Stochastic Analysis and Control of Fluid Flows

Lecture 6

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Estimation – Model Reduction

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1. Introduction to model reduction

We have a noisy control system and a noisy measurement

$$z' = Az + Bu + \mu, \quad z(0) = z_0 + \mu_0,$$
$$y_{obs}(t) = Hz(t) + \eta(t) \in Y_o.$$

Under suitable assumptions on (A, B, H), we have seen that we can determine a feedback control with partial information by solving a finite dimensional Riccati equation for the feedback control, e.g.

$$P_{\omega,u} \in \mathcal{L}(Z_{\omega,u}, Z^*_{\omega,u}), \quad P_{\omega,u} = P^*_{\omega,u} > 0,$$
$$P_{\omega,u}A_{\omega,u} + A^*_{\omega,u}P_{\omega,u} - P_{\omega,u}\,B_{\omega,u}B^*_{\omega,u}P_{\omega,u} \;=\; 0,$$

an infinite dimensional Riccati equation (or a finite dimensional approximation of high dimension) to determine the filtering gain

$$P_e = P_e^* \geq 0, \quad AP_e + P_eA^* - P_eH^*R_o^{-1}HP_e + Q_o = 0,$$

and the control is obtained by solving the system

$$z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0,$$

with

$$K = -B^*_{\omega,u}P_{\omega,u}\pi_{\omega,u} \quad \text{and} \quad L = -P_eH^*R_o^{-1}.$$

Next the control

$$u(t) = -B^*_{\omega,u}\, P_{\omega,u}\, \pi_{\omega,u}\, z_e(t),$$

is inserted in the dynamical system

$$z' = Az + Bu + \mu, \quad z(0) = z_0 + \mu_0.$$

Since the equation for the estimator is of high dimension, it cannot be solved fast enough to determine the control which is inserted in the real system.

Our goal is to replace the estimator by an equation of small dimension and try to keep the good stability properties of the full system coupling z and z_e.

2. Model reduction for finite dimensional systems

2.1. Balancing transformation

Consider

$$z' = Az + Bu, \quad z(0) = z_0, \quad y_{obs}(t) = Hz(t) \in \mathbb{R}^p,$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times p}$. We assume that A is stable.

Let $\mathcal{T} \in \mathcal{L}(\mathbb{R}^n)$ be invertible, and set $\widetilde{z} = \mathcal{T}z$. With transformation $\mathcal{T}$, the system becomes

$$\widetilde{z}' = \widetilde{A}\widetilde{z} + \widetilde{B}u, \quad \widetilde{z}(0) = \widetilde{z}_0, \quad y_{obs}(t) = \widetilde{H}\widetilde{z}(t) \in \mathbb{R}^p,$$

with

$$\widetilde{A} = \mathcal{T} A \mathcal{T}^{-1}, \quad \widetilde{B} = \mathcal{T} B, \quad \widetilde{H} = H \mathcal{T}^{-1}.$$

The extended controllability Gramians for (A, B) is

$$\mathcal{P} = \int_0^\infty e^{\tau A} BB^* e^{\tau A^*} d\tau,$$

and the extended observability Gramians for (A, H) is

$$\mathcal{Q} = \int_0^\infty e^{\tau A^*} H^* H e^{\tau A} d\tau.$$

Since $\sigma(\widetilde{A}) = \sigma(A)$, $\widetilde{A}$ is also stable.

