Embedded Surfaces as a 3-category

J. Scott Carter

University of South Alabama

Jim Hoste's Retirement Party
April 21, 2018

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore . . .

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore … maybe 20 years old

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore ... maybe 20 years old
3. There are lots of adjectives in the definition

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore . . . maybe 20 years old
3. There are lots of adjectives in the definition
    - The point,

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore ... maybe 20 years old
3. There are lots of adjectives in the definition
    - The point, then,

Preliminaries

1. Based upon on-going talks and initial research with Masahico Saito
2. Result is folklore ... maybe 20 years old
3. There are lots of adjectives in the definition
    - The point, then, is to describe the adjectives.

This talk was supported by the Simons Foundation.

Theorem

The naturally monoidal,

Theorem

The naturally monoidal, strictly 2-pivotal,

Theorem

The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, Theorem

The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative

Theorem

The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile*

Theorem

The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative strictly 3-tortile 3-category with one object

Theorem

The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible*

Theorem

The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible non-identity* 1*-morphism is equivalent to the*

Theorem

The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible non-identity* 1*-morphism is equivalent to the* 3*-category of isotopy classes of surfaces embedded*

Theorem

The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible non-identity* 1*-morphism is equivalent to the* 3*-category of isotopy classes of surfaces embedded in* 3*-dim'l space.*

The statement is really part of the Baez-Dolan cobordism hypothesis.

The statement is really part of the Baez-Dolan cobordism hypothesis. That was formulated in the early days of categorification as part of the program proposed by Crane and Frenkel in order to define higher TQFTs.

The statement is really part of the Baez-Dolan cobordism hypothesis. That was formulated in the early days of categorification as part of the program proposed by Crane and Frenkel in order to define higher TQFTs.
I like to re-experience simple stuff.

Really small n-cats

- Collection of objects is a set.

Really small n-cats

- Collection of objects is a set.
- collection of k-morphs is a really small $(k-1)$-cat.

Really small n-cats

- Collection of objects is a set.
- collection of k-morphs is a really small $(k-1)$-cat. So it's a set.

Really small n-cats

- Collection of objects is a set.
- collection of k-morphs is a really small $(k-1)$-cat. So it's a set.
- a 0-cat is a set.

Really small n-cats

- Collection of objects is a set.
- collection of k-morphs is a really small $(k-1)$-cat. So it's a set.
- a 0-cat is a set.

Here $k \leq n \leq 4$.

Really small n-cats

- Collection of objects is a set.
- collection of k-morphs is a really small $(k-1)$-cat. So it's a set.
- a 0-cat is a set.

Here $k \leq n \leq 4$.

A 0-morphism is also called *an object*.

A 0-morphism is also called *an object*. A 1-morphism between a pair of objects is called a *(single) arrow*.

A 0-morphism is also called *an object.* A 1-morphism between a pair of objects is called a *(single) arrow.* In general, a k-morphism will be also called a *double, triple, or quadruple* arrow, for the obvious values of k.

A 0-morphism is also called *an object*. A 1-morphism between a pair of objects is called a *(single) arrow*. In general, a k-morphism will be also called a *double, triple, or quadruple* arrow, for the obvious values of k. An overly simplistic definition of a *really small n-category* is that it is a category in which the set of morphisms between $(n-1)$-morphisms is a category.

A 0-morphism is also called *an object.* A 1-morphism between a pair of objects is called a *(single) arrow.* In general, a k-morphism will be also called a *double, triple, or quadruple* arrow, for the obvious values of k. An overly simplistic definition of a *really small* n*-category* is that it is a category in which the set of morphisms between $(n-1)$-morphisms is a category. Composition is associative, and there are identity n-morphs.

Let $a, b \in \mathrm{Obj}$.

Let $a, b \in \text{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with $source\ a = s(f)$, Let $a, b \in \mathrm{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with *source* $a = s(f)$, *target* $b = t(f)$;

Let $a, b \in \mathrm{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with *source* $a = s(f)$, *target* $b = t(f)$; let $c \xleftarrow{g} b$ be an arrow s.t. $s(g) = t(f)$, Let $a, b \in \text{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with *source* $a = s(f)$, *target* $b = t(f)$; let $c \xleftarrow{g} b$ be an arrow s.t. $s(g) = t(f)$, Comp: $c \xleftarrow{g \circ f} a$ with source $s(g \circ f) = a$ and target $t(g \circ f) = b$.

Let $a, b \in \mathrm{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with *source* $a = s(f)$, *target* $b = t(f)$; let $c \xleftarrow{g} b$ be an arrow s.t. $s(g) = t(f)$, Comp: $c \xleftarrow{g \circ f} a$ with source $s(g \circ f) = a$ and target $t(g \circ f) = b$.

Assoc.

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

Let $a, b \in \mathrm{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with *source* $a = s(f)$, *target* $b = t(f)$; let $c \xleftarrow{g} b$ be an arrow s.t. $s(g) = t(f)$, Comp: $c \xleftarrow{g \circ f} a$ with source $s(g \circ f) = a$ and target $t(g \circ f) = b$.

