Isotopy Classes of Embedded Surfaces in 3-space as a 3-category

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TAPU-KOOK, Busan Jul 2018

Preliminaries

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3. There are lots of adjectives in the definition
    - The point, then, is to describe the adjectives.

Theorem

The naturally monoidal,

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The naturally monoidal, strictly 2-pivotal,

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The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal,

Theorem

The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative

Theorem

The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile*

Theorem

The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative strictly 3-tortile 3-category with one object

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The naturally monoidal, strictly 2*-pivotal, weakly* 3*-pivotal, rotationally commutative strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible*

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I like to re-experience simple stuff.

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- There is no sychronicity i.e. different events do NOT occur simultaneously.

Really small n-cats

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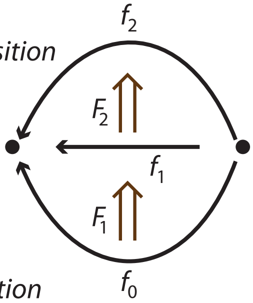
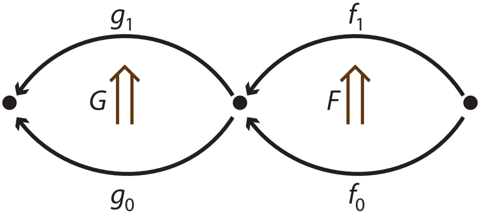


This composition is ambiguous.



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- $\mathbb{N} = \{0, 1, 2, \ldots\}$, in unary notation. This is the *free monoid on a single generator*.
- Call $(x \xleftarrow{-\bullet-} x)$ *rock*.

Dim'lity of comp.

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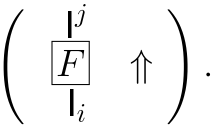

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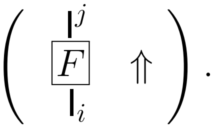

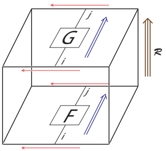

Super Id/Def Jam

- $|$ ___ is $\overline{\underline{\square}}$.

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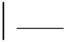
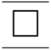
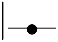
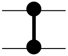
-  is  .  is  .

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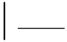
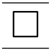
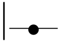
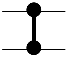
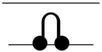
-  is  .  is  .
- Def  .

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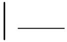
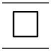
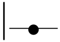
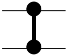
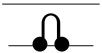
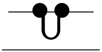
-  is  .  is  .
- Def  . Def  .

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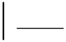
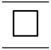
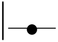
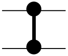
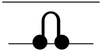
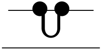
-  is  .  is  .
- Def  . Def  .
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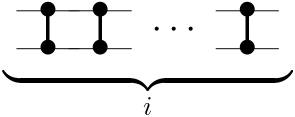
$$\mathsf{I}_i =$$  .

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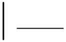
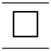
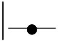
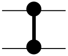
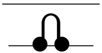
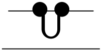
-  is  .  is  .
- Def  . Def  .
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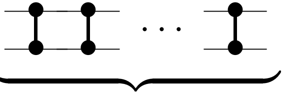
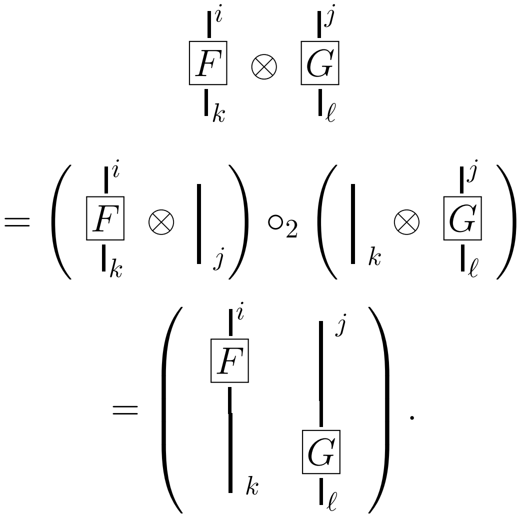
$$\mathsf{I}_i = $$ 