The extended controllability Gramians for $(\widetilde{A}, \widetilde{B})$ is

$$\widetilde{\mathcal{P}} = \int_0^\infty e^{\tau \widetilde{A}} \widetilde{B} \widetilde{B}^* e^{\tau \widetilde{A}^*} \, d\tau = \mathcal{T} \mathcal{P} \mathcal{T}^*,$$

and the extended observability Gramians for $(\widetilde{A}, \widetilde{H})$ is

$$\widetilde{\mathcal{Q}} = \int_0^\infty e^{\tau \widetilde{A}^*} \widetilde{H}^* \widetilde{H} e^{\tau \widetilde{A}} d\tau = \mathcal{T}^{-*} \mathcal{Q} \mathcal{T}^{-1}.$$

The *balancing transformation* consists in finding $\mathcal{T} \in \mathcal{L}(\mathbb{R}^n)$, invertible, such that

$$\widetilde{\mathcal{P}} = \widetilde{\mathcal{Q}}.$$

In that way the degree of controllability of a state z_0, that is the ratio

$$d_c(z_0) = \frac{\left(\widetilde{\mathcal{P}} z_0, z_0\right)_Z}{\|z_0\|_Z^2},$$

is equal to its degree of observability

$$d_o(z_0) = \frac{\left(\widetilde{\mathcal{Q}} z_0, z_0\right)_Z}{\|z_0\|_Z^2}.$$

To find a balancing transformation, we can use the Cholesky factor of $\mathcal{P}$

$$\mathcal{P} = V\,V^*,$$

and the eigenvalue decomposition of $V^*\mathcal{P}V$

$$V^*\mathcal{P}V = W\,\Sigma^2\,W^*.$$

Assume that A is stable, (A, B) is controllable, (A, H) is observable, then a balancing transformation is given by

$$\mathcal{T} = \Sigma^{1/2}W^*\,V^{-1} \quad \text{and} \quad \mathcal{T}^{-1} = V\,W\,\Sigma^{-1/2}.$$

This transformation leads to

$$\mathcal{T}\mathcal{P}\,\mathcal{T}^* = \widetilde{\mathcal{P}} = \Sigma = \mathcal{T}^{-*}\mathcal{Q}\mathcal{T}^{-1} = \widetilde{\mathcal{Q}}.$$

2.2. Reduction of stable systems by balanced truncation

Let us assume that $(\widetilde{A}, \widetilde{B}, \widetilde{H})$ is a balanced realization of (A, B, H) and that the corresponding extended controllability and observability Gramians are equal and diagonal

$$\widetilde{\mathcal{P}} = \widetilde{\mathcal{Q}} = \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}.$$

Here Σ_1 and Σ_2 are two diagonal matrices corresponding to a partition of Σ.

With each partition (Σ_1, Σ_2) of Σ is associated a partition of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{H}$ as follows

$$\widetilde{A} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} \widetilde{H}_1 & \widetilde{H}_2 \end{pmatrix}.$$

Assume that $\Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ and $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \cdots, \sigma_n)$ and that $\sigma_{i-1} \geq \sigma_i > 0$ for $1 \leq i \leq n-1$ (the eigenvalues are repeated according to their multiplicity). Assume in addition that $\sigma_r > \sigma_{r+1}$.

Let $\widetilde{G} = G_{\widetilde{A},\widetilde{B},\widetilde{H}}$ be the transfer function of the system $(\widetilde{A}, \widetilde{B}, \widetilde{H})$, that is

$$\widetilde{G}(s) = \widetilde{H}(sI - \widetilde{A})^{-1}\widetilde{B},$$

and $\widetilde{G}_1 = G_{\widetilde{A}_{11},\widetilde{B}_1,\widetilde{H}_1}$ be the transfer function of the system $(\widetilde{A}_{11}, \widetilde{B}_1, \widetilde{H}_1)$

$$\widetilde{G}_1(s) = \widetilde{H}_1(sI - \widetilde{A}_{11})^{-1}\widetilde{B}_1.$$

Then

$$\|\widetilde{G} - \widetilde{G}_1\|_{\mathcal{H}_\infty} \leq 2 \sum_{\substack{i \geq r+1 \\ \text{without repetition}}} \sigma_i,$$

where

$$\|\widetilde{G}\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \max_j \sigma_j(\widetilde{G}(i\omega)),$$

$\sigma_j(\widetilde{G}) = \left(\lambda_j(\widetilde{G}^*\widetilde{G})\right)^{1/2}$ and λ_j denotes the j-th eigenvalue.