Assoc.

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

Id. $\forall a \in \mathrm{Obj}$ $\exists$ an arrow $a \longrightarrow a$ s.t.

Let $a, b \in \mathrm{Obj}$. Let $b \xleftarrow{f} a$ be an arrow with *source* $a = s(f)$, *target* $b = t(f)$; let $c \xleftarrow{g} b$ be an arrow s.t. $s(g) = t(f)$, Comp: $c \xleftarrow{g \circ f} a$ with source $s(g \circ f) = a$ and target $t(g \circ f) = b$.

Assoc.

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

Id. $\forall a \in \mathrm{Obj}$ $\exists$ an arrow $a \longrightarrow a$ s.t.

$$\left(b \longrightarrow b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a \longrightarrow a\right).$$

Assoc.

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

Assoc.

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

Id. $\forall a \in \mathrm{Obj}$ $\exists$ an arrow $a \longrightarrow a$ s.t.

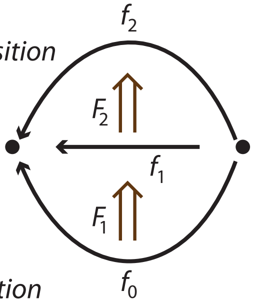
Assoc.

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

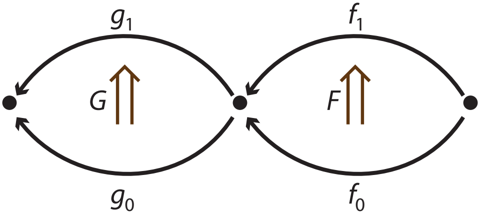
Id. $\forall a \in \mathrm{Obj}$ $\exists$ an arrow $a \,\rule{1em}{0.4pt}\, a$ s.t.

$$\left(b \,\rule{1em}{0.4pt}\, b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a \,\rule{1em}{0.4pt}\, a\right).$$

This composition is OK



This composition is ambiguous.



- $\exists! x \in \mathrm{Obj}.$

- $\exists! x \in \mathrm{Obj}.$
- $\exists x \xleftarrow{\bullet} x.$

- $\exists! x \in \mathrm{Obj}.$
- $\exists x \xleftarrow{-\!\bullet\!-} x.$
- def $(-\!\bullet\!-)^k = (-\!\bullet\!-)^{k-1} \circ -\!\bullet\!-.$

- $\exists! x \in \mathrm{Obj}.$
- $\exists x \xleftarrow{\;-\bullet-\;} x.$
- def $(\!-\!\bullet\!-\!)^{k} = (\!-\!\bullet\!-\!)^{k-1} \circ \;-\!\bullet\!-.$
- Of course, $(\!-\!\bullet\!-\!)^{0} = \;\text{——}\; = x \,\text{——}\, x.$

- $\exists! x \in \mathrm{Obj}$.
- $\exists x \xleftarrow{-\!\bullet\!-} x$.
- def $(-\!\bullet\!-)^k = (-\!\bullet\!-)^{k-1} \circ -\!\bullet\!-$.
- Of course, $(-\!\bullet\!-)^0 = \text{——} = x \text{——} x$.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$,

- $\exists! x \in \mathrm{Obj}.$
- $\exists x \xleftarrow{-\bullet-} x.$
- def $(-\bullet-)^k = (-\bullet-)^{k-1} \circ -\bullet-.$
- Of course, $(-\bullet-)^0 = \text{——} = x \text{——} x.$
- $\mathbb{N} = \{0, 1, 2, \ldots\}$, in unary notation.

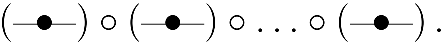
- $\exists! x \in \mathrm{Obj}$.
- $\exists x \xleftarrow{-\bullet-} x$.
- def $(-\bullet-)^k = (-\bullet-)^{k-1} \circ -\bullet-$.
- Of course, $(-\bullet-)^0 = \text{——} = x \,\text{——}\, x$.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$, in unary notation. This is the *free monoid on a single generator*.

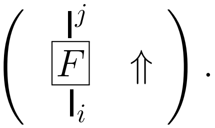
- $\exists! x \in \mathrm{Obj}$.
- $\exists x \xleftarrow{\;\bullet\;} x$.
- def $(\!-\!\bullet\!-\!)^k = (\!-\!\bullet\!-\!)^{k-1} \circ -\!\bullet\!-$.
- Of course, $(\!-\!\bullet\!-\!)^0 = \text{——} = x \text{——} x$.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$, in unary notation. This is the *free monoid on a single generator.*
- Call $(x \xleftarrow{\;\bullet\;} x)$ *rock.*

Dim'lity of comp.

$$\left(-\!\!\bullet\!\!-\right) \circ \left(-\!\!\bullet\!\!-\right) \circ \ldots \circ \left(-\!\!\bullet\!\!-\right) .$$

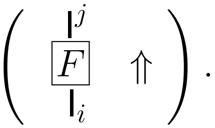
Dim'lity of comp.