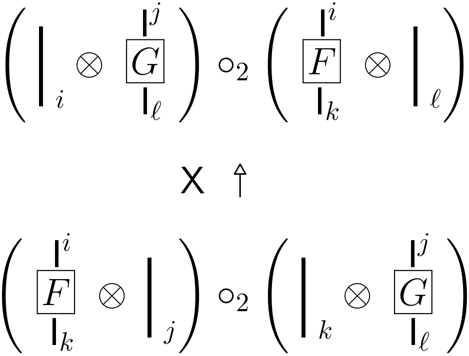
- Let $\mathsf{U}^{i,j} = \mathsf{U}(i,j) = \mathsf{I}_i \otimes \mathsf{U} \otimes \mathsf{I}_j$, $\cap_{i,j} = \cap(i,j) = \mathsf{I}_i \otimes \cap \otimes \mathsf{I}_j$.

S'pose  and  are given.



The Exchanger

Def a natural isom:



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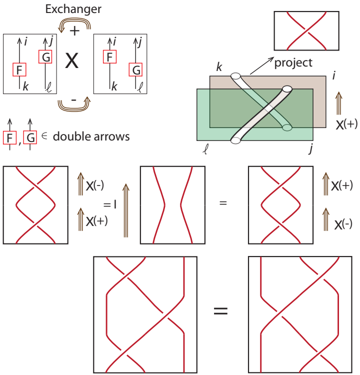
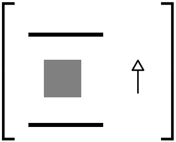

Exchanger
+
X
-
F
G
i
j
k
ℓ
∈ double arrows
project
X(+)
X(-)
= I
=
=

More triple arrows. Pt 1: Ids



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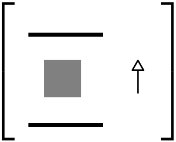

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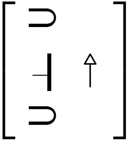

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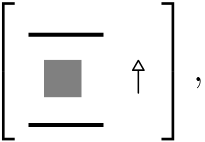
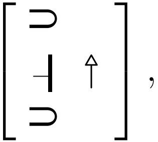
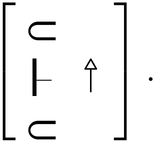
More triple arrows. Pt 1: Ids



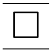


and



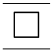
More 3-morphisms. Pt. 2: not isoms:

Birth $[\dot{\smile}]$:

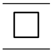
$(\cap) \circ_2 (\cup) \leftarrow \Bigg($  $\Bigg)$,

More 3-morphisms. Pt. 2: not isoms:

Birth $[\dot{\smile}]$:

$$(\cap) \circ_2 (\cup) \leftarrow \Big($$  $$\Big),$$

Death $[\underset{\cdot}{\frown}]$:

$$\Big($$  $$\Big) \leftarrow (\cap) \circ_2 (\cup),$$

More 3-morphisms. Pt. 2: not isoms:

Birth $[\dot{\smile}]$:

$$(\cap) \circ_2 (\cup) \leftarrow \left(\overline{\underline{\square}} \right),$$

Death $[\,\dot{\frown}\,]$:

$$\left(\overline{\underline{\square}} \right) \leftarrow (\cap) \circ_2 (\cup),$$

Saddle $[\dot{\cup}]$:

$$(\cup) \circ_2 (\cap) \leftarrow (\mathsf{I} \otimes \mathsf{I}),$$

More 3-morphisms. Pt. 2: not isoms:

Birth $[\dot{\smile}]$:

$$(\cap) \circ_2 (\cup) \leftarrow \left(\overline{\underline{\square}} \right),$$

Death $[\underset{\cdot}{\frown}]$:

$$\left(\overline{\underline{\square}} \right) \leftarrow (\cap) \circ_2 (\cup),$$

Saddle $[\underset{\cdot}{\cup}]$:

$$(\cup) \circ_2 (\cap) \leftarrow (\mathsf{I} \otimes \mathsf{I}),$$

Crotch $[\dot{\cap}]$:

$$(\mathsf{I} \otimes \mathsf{I}) \leftarrow (\cup) \circ_2 (\cap),$$

So that these are not isoms. is why the $\cap$ and $\cup$ maps are only **weak** inversions on —•—.

