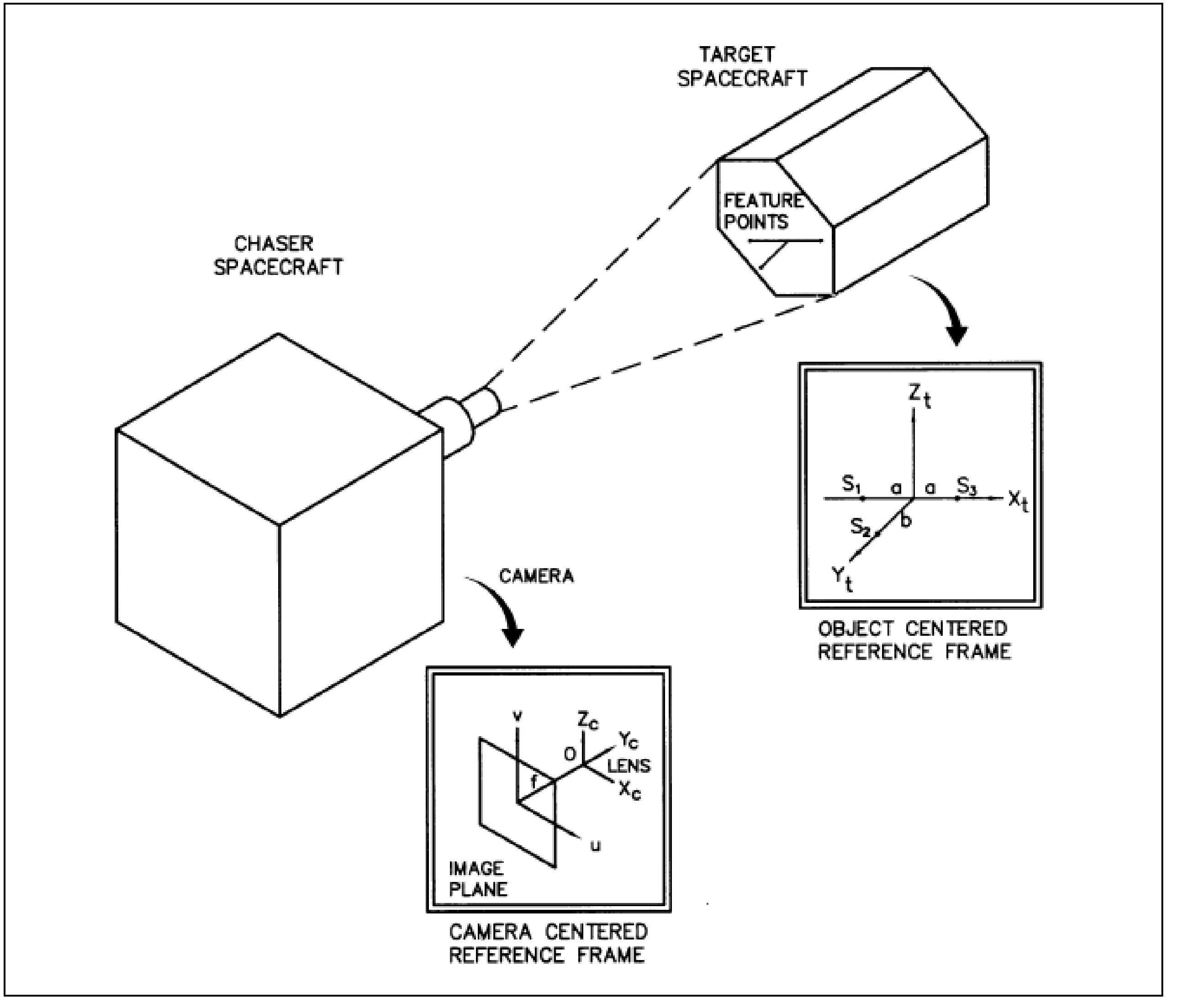
Fig. 16 Comparison of the estimated time varying debris number density with TLE values in the bin (7150,7160) km during a year.

Further Examples of Applying Constant Gain Kalman Filter

- **Rendezvous and Docking (Philip)**
- **Reentry of a Space Debris (Anilkumar and SubbaRao)**
- **Evaluation of the INS errors in the GPS/INS Coupling (Helen Basil and Puri)**
- **Determination of the Atmospheric Total Electron Content (Anandraj, Soma and Mahapatra)**

  The subsequent figures illustrate the results in a brief way.