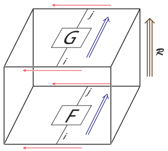




Dim'lity of comp.

$$(-\bullet-) \circ (-\bullet-) \circ \ldots \circ (-\bullet-) \,.$$

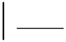
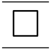
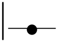
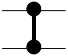




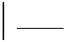
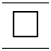
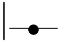
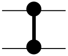
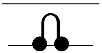
Super Id/Def Jam

- $|\underline{\quad}$ is $\overline{\underline{\square}}$.

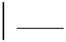
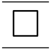
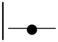
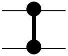
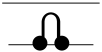
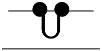
Super Id/Def Jam

-  is  .  is  .

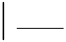
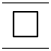
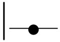
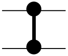
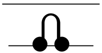
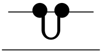
Super Id/Def Jam

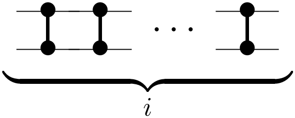
-  is  .    is  .
- Def  .

Super Id/Def Jam

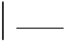
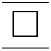
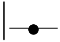
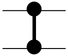
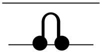
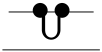
-  is  .  is  .
- Def  . Def  .

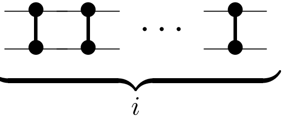
Super Id/Def Jam

-  is  .  is  .
- Def  . Def  .
- Def

$\mathsf{I}_i =$  .

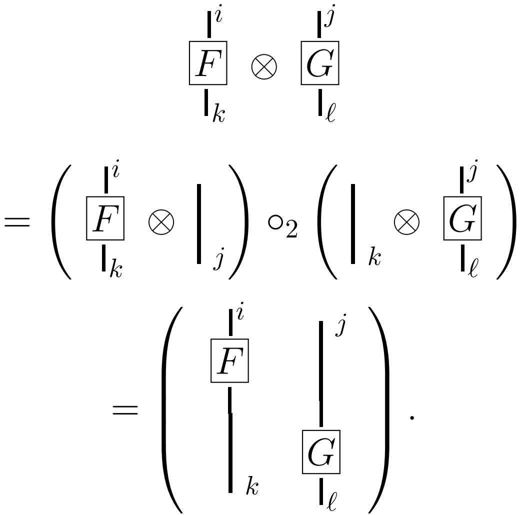
Super Id/Def Jam

-  is  .  is  .
- Def  . Def  .
- Def

$$\mathsf{I}_i =$$ 

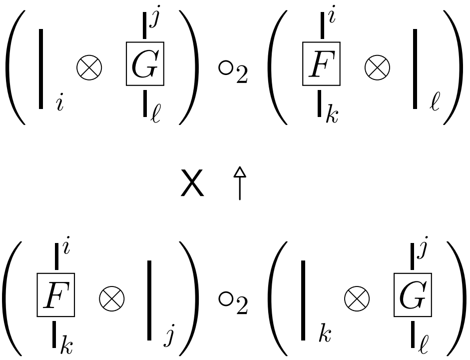
- Let $\mathsf{U}^{i,j} = \mathsf{U}(i,j) = \mathsf{I}_i \otimes \mathsf{U} \otimes \mathsf{I}_j$,
$\cap_{i,j} = \cap(i,j) = \mathsf{I}_i \otimes \cap \otimes \mathsf{I}_j$.

S'pose  and  are given.

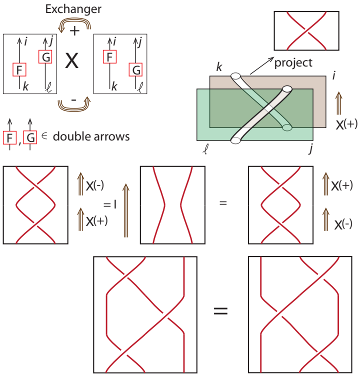


The Exchanger

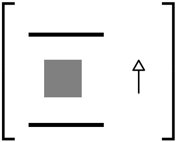
Def a natural isom:



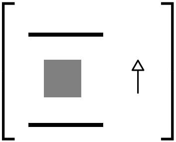
the *exchanger*.

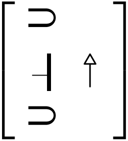

Exchanger
+
X
-
i
j
F
G
k
ℓ
F , G ∈ double arrows
project
X(+)
X(-)
= I
=

More triple arrows. Pt 1: Ids

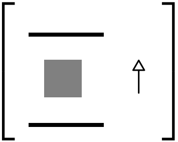

,

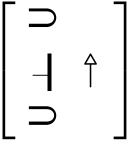
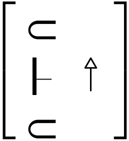
More triple arrows. Pt 1: Ids


,


,

More triple arrows. Pt 1: Ids

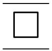
,

, and .