More 3-morphs: Pt 3: isoms

Left cusp $[\curlyvee_L]$:

$$(\mathsf{\cap} \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \mathsf{\cup}) \leftrightarrows (\mathsf{I}),$$

Right cusp $[\curlyvee_R]$:

$$(\mathsf{I} \otimes \mathsf{\cap}) \circ_2 (\mathsf{\cup} \otimes \mathsf{I}) \leftrightarrows (\mathsf{I}).$$

In a moment, I'll let you know in what way these are isoms.

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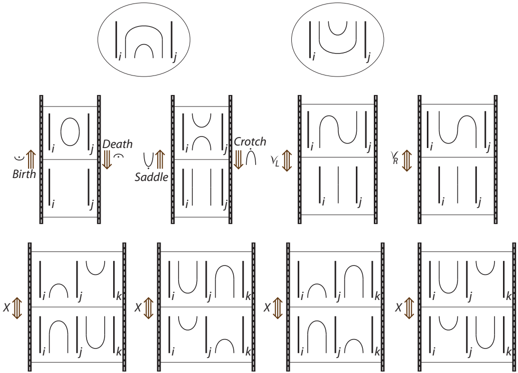
In a moment, I'll let you know in what way these are isoms. That they are isoms, is what I am calling **strictly 2-pivotal**. First, let's examine how to 2-compose.



Birth
Death
Saddle
Crotch
γ_L
γ_R
X

Strongly 2-pivotal

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left cusp down: $\curlyvee_L$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$.

left cusp down: $\curlyvee_L$ *right cusp down*: $\curlyvee_R$. def *left cusp up*: $\curlywedge^L = \curlyvee_L^{-1}$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$.

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$. Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$. Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

$$\mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I},$$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$. Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

$$\mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I},$$

$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$.
Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

$$\mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I},$$

$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

$$\mathsf{I} \xleftarrow{\curlywedge^R} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \xleftarrow{\curlyvee_R} \mathsf{I},$$

left cusp down: $\curlyvee_L$ *right cusp down:* $\curlyvee_R$. def *left cusp up:* $\curlywedge^L = \curlyvee_L^{-1}$ *right cusp up:* $\curlywedge^R = \curlyvee_R^{-1}$.
Assert $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses:

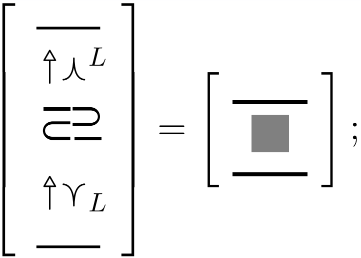
$$\mathsf{I} \overset{\curlywedge^L}{\leftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\leftarrow} \mathsf{I},$$

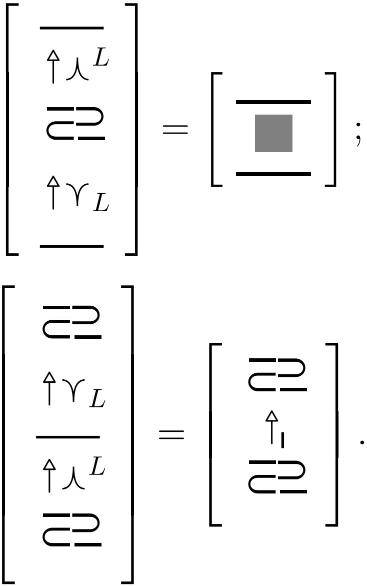
$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\leftarrow} \mathsf{I} \overset{\curlywedge^L}{\leftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

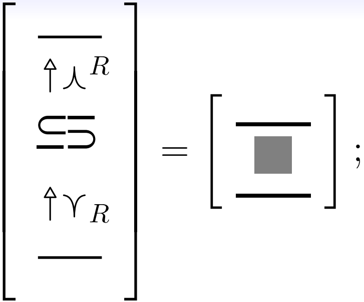
$$\mathsf{I} \overset{\curlywedge^R}{\leftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\leftarrow} \mathsf{I},$$