The substitution of the system (A, B, H) by $(\widetilde{A}_{11}, \widetilde{B}_1, \widetilde{H}_1)$ is called a model reduction by balanced truncation.

Let us set $\widetilde{z} = (\widetilde{z}_1, \widetilde{z}_2)$. The system for $\widetilde{z}$ is

$$\widetilde{z}_1' = \widetilde{A}_{11}\widetilde{z}_1 + \widetilde{A}_{12}\widetilde{z}_2 + \widetilde{B}_1 u + \widetilde{\mu}_1,$$
$$\widetilde{z}_2' = \widetilde{A}_{21}\widetilde{z}_1 + \widetilde{A}_{22}\widetilde{z}_2 + \widetilde{B}_2 + \widetilde{\mu}_2$$
$$y_{obs}(t) = \widetilde{H}_1\widetilde{z}_1(t) + \widetilde{H}_2\widetilde{z}_2(t) + \eta(t),$$

The reduced model is

$$\widetilde{z}_{1,e}' = \widetilde{A}_{11}\widetilde{z}_{1,e} + \widetilde{B}_1 u + \widetilde{L}_1(\widetilde{H}_1\widetilde{z}_1 - y_{obs}),$$

$$z' = Az + B\widetilde{K}_1\widetilde{z}_{1,e} + \mu, \quad z(0) = z_0 + \mu_0.$$

2.3. Reduction of unstable systems using balanced truncation for stable systems Rowley et al. 09, Sipp et al. 10.

Now we consider the unstable system

$$\begin{pmatrix} z_u \\ z_s \end{pmatrix}' = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix} + \begin{pmatrix} B_u \\ B_s \end{pmatrix} u.$$

The stable part is rewritten by using a balancing transformation. Introducing $B_s = \pi_s B$, $H_s = H\pi_s$,

$$\mathcal{P}_c = \int_0^\infty e^{tA_s} B_s B_s^* e^{tA_s^*}\, dt \quad \text{and} \quad \mathcal{Q}_o = \int_0^\infty e^{tA_s^*} H_s^* H_s e^{tA_s}\, dt,$$

the balancing transformation consists of finding a basis J in which $\mathcal{P}_c$ is diagonal, a basis K in which $\mathcal{Q}_o$ is diagonal, the two diagonals being equal and the two bases being bi-orthogonal

$$\mathcal{P}_c \mathcal{Q}_o = J\Sigma^2 \quad \text{and} \quad \mathcal{Q}_o \mathcal{P}_c = K\Sigma^2.$$

Let $(\widetilde{A}_s, \widetilde{B}_s, \widetilde{H}_s)$ be the corresponding balanced realization.

To each partition of $\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$ corresponds partitions of $\widetilde{A}_s$, $\widetilde{B}_s$, and $\widetilde{H}_s$, of the form

$$\widetilde{A}_s = \begin{pmatrix} \widetilde{A}_{s,11} & \widetilde{A}_{s,12} \\ \widetilde{A}_{s,21} & \widetilde{A}_{s,22} \end{pmatrix}, \quad \widetilde{B}_s = \begin{pmatrix} \widetilde{B}_{s,1} \\ \widetilde{B}_{s,2} \end{pmatrix}, \quad \widetilde{H}_s = (\widetilde{H}_{s,1} \ \widetilde{H}_{s,2}).$$

The reduced system is

$$\begin{aligned} z_u' &= A_u z_u + \pi_u B u, \qquad & z_u(0) &= \pi_u z_0, \\ \widetilde{z}_{s,1}' &= \widetilde{A}_{s,11}\widetilde{z}_{s,1} + \widetilde{B}_{s,1} u, \qquad & \widetilde{z}_{s,1}(0) &= \widetilde{z}_{0,s,1}, \end{aligned}$$

that is

$$\widehat{z} = \widehat{A}\widehat{z} + \widehat{B}u,$$

where

$$\widehat{z} = \begin{pmatrix} z_u \\ \widetilde{z}_{s,1} \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} A_u & 0 \\ 0 & \widetilde{A}_{s,11} \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} B_u \\ \widetilde{B}_{s,1} \end{pmatrix}.$$