More 3-morphisms. Pt. 2: not isoms:

Birth [$\smile$ with a dot above]:

$$(\cap) \circ_2 (\cup) \leftarrow \Big(\quad\Big),$$



More 3-morphisms. Pt. 2: not isoms:

Birth $[\smile]$:

$$(\cap) \circ_2 (\cup) \leftarrow \left(\frac{\overline{\quad\square\quad}}{} \right),$$

Death $[\frown]$:

$$\left(\frac{\overline{\quad\square\quad}}{} \right) \leftarrow (\cap) \circ_2 (\cup),$$

More 3-morphisms. Pt. 2: not isoms:

Birth $[\smile\!\!\!\!\cdot\;]$:

$$(\cap)\circ_2(\cup)\leftarrow\left(\;\overline{\underline{\;\square\;}}\;\right),$$

Death $[\;\frown\!\!\!\!\cdot\;]$:

$$\left(\;\overline{\underline{\;\square\;}}\;\right)\leftarrow(\cap)\circ_2(\cup),$$

Saddle $[\dot{\cup}]$:

$$(\cup)\circ_2(\cap)\leftarrow(\mathsf{I}\otimes\mathsf{I}),$$

More 3-morphisms. Pt. 2: not isoms:

Birth [$\overset{\smile}{\cdot}$]:

$$(\cap) \circ_2 (\cup) \leftarrow \left(\overline{\underline{\;\square\;}} \right),$$

Death [$\overset{\frown}{\cdot}$]:

$$\left(\overline{\underline{\;\square\;}} \right) \leftarrow (\cap) \circ_2 (\cup),$$

Saddle [$\underset{\cdot}{\cup}$]:

$$(\cup) \circ_2 (\cap) \leftarrow (\mathsf{I} \otimes \mathsf{I}),$$

Crotch [$\dot{\cap}$]:

$$(\mathsf{I} \otimes \mathsf{I}) \leftarrow (\cup) \circ_2 (\cap),$$

So that these are not isoms. is why the $\cap$ and $\cup$ maps are only **weak** inversions on $-\!\bullet\!-$.

More 3-morphs: Pt 3: isoms

Left cusp $[\curlyvee_L]$:

$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \leftrightarrows (\mathsf{I}),$$

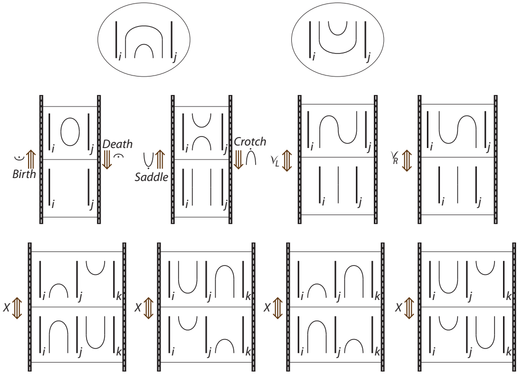
Right cusp $[\curlyvee_R]$:

$$(\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \leftrightarrows (\mathsf{I}).$$

In a moment, I'll let you know in what way these are isoms.

In a moment, I'll let you know in what way these are isoms. That they are isoms, is what I am calling **strictly 2-pivotal**.

In a moment, I'll let you know in what way these are isoms. That they are isoms, is what I am calling **strictly 2-pivotal**. First, let's examine how to 2-compose.


Birth
Death
Saddle
Crotch
γ_L
γ_R
X
X
X
X

Strongly 2-pivotal

This page is intentionally blank.

left cusp down: $\curlyvee_L$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$.

left cusp down: $\curlyvee_L$ *right cusp down*: $\curlyvee_R$. def *left cusp up*: $\curlywedge^L = \curlyvee_L^{-1}$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$.

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$. Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$. Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

$$\mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I},$$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$.
Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

$$\mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I},$$

$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$. Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

$$\mathsf{I} \overset{\curlywedge^L}{\leftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\leftarrow} \mathsf{I},$$

$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\leftarrow} \mathsf{I} \overset{\curlywedge^L}{\leftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

$$\mathsf{I} \overset{\curlywedge^R}{\leftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\leftarrow} \mathsf{I},$$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$.
Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

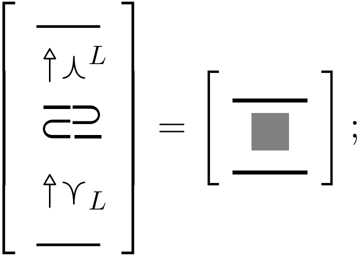
$$\mathsf{I} \overset{\curlywedge^L}{\leftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\leftarrow} \mathsf{I},$$