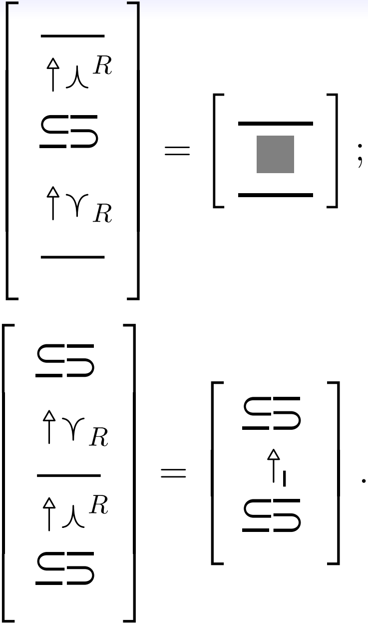
$$(\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\leftarrow} \mathsf{I} \overset{\curlywedge^R}{\leftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I})$$

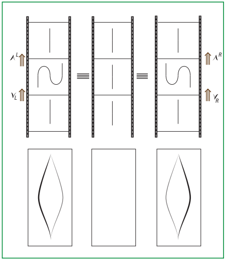
are the identity 3-morphisms on their (coincident) sources and targets.

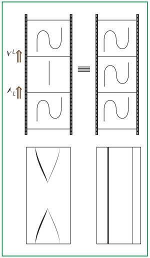


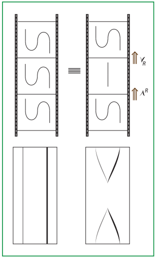

L
L
L
L


R
R

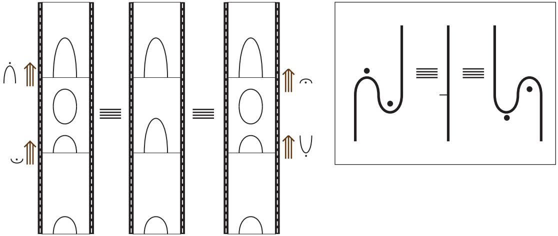

R
R
R
R

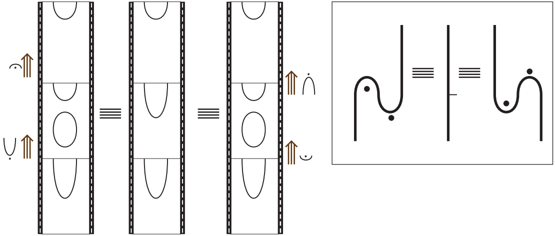






Weakly 3-pivotal





Weakly 3-pivotal

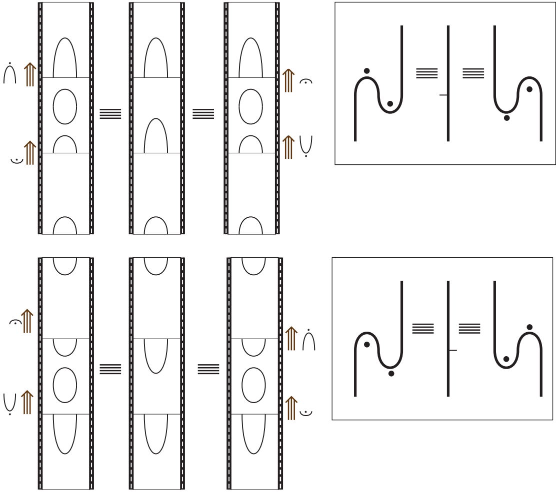
$$\left(\dashv \otimes \dot{\cap}\right) \circ_3 \left(\dot{\smile} \otimes \dashv\right) = \dashv = \left(\dashv \otimes \dot{\frown}\right) \circ_3 \left(\underset{\cdot}{\cup} \otimes \dashv\right),$$

Weakly 3-pivotal

$$\left(\dashv \otimes \dot{\cap}\right) \circ_3 \left(\dot{\smile} \otimes \dashv\right) = \dashv = \left(\dashv \otimes \underset{\cdot}{\frown}\right) \circ_3 \left(\underset{\cdot}{\cup} \otimes \dashv\right),$$

$$\left(\underset{\cdot}{\frown} \otimes \vdash\right) \circ_3 \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \vdash = \left(\vdash \otimes \dot{\cap}\right) \circ_3 \left(\dot{\smile} \otimes \vdash\right).$$

Weakly 3-pivotal



Rotationally commutativity

Modify notation so that it is more transparent.