RENDEZVOUS AND DOCKING



EXAMPLE OF A REENTRY STUDY

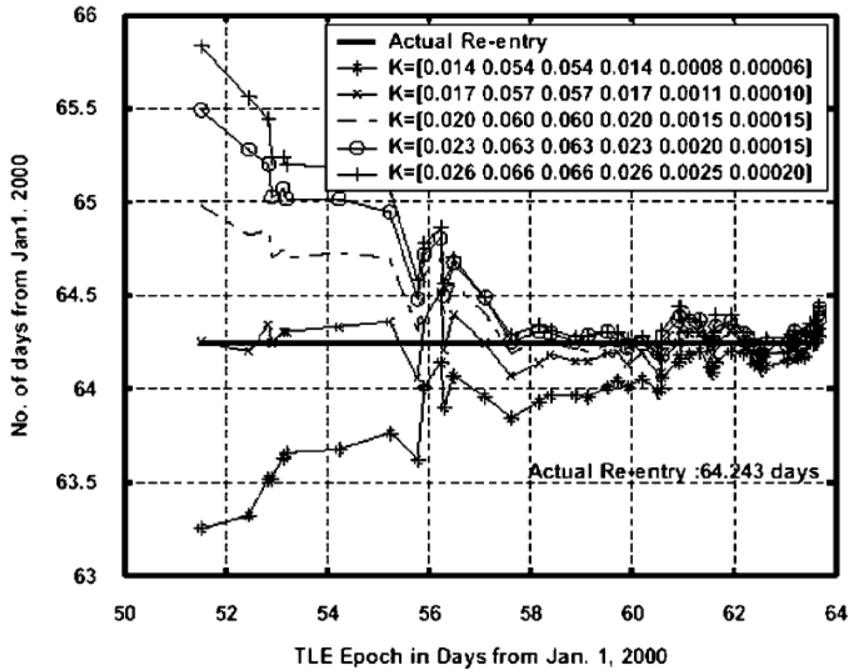


Fig. 3. Re-entry prediction with different sets of Kalman gains (Sat. No. 25947).

EXAMPLE OF ESTIMATING INS ERRORS

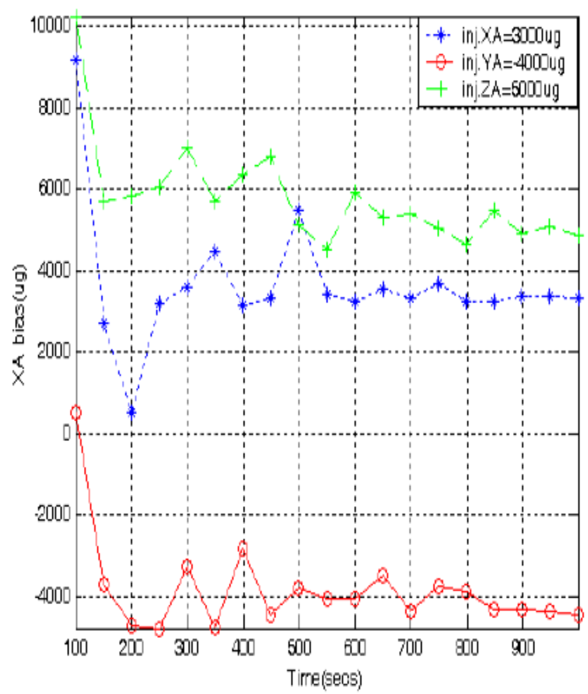


Fig. 9 Estimated Accelerometer Bias using Time Varying Kalman Gains.

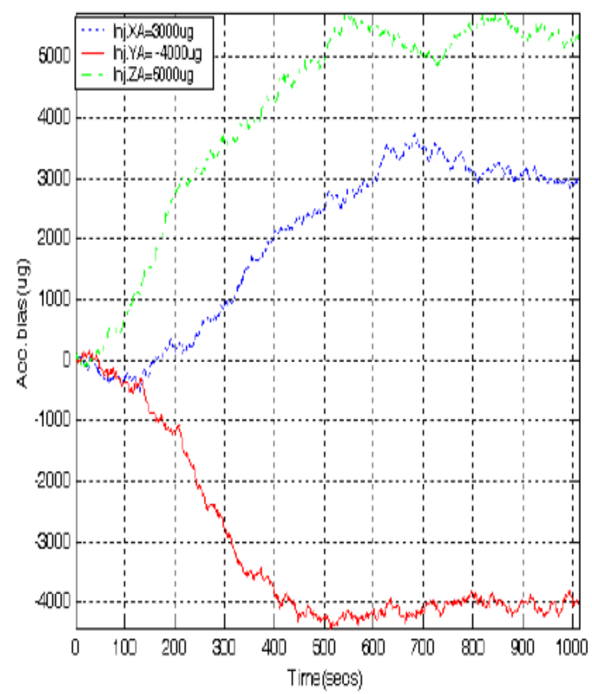


Fig.10 Estimated Accelerometer Bias using Constant Kalman Gains.