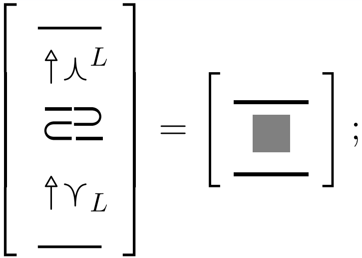
$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\leftarrow} \mathsf{I} \overset{\curlywedge^L}{\leftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

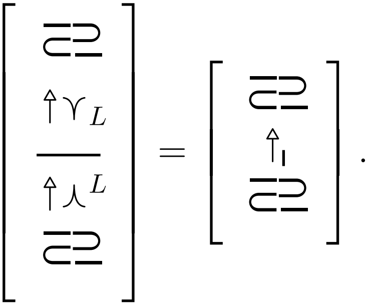
$$\mathsf{I} \overset{\curlywedge^R}{\leftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\leftarrow} \mathsf{I},$$

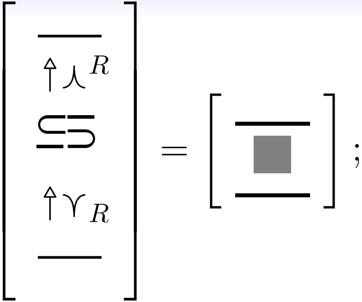
$$(\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\leftarrow} \mathsf{I} \overset{\curlywedge^R}{\leftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I})$$

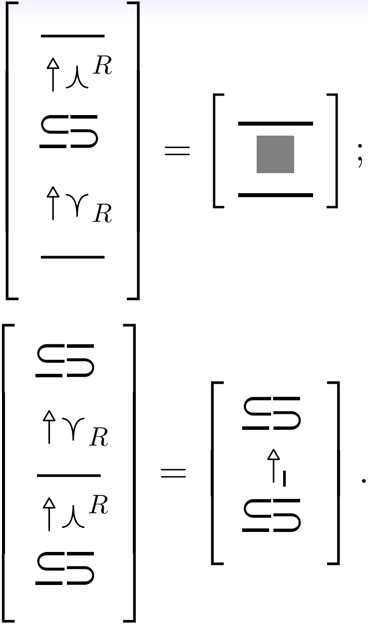
are the identity 3-morphisms on their (coincident) sources and targets.

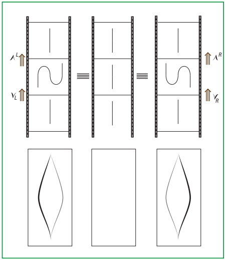


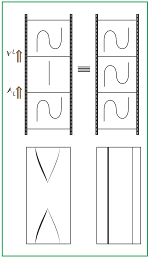


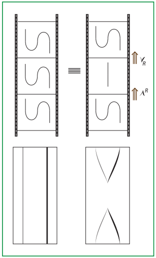



R
R

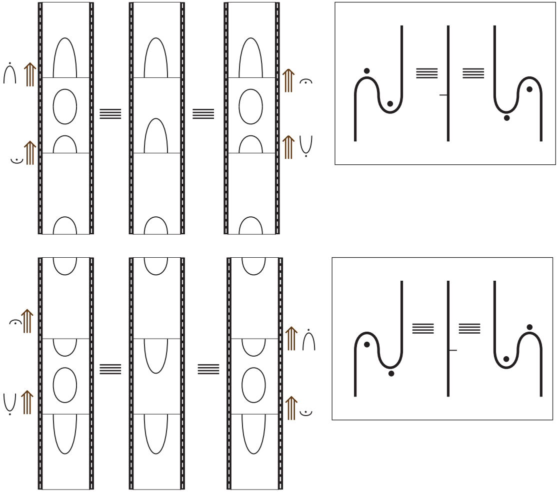

R
R
R
R







Weakly 3-pivotal



Weakly 3-pivotal

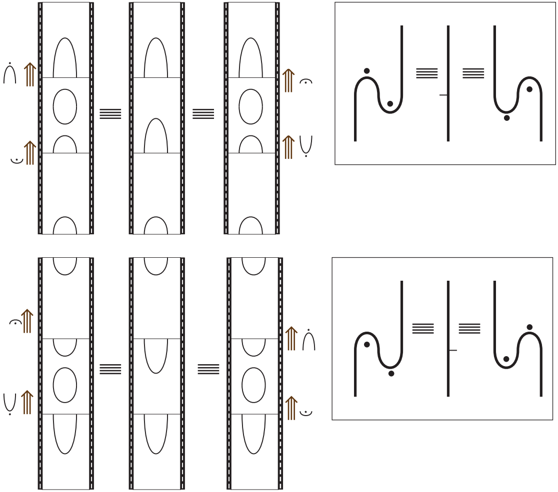
$$\left(\dashv \otimes \dot{\cap}\right) \circ_3 \left(\dot{\smile} \otimes \dashv\right) = \dashv = \left(\dashv \otimes \underset{\cdot}{\frown}\right) \circ_3 \left(\underset{\cdot}{\cup} \otimes \dashv\right),$$

Weakly 3-pivotal

$$\left(\dashv \otimes \dot{\cap}\right) \circ_3 \left(\overset{\cdot}{\smile} \otimes \dashv\right) = \dashv = \left(\dashv \otimes \underset{\cdot}{\frown}\right) \circ_3 \left(\underset{\cdot}{\cup} \otimes \dashv\right),$$

$$\left(\underset{\cdot}{\frown} \otimes \vdash\right) \circ_3 \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \vdash = \left(\vdash \otimes \dot{\cap}\right) \circ_3 \left(\overset{\cdot}{\smile} \otimes \vdash\right).$$

Weakly 3-pivotal



Rotationally commutativity

Modify notation so that it is more transparent.