Rotationally commutativity

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$\curlywedge^{R} = \curlywedge^{\bullet}$, $\curlywedge^{L} = {}^{\bullet}\curlywedge$, $\curlyvee_{R} = \curlyvee_{\bullet}$, $\curlyvee_{L} = {}_{\bullet}\curlywedge$, and
$\circ_3 = \circ$.

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

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$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_\bullet \otimes \vdash\right) = \left({}^\bullet\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

$$\left(\curlywedge^\bullet \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_\bullet\curlyvee \otimes \vdash\right)$$

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$$\left(\curlywedge^{\bullet} \otimes \vdash \right) \circ \left(\vdash \otimes \underset{\cdot}{\cup} \right) = \left(\vdash \otimes \dot{\cap} \right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash \right)$$

$$\left(\dashv \otimes \curlywedge^{\bullet} \right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv \right) = \left(\dot{\cap} \otimes \dashv \right) \circ \left(\dashv \otimes {}_{\bullet}\curlyvee \right)$$

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$$\left(\curlywedge^\bullet \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_\bullet\curlyvee \otimes \vdash\right)$$

$$\left(\dashv \otimes \curlywedge^\bullet\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_\bullet\curlyvee\right)$$

$$\left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee_\bullet\right) = \left(\dashv \otimes {}^\bullet\curlywedge\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right)$$

Rotationally commutativity

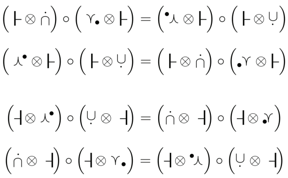
$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

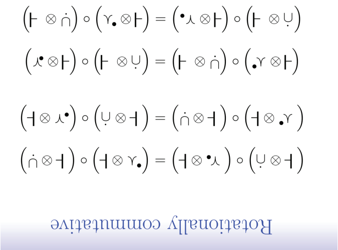
$$\left(\curlywedge^{\bullet} \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash\right)$$

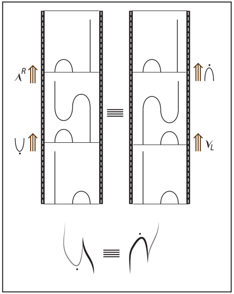
$$\left(\dashv \otimes \curlywedge^{\bullet}\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_{\bullet}\curlyvee\right)$$

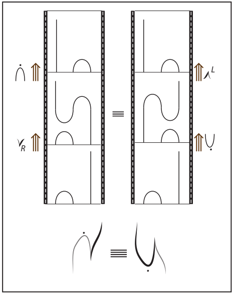
$$\left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee_{\bullet}\right) = \left(\dashv \otimes {}^{\bullet}\curlywedge\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right)$$

Rotationally commutative



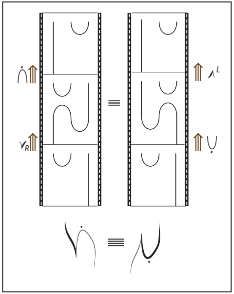


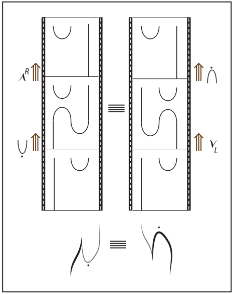




$(\lambda^R)\circ_3(\underset{\cdot}{\cup}) = (\dot{\cap})\circ_3(\curlyvee_L)$ $\qquad$ $(\dot{\cap})\circ_3(\curlyvee_R) = (\lambda^L)\circ_3(\underset{\cdot}{\cup})$

$(\dashv\otimes\lambda^R)\circ_3(\underset{\cdot}{\cup}\otimes\dashv) = (\dot{\cap}\otimes\dashv)\circ_3(\dashv\otimes\curlyvee_L)$ $\qquad$ $(\dot{\cap}\otimes\dashv)\circ_3(\dashv\otimes\curlyvee_R) = (\vdash\otimes\lambda^L)\circ_3(\underset{\cdot}{\cup}\otimes\dashv)$





$$(\dot\cap)\circ_3(\curlyvee_R) = (\curlywedge^L)\circ_3(\underset{\cdot}{\cup}) \qquad (\curlywedge^R)\circ_3(\underset{\cdot}{\cup}) = (\dot\cap)\circ_3(\curlyvee_L)$$

$$(\vdash\otimes\dot\cap)\circ_3(\curlyvee_R\otimes\vdash) = (\curlywedge^L\otimes\vdash)\circ_3(\vdash\otimes\underset{\cdot}{\cup}) \qquad (\curlywedge^R\otimes\vdash)\circ_3(\vdash\otimes\underset{\cdot}{\cup}) = (\vdash\otimes\dot\cap)\circ_3(\curlyvee_L\otimes\vdash)$$

$$\boxed{\cap \circ \Upsilon = \lambda \circ \cup}$$

Strongly 3-tortile

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv \,,$$

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv ,$$

$$\dashv = \left(\curlywedge^{R} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) ,$$