As in the stable case, the control law has to be determined for the reduced system and the estimator is determined with the reduced system and the measure operator

$$\widehat{H} = \begin{pmatrix} H_u & \widetilde{H}_{s,1} \end{pmatrix}.$$

Advantage. The transfer function is well approximated.

Drawback. The measure $\widetilde{H}_{s,2}\widetilde{z}_{s,2}$ is not taken into account.

2.4. Spectral reduction

As before we start with the system

$$\begin{pmatrix} z_u \\ z_s \end{pmatrix}' = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix} + \begin{pmatrix} B_u \\ B_s \end{pmatrix} u.$$

We use the following decomposition of Z_s

$$Z_s = \sum_{j=1}^{K} G_{\mathbb{R}}(\lambda_j),$$

where Z_s is the stable subspace of $Z = \mathbb{R}^n$, $(\lambda_j)_{1\leq j\leq K}$ and $(\overline{\lambda}_j)_{1\leq j\leq K}$ are the eigenvalues of A_s and $G_{\mathbb{R}}(\lambda_j)$ are the associated real generalized eigenspaces.

We look for a decomposition of the form

$$Z_s = \sum_{j\in J} G_{\mathbb{R}}(\lambda_j) + \sum_{j\notin J} G_{\mathbb{R}}(\lambda_j) = Z_{s,1} \oplus Z_{s,2},$$

where the set $J \subset \{1, \cdots, K\}$ is selected so that $(\lambda_j)_{j\in J}$ and $(\overline{\lambda}_j)_{j\in J}$ correspond to the *most stabilizable (or controllable) and detectable (or observable) modes*.

If we introduce the variables $z_{s,1}$ and $z_{s,2}$ corresponding to the decomposition of z_s, the original system may be written as

$$\begin{pmatrix} z_u \\ z_{s,1} \\ z_{s,2} \end{pmatrix}' = \begin{pmatrix} A_u & 0 & 0 \\ 0 & A_{s,1} & 0 \\ 0 & 0 & A_{s,2} \end{pmatrix} \begin{pmatrix} z_u \\ z_{s,1} \\ z_{s,2} \end{pmatrix} + \begin{pmatrix} B_u \\ B_{s,1} \\ B_{s,2} \end{pmatrix} u,$$

with $A_{s,1} = \pi_1 A_s$, $A_{s,2} = \pi_2 A_s$, $B_{s,1} = \pi_1 B_s$, $B_{s,2} = \pi_2 B_s$, π_1 is the projection onto Z_1 along Z_2 and π_2 is the projection onto Z_2 along Z_1.

We can choose

$$\begin{pmatrix} z_u \\ z_{s,1} \end{pmatrix}' = \begin{pmatrix} A_u & 0 \\ 0 & A_{s,1} \end{pmatrix} \begin{pmatrix} z_u \\ z_{s,1} \end{pmatrix} + \begin{pmatrix} B_u \\ B_{s,1} \end{pmatrix} u,$$

as reduced system.

Advantage. We can a priori determine a feedback control law K, for e.g. for the unstable system. If $A_u + B_u K \pi_u$ is stable, the reduced closed loop system

$$\begin{pmatrix} A_u & 0 \\ 0 & A_{s,1} \end{pmatrix} + \begin{pmatrix} B_u \\ B_{s,1} \end{pmatrix} K \pi_u$$

is also stable.