Rotationally commutativity

Modify notation so that it is more transparent.
$\curlywedge^R = \curlywedge^{\bullet}$, $\curlywedge^L = {}^{\bullet}\curlywedge$, $\curlyvee_R = \curlyvee_{\bullet}$, $\curlyvee_L = {}_{\bullet}\curlywedge$, and $\circ_3 = \circ$.

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

Rotationally commutativity

Modify notation so that it is more transparent. $\curlywedge^R = \curlywedge^{\bullet}$, $\curlywedge^L = {}^{\bullet}\curlywedge$, $\curlyvee_R = \curlyvee_{\bullet}$, $\curlyvee_L = {}_{\bullet}\curlywedge$, and $\circ_3 = \circ$.

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

$$\left(\curlywedge^{\bullet} \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash\right)$$

Rotationally commutativity

Modify notation so that it is more transparent. $\curlywedge^R = \curlywedge^\bullet$, $\curlywedge^L = {}^\bullet\curlywedge$, $\curlyvee_R = \curlyvee_\bullet$, $\curlyvee_L = {}_\bullet\curlywedge$, and $\circ_3 = \circ$.

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_\bullet \otimes \vdash\right) = \left({}^\bullet\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

$$\left(\curlywedge^\bullet \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_\bullet\curlyvee \otimes \vdash\right)$$

$$\left(\dashv \otimes \curlywedge^\bullet\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_\bullet\curlyvee\right)$$

Rotationally commutativity

Modify notation so that it is more transparent.
$\curlywedge^{R} = \curlywedge^{\bullet}$, $\curlywedge^{L} = {}^{\bullet}\curlywedge$, $\Upsilon_{R} = \Upsilon_{\bullet}$, $\Upsilon_{L} = {}_{\bullet}\curlywedge$, and
$\circ_3 = \circ$.

$$\left(\vdash \otimes \dot{\cap} \right) \circ \left(\Upsilon_{\bullet} \otimes \vdash \right) = \left({}^{\bullet}\curlywedge \otimes \vdash \right) \circ \left(\vdash \otimes \underset{\cdot}{\cup} \right)$$

$$\left(\curlywedge^{\bullet} \otimes \vdash \right) \circ \left(\vdash \otimes \underset{\cdot}{\cup} \right) = \left(\vdash \otimes \dot{\cap} \right) \circ \left({}_{\bullet}\Upsilon \otimes \vdash \right)$$

$$\left(\dashv \otimes \curlywedge^{\bullet} \right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv \right) = \left(\dot{\cap} \otimes \dashv \right) \circ \left(\dashv \otimes {}_{\bullet}\Upsilon \right)$$

$$\left(\dot{\cap} \otimes \dashv \right) \circ \left(\dashv \otimes \Upsilon_{\bullet} \right) = \left(\dashv \otimes {}^{\bullet}\curlywedge \right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv \right)$$

Rotationally commutativity

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

$$\left(\curlywedge^{\bullet} \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash\right)$$

$$\left(\dashv \otimes \curlywedge^{\bullet}\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_{\bullet}\curlyvee\right)$$

$$\left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee_{\bullet}\right) = \left(\dashv \otimes {}^{\bullet}\curlywedge\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right)$$

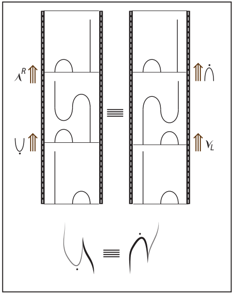
Rotationally commutative

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

$$\left(\curlywedge^{\bullet} \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash\right)$$

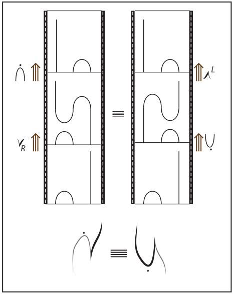
$$\left(\dashv \otimes \curlywedge^{\bullet}\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_{\bullet}\curlyvee\right)$$

$$\left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee_{\bullet}\right) = \left(\dashv \otimes {}^{\bullet}\curlywedge\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right)$$



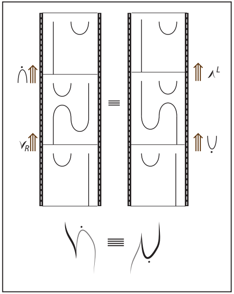
$(\lambda^R)\circ_3(\underset{\cdot}{\cup}) = (\dot{\cap})\circ_3(\vee_L)$