$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv ,$$

$$\dashv = \left(\curlywedge^{R} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) ,$$

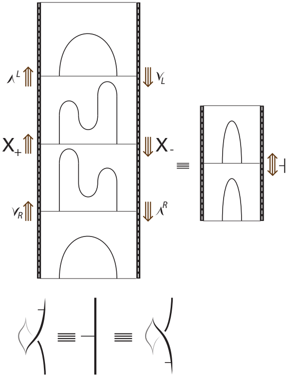
$$\left(\vdash \otimes \curlywedge^{L} \right) \circ_3 \left(\mathsf{X}(+) \otimes \dashv \right) \circ_3 \left(\dashv \otimes \curlyvee^{L} \right) = \vdash ,$$

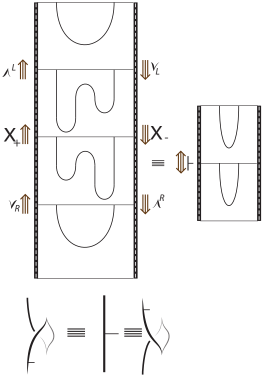
$$\left(\curlywedge^{L} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) = \dashv ,$$

$$\dashv = \left(\curlywedge^{R} \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ_3 \left(\curlyvee^{L} \otimes \dashv \right) ,$$

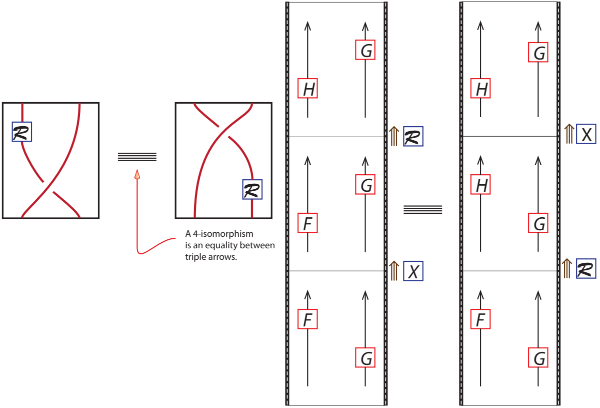
$$\left(\vdash \otimes \curlywedge^{L} \right) \circ_3 \left(\mathsf{X}(+) \otimes \dashv \right) \circ_3 \left(\dashv \otimes \curlyvee^{L} \right) = \vdash ,$$

$$\vdash = \left(\vdash \otimes \curlywedge^{L} \right) \circ_3 \left(\mathsf{X}(-) \otimes \dashv \right) \circ_3 \left(\dashv \otimes \curlyvee^{L} \right) .$$


X+
X-

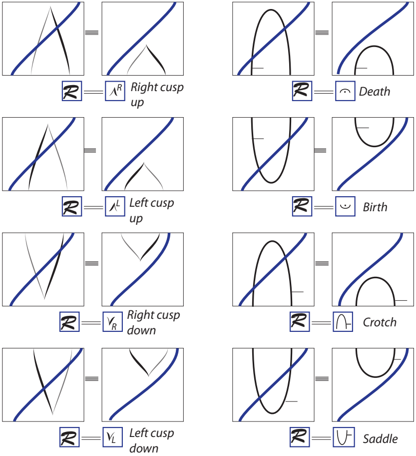

X+
X-

Naturality



R
R
A 4-isomorphism
is an equality between
triple arrows.
H
G
F
G
F
G
R
X
H
G
H
G
F
G
X
R