The estimator may be determined for the reduced system

$$\begin{pmatrix} z_{u,e} \\ z_{s,1,e} \end{pmatrix}' = \begin{pmatrix} A_u & 0 \\ 0 & A_{s,1} \end{pmatrix} \begin{pmatrix} z_{u,e} \\ z_{s,1,e} \end{pmatrix} + \begin{pmatrix} B_u \\ B_{s,1} \end{pmatrix} u$$
$$+\widehat{L}(H_u z_{u,e} + H_{s,1} z_{s,1,e} - y_{obs}),$$

where $\widehat{L}$ is a filtering operator for the system

$$\widehat{A} = \begin{pmatrix} A_u & 0 \\ 0 & A_{s,1} \end{pmatrix}, \qquad \widehat{H} = \begin{pmatrix} H_u z_u & H_{s,1} \end{pmatrix}$$

where $H_u = H\pi_u$ and $H_{s,1} = H\pi_1\,\pi_s$.

Drawback. There is no common criteria to detect the most stabilizable and the most detectable modes since the realization is not balanced.

An empirical criterion has to be defined (see e.g. Bagheri etal. 2009 or Akervik etal. 2007).

3. Model reduction of infinite dimensional systems

We are in the case when

$$Z = Z_u \oplus Z_s,$$

where Z_u is of finite dimension, while Z_s is of infinite dimension.

We denote by $\mathbb{A}$, $\mathbb{A}_u$, and $\mathbb{A}_s$, a finite element approximation of A, A_u, and A_s respectively.

We can assume that $\mathbb{Z}_u$ the F.E.M. approximation of Z_u is accurate, while for Z_s we have

$$Z_s = Z_{s,a} + Z_{s,r} \quad \text{and} \quad \mathbb{Z}_s = \mathbb{Z}_{s,a} + \mathbb{Z}_{s,r},$$

where $\mathbb{Z}_{s,a}$ is a finite dimensional subspace of $\mathbb{Z}_s$, invariant by $\mathbb{A}_s$, while $Z_{s,r}$ is of infinite dimension and is badly approximated by the finite element model $\mathbb{Z}_{s,r}$, which is of finite but huge dimension. Thus the approximate model is a dynamical system on the space

$$\mathbb{Z} = \mathbb{Z}_u + \mathbb{Z}_s.$$

Associated with this decomposition we introduce three projection operators $\pi_{u,f}$, $\pi_{s,f}$ and π_r, where $\pi_{u,f}$ is the projection onto $Z_{u,f}$ parallel to the sum of the two other ones, and so on.

The vector spaces $\mathbb{Z}_u$, $\mathbb{Z}_{s,a}$, $\mathbb{Z}_{s,r}$ are invariant under $\mathbb{A}$ (the F.E. approximation of A), and that $\pi_{u,f}A$ is unstable while $\pi_{s,f}A$ is stable. Typical dimensions are $2 \leq \dim\mathbb{Z}_u \leq 20$ and $\dim\mathbb{Z}_s \simeq 100$.

For the 1D heat equation on $[0,1]$ with Homogeneous Dirichlet B.C., the eigenvalues of $\Delta = \frac{d^2}{dx^2}$ are

$$\lambda_j = -\pi^2 j^2.$$

The approximate values (by a P_1-F.E.M.) are

$$\lambda_{j,n} = \frac{4}{h^2} \sin^2\left(\frac{j\pi}{2(n+1)}\right) = \frac{4}{h^2} \sin^2\left(\frac{j\,\pi\,h}{2}\right), \quad h = \frac{1}{n+1}.$$

Thus

$$\left|\lambda_j - \lambda_{j,n}\right| \leq \frac{1}{6}\lambda_j^2\, h^2.$$

Thus for $j = 100$, we have

$$\left|\lambda_j - \lambda_{j,n}\right| \leq \frac{1}{6}\lambda_j^2\, h^2 \leq 10^{-2} \quad \text{if} \quad h \leq \frac{1}{\sqrt{2}}10^{-3}.$$

Since the approximation $Z_{s,r}$ by $\mathbb{Z}_{s,r}$ is bad, we definitely forget that part. We would like to construct an estimator for the projected system living in the space $\mathbb{Z}_u \oplus \mathbb{Z}_{s,a}$.