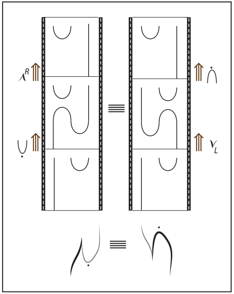
$(|\otimes\lambda^R)\circ_3(\underset{\cdot}{\cup}\otimes|) = (\dot{\cap}\otimes|)\circ_3(|\otimes\vee_L)$



$(\dot{\cap})\circ_3(\vee_R) = (\lambda^L)\circ_3(\underset{\cdot}{\cup})$

$(\dot{\cap}\otimes|)\circ_3(|\otimes\vee_R) = (|\otimes\lambda^L)\circ_3(\underset{\cdot}{\cup}\otimes|)$





$$(\dot{\cap}) \circ_3 (\curlyvee_R) = (\curlywedge^L) \circ_3 (\underset{\cdot}{\cup})$$

$$(\vdash \otimes \dot{\cap}) \circ_3 (\curlyvee_R \otimes \vdash) = (\curlywedge^L \otimes \vdash) \circ_3 (\vdash \otimes \underset{\cdot}{\cup})$$

$$(\curlywedge^R) \circ_3 (\underset{\cdot}{\cup}) = (\dot{\cap}) \circ_3 (\curlyvee_L)$$

$$(\curlywedge^R \otimes \vdash) \circ_3 (\vdash \otimes \underset{\cdot}{\cup}) = (\vdash \otimes \dot{\cap}) \circ_3 (\curlyvee_L \otimes \vdash)$$

$$\boxed{\cap \circ \Upsilon = \lambda \circ \cup}$$

Strongly 3-tortile

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv ,$$

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv ,$$

$$\dashv = \left(\curlywedge^{R} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) ,$$

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv \,,$$

$$\dashv = \left(\curlywedge^{R} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right),$$

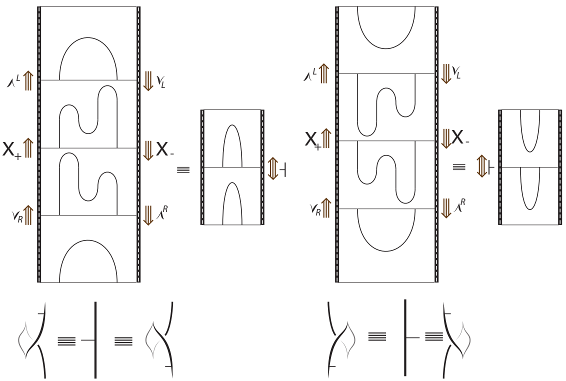
$$\left(\vdash \otimes \curlywedge^{L} \right) \circ_3 \left(\mathsf{X}(+) \otimes \dashv \right) \circ_3 \left(\dashv \otimes \curlyvee^{L} \right) = \vdash \,,$$

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv ,$$

$$\dashv = \left(\curlywedge^{R} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right),$$

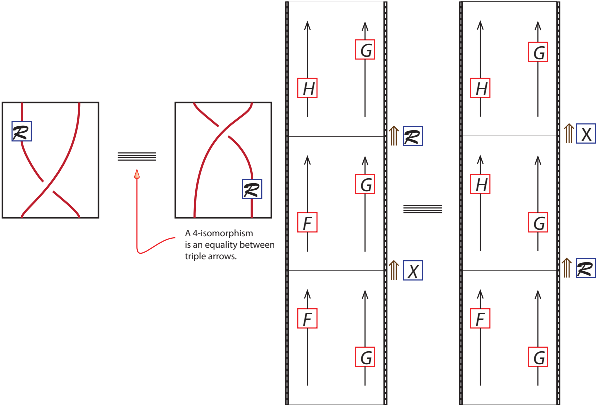
$$\left(\vdash \otimes \curlywedge^{L} \right) \circ_3 \left(\mathsf{X}(+) \otimes \dashv \right) \circ_3 \left(\dashv \otimes \curlyvee^{L} \right) = \vdash ,$$

$$\vdash = \left(\vdash \otimes \curlywedge^{L} \right) \circ_3 \left(\mathsf{X}(-) \otimes \dashv \right) \circ_3 \left(\dashv \otimes \curlyvee^{L} \right).$$



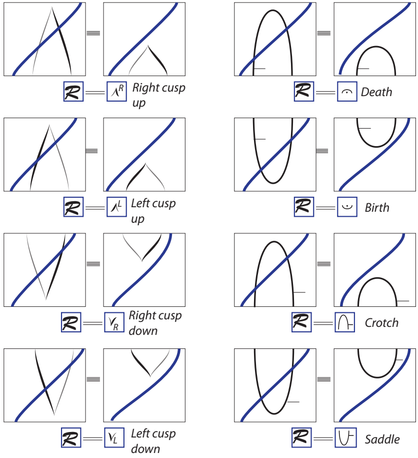
λ^L
ν_L
X+
X-
ν_R
λ^R

Naturality



R
R
A 4-isomorphism
is an equality between
triple arrows.
G
H
R
G
F
X
F
G
G
H
X
H
G
F
R
F
G