EXAMPLE OF TEC ESTIMATE

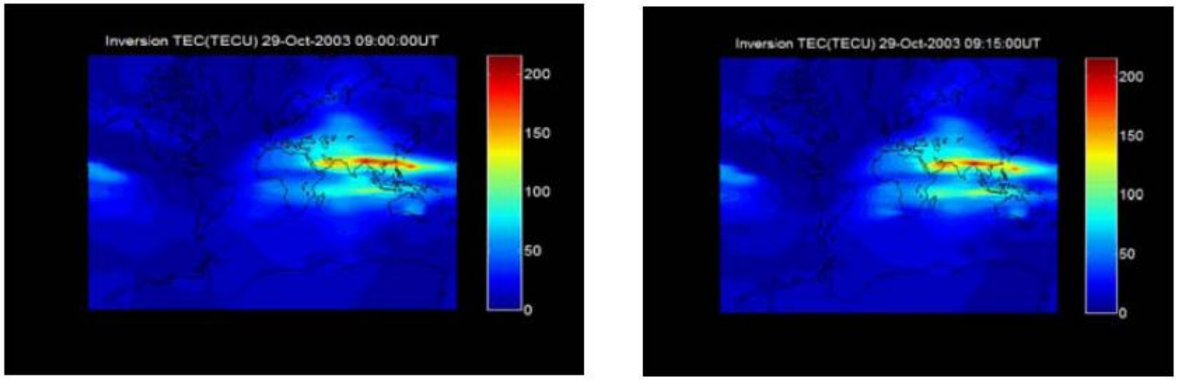


Figure 3. TEC enhancement over Indian region during the ionospheric storm on October 29, 2003. (Obtained from Hhttp://www.ngs.noaa.govH)

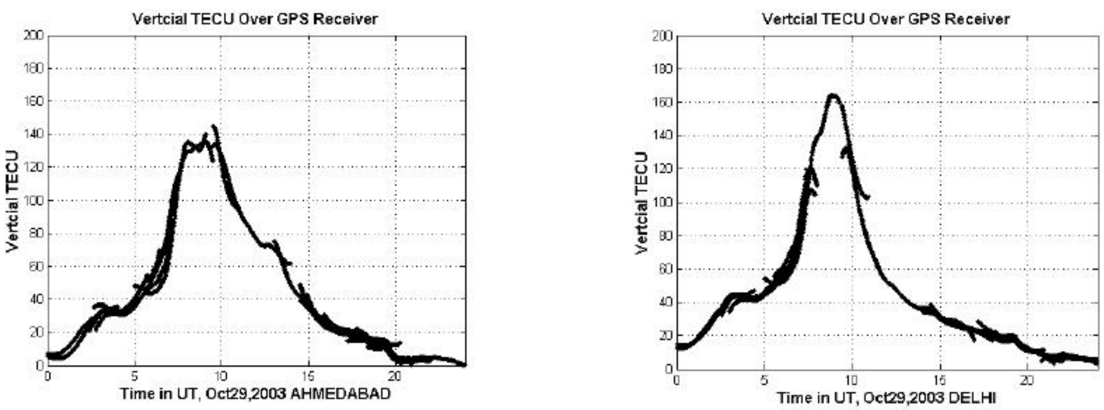


Figure 4. Vertical TEC over Ahmedabad and Delhi during the Ionospheric Storm on October 29, 2003.

CONCLUSIONS

The heart of the technique is to use a constant Kalman gain, which is nearly optimal and able to track the dynamically evolving fragment scenario and further expand the scenario to provide appropriate time varying EQB for the EQF in the various bins. The present approach of constant gain Kalman filter involving thousands of debris fragments can be fruitfully used even in massive atmospheric data assimilation problems that have tens of thousands of states and measurements.

THANK YOU