When we look for a filtering gain of the form

$$L(H_u z_{u,e} + H_s z_{s,a,e} - y_{obs}),$$

it means that we neglect

$$H_{s,2}\, z_{s,r}.$$

Is the estimator accurate if we neglect high frequency observations ?

Let us make some calculations for the k-th mode of the 1D heat equation. We re-use the model of Lecture 2

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1)\times(0,\infty),$$
$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t \in (0,\infty),$$
$$z(x,0) = z_0(x) \quad \text{in } (0,1).$$

We choose
$$Hz(t) = \frac{\partial z}{\partial x}(0,t).$$

Recall that
$$\lambda_j = -\pi^2 j^2, \quad e_j(x) = \sqrt{2}\sin(j\pi x),$$

and that we use the expansion

$$z(x,t) = \sum_{j=1}^{\infty} z_j(t)\, e_j(x).$$

We would like to answer the following unrealistic situation.

• Assume that the model is exact, except for the mode k, which is the only one for which there is an error. Thus the estimation problem reduces to estimate only z_k.

• We would like to know if, for k large enough, the filtering gain corresponding to this frequency may be neglected.

Since we estimate only z_k, the Riccati equation in the space $E(\lambda_k)$ is of the form

$$\mathcal{P}_k > 0, \quad A\mathcal{P}_k + \mathcal{P}_k A^* - \mathcal{P}_k H^* R_o^{-1} H\mathcal{P}_k + Q_o = 0.$$

We can look for $\mathcal{P}_k = P_k\, e_k$, $P_k \in \mathbb{R}^+$.

The equation for z_k is

$$z_k'(t) = -\pi^2 k^2 z_k(t) + u(t)e_k'(1) = -\pi^2 k^2 z_k(t) + u(t)k\pi\sqrt{2}(-1)^k.$$

The measure is

$$H(z_k(t)\, e_k) = z_k(t) He_k = z_k(t)\, e_k'(0) = z_k(t)\, k\pi\sqrt{2}.$$

Assume that we have to construct an estimator with $Q_o = I_{L^2(0,1)}$ and $R_o = 1$.

We are going to estimate $L_k H(z_k\, e_k)$, where L_k is the estimator for the equation satisfied by z_k. We know that $L_k = -P_k H^*$, where P_k is the solution to

$$P_k \in \mathbb{R}^+, \quad -2\pi^2 k^2 P_k - P_k^2 (He_k)^2 + 1 = 0.$$

Thus

$$P_k^2 + P_k - \frac{1}{2\pi^2 k^2} = 0,$$

that is

$$\left(P_k + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{2\pi^2 k^2}}\right)\left(P_k + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2\pi^2 k^2}}\right) = 0.$$

The only positive solution is

$$P_k = \sqrt{\frac{1}{4} + \frac{1}{2\pi^2 k^2}} - \frac{1}{2} = \frac{1}{2}\left(\sqrt{1 + \frac{2}{\pi^2 k^2}} - 1\right) = \frac{1}{2\pi^2 k^2} + O\left(\frac{1}{k^4}\right).$$

The estimator equation for $z_k e_k$ is

$$z_k' e_k = A(z_k e_k) + B_k u - P_k H^*(H z_k e_k - y_{obs}).$$

We would like to know if $P_k H^* H e_k$ is negligible. We have

$$(P_k H^* H e_k, e_k)_{L^2(0,1)} = P_k (H e_k)^2 \simeq \frac{1}{2\pi^2 k^2} 2\pi^2 k^2 = 1.$$

If we neglect $P_k H^* H e_k$, the convergence of the estimator follows from the exponential decrease of z_k, but we cannot say that $P_k H^* H e_k$ is negligible.