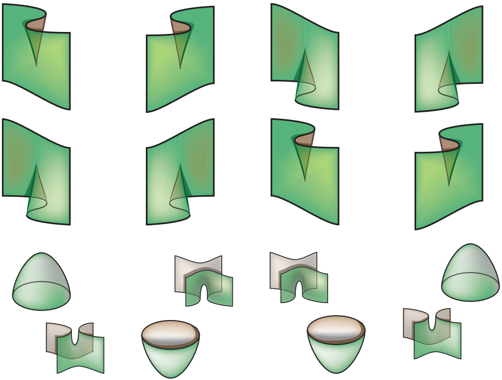
Naturality

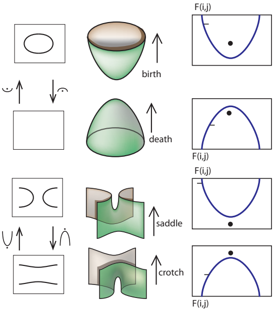

Right cusp up
Death
Left cusp up
Birth
Right cusp down
Crotch
Left cusp down
Saddle

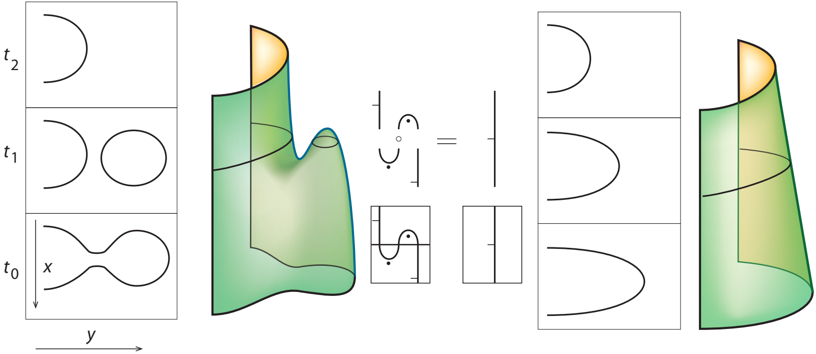
Theorem

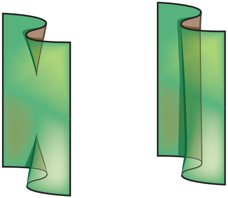
The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative strictly 3-tortile 3-category with one object that is freely generated by a weakly self-invertible non-identity 1-morphism is equivalent to the 3-category of isotopy classes of surfaces embedded in 3-dim'l space.

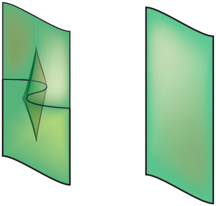
OK. So now here is the sketch of the proof.

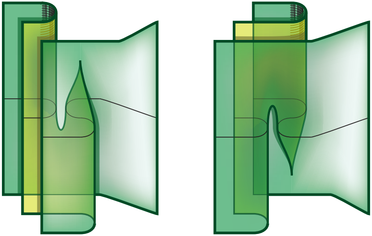


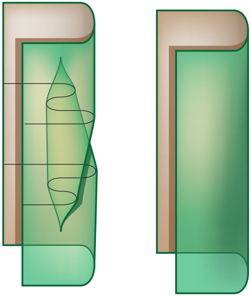

F(i,j)
birth
death
F(i,j)
F(i,j)
saddle
crotch
F(i,j)

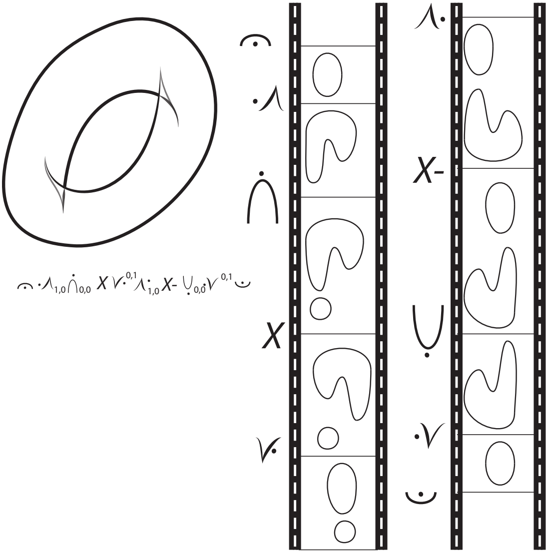

t_2
t_1
t_0
x
y











Thank you

That’s my story.

Thank you

That's my story. And I am sticking to it.

Thank you

That's my story. And I am sticking to it. Thank you!

Thank you

That's my story. And I am sticking to it. Thank you! Congratulations, Jim!