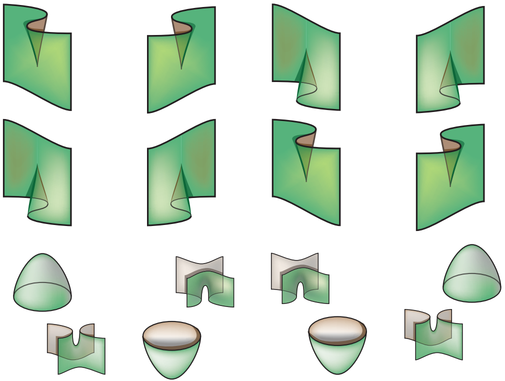
Naturality

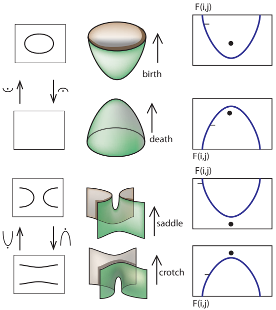

Right cusp up
Death
Left cusp up
Birth
Right cusp down
Crotch
Left cusp down
Saddle

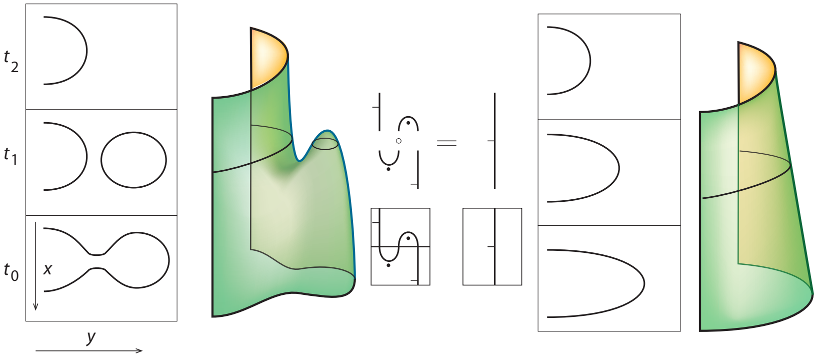
Theorem

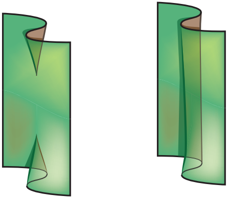
The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative strictly 3-tortile 3-category with one object that is freely generated by a weakly self-invertible non-identity 1-morphism is equivalent to the 3-category of isotopy classes of surfaces properly embedded in $\mathbb{R}^2 \times [0, 1]$.

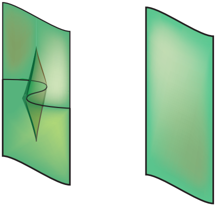
OK. So now here is the sketch of the proof.

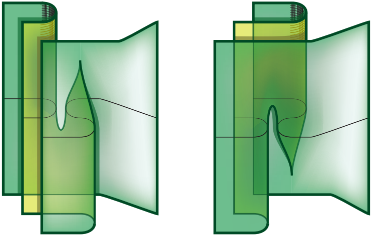


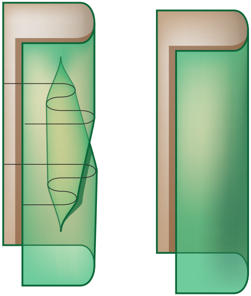

F(i,j)
birth
death
F(i,j)
F(i,j)
saddle
crotch
F(i,j)

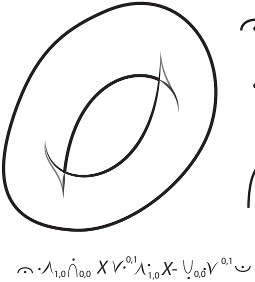

t_2
t_1
t_0
x
y
=

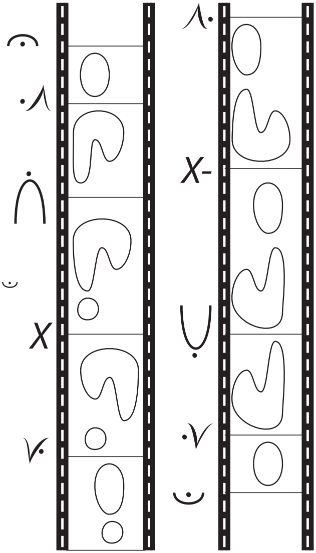












Fundamental 3-goupoid

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Thank you

That's my story.

Thank you

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