If we replace $(Q_o e_k, e_k)$ by q_k and R_o by r_k, the equation for P_k is

$$P_k \in \mathbb{R}^+, \quad -2\pi^2 k^2 P_k - 2\pi^2 k^2 \frac{1}{r_k} P_k^2 + q_k = 0.$$

The only positive solution to the equation

$$P_k^2 + r_k\, P_k - \frac{q_k\, r_k}{2\pi^2\, k^2} = 0,$$

is

$$P_k = \sqrt{\frac{r_k^2}{4} + \frac{q_k\, r_k}{2\pi^2\, k^2}} - \frac{r_k}{2} = \frac{r_k}{2}\left(\sqrt{1 + \frac{q_k}{\pi^2\, k^2}} - 1\right) = \frac{r_k\, q_k}{2\pi^2\, k^2} + O\left(\frac{r_k}{k^4}\right).$$

In that case

$$(P_k H^* H e_k, e_k)_{L^2(0,1)} = P_k (H\, e_k)^2 \simeq \frac{r_k\, q_k}{2\pi^2\, k^2}\, 2\pi^2\, k^2 = r_k\, q_k.$$

Assuming that

$$r_k\, q_k \longrightarrow 0 \quad \text{as} \quad k \to \infty,$$

corresponds to a regularity assumption on the noises.

• The estimator is not accurate for high frequencies.

Why the filtering equation provides a stabilizing feedback with partial information ?

• It is because high high frequencies are naturally damped.

Why reduction by balanced truncation works well ?

• One reason is that all the numerical experiments are done for the same discrete model.

What can be the remedy ?

• Try to choose Q_o in the filtering equation so that in the term $L\,y_{obs}$, the image by L of the observation coming from high frenquencies are small.

4. A new approach for finding an estimator of finite dimension

We start with the system split as follows

$$z_u' = A_u z_u + \pi_u B u, \qquad z_u(0) = \pi_u z_0,$$
$$z_s' = A_s z_s + \pi_s B u, \qquad z_s(0) = \pi_s z_0.$$

where $\pi_u : Z \mapsto Z_u$, $\pi_s = I - \pi_u$,

$$A_u = \pi_u A, \quad B_u = \pi_u B, \quad A_s = \pi_s A, \; B_s = \pi_s B.$$

We choose the best control law for the unstable system

$$K = -B_u^* P_u = -B^* \pi_u^* P_u,$$

and the best estimator for the unstable system

$$L = -P_e H_u^* R^{-1}, \quad H_u = H\pi_u,$$

with $P_e \in \mathcal{L}(Z_u^*, Z_u)$

$$P_e = P_e^* \geq 0, \quad P_e A_u^* + A_u P_e - P_e H_u^* R^{-1} H_u P_e + Q = 0,$$

and

$$H_u = H\pi_u, \quad H_s = H\pi_s.$$

At that stage, the estimator is of the form

$$z'_{u,e} = A_u z_{u,e} + B_u K z_{u,e} - P_e H_u^* R^{-1}(H_u z_{u,e} + H_s z_{s,e} - y_{obs}),$$
$$z_{u,e}(0) = \pi_u z_{0,e} = \pi_u z_0,$$
$$z'_{s,e} = A_s z_{s,e} + B_s K z_{u,e}, \qquad z_s(0) = \pi_s z_{0,e} = \pi_s z_0.$$

The estimator is still of infinite dimension.

To obtain a finite dimensional estimator, we use the exact expressions of the projectors π_u and π_u^* to determine $P_e H_u^* R^{-1} H_s z_{s,e}$.

There exists a basis $(e_i)_{1\le i\le N}$ of Z_u and a basis $(\phi_i)_{1\le i\le N}$ of Z_u^*, which are bi-orthogonal and

$$\pi_u f = \sum_{i=1}^{N} (f, \phi_i) e_i, \quad \pi_u^* f = \sum_{i=1}^{N} (f, e_i) \phi_i.$$

We have

$$z_{s,e}(t) = z_{s,e}^1(t) + z_{s,e}^2(t) = e^{tA_s} \pi_s z_0 + \int_0^t e^{(t-\tau)A_s} B_s K z_{u,e}\, d\tau.$$

We have to determine

$$P_e H_u^* R^{-1} H_s z_{s,e}(t) = P_e H_u^* R^{-1} H_s z_{s,e}^1(t) + P_e H_u^* R^{-1} H_s z_{s,e}^2(t).$$

We have

$$P_e H_u^* R^{-1} H_s z_{s,e}^1(t) = P_e H_u^* R^{-1} H_s e^{tA_s} \pi_s z_0 = \sum_{i=1}^{N} (\pi_s z_0, \mathcal{E}_i(t))\, P_e \phi_i,$$

where

$$\mathcal{E}_i(t) = e^{tA_s^*} H^* R^{-1} H e_i, \quad \text{for } 1 \leq i \leq N.$$

For $z_{s,e}^2$, we write

$$P_e H_u^* R^{-1} H_s z_{s,e}^2(t) = \sum_{j=1}^{N} \sum_{i=1}^{N} \int_0^t \big(z_{u,e}(\tau), E_{i,j}(t-\tau)\phi_j\big)\, P_e \phi_i \, d\tau,$$

with, for $1 \leq i,\ j \leq N$,

$$E_{i,j}(t) = \Big(R^{-1} H e_i, H\, e^{tA_s} B_s B_u^* P_u e_j\Big) \quad \in \mathbb{R}.$$

We choose the estimator

$$z'_{u,e} = A_u z_{u,e} + B_u K z_{u,e} - P_e H_u^* R^{-1}(H_u z_{u,e} - y_{obs})$$
$$- \sum_{j=1}^{N}\sum_{i=1}^{N}\int_0^t \left(z_{u,e}(\tau), E_{i,j}(t-\tau)\phi_j\right) P_e\phi_i \, d\tau - \sum_{i=1}^{N}(\pi_s z_0, \mathcal{E}_i(t))\, P_e\phi_i,$$
$$z_{u,e}(0) = \pi_u z_{0,e} = \pi_u z_0.$$

Therefore, the estimator is an integro-differential equation.

The system for (z_u, e_u), with $e_u = z_u - z_{u,e}$, is

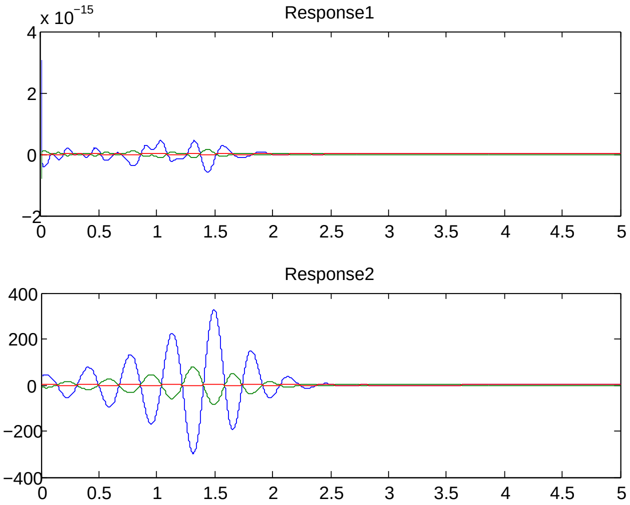
$$\begin{pmatrix} z_u \\ e_u \end{pmatrix}' = \begin{pmatrix} A_u + B_u K_u & -B_u K_u \\ 0 & A_u - P_e H_u^* H_u \end{pmatrix}\begin{pmatrix} z_u \\ e_u \end{pmatrix}$$
$$+ \begin{pmatrix} \mu \\ L H_s e_s + L\eta \end{pmatrix},$$
$$z_u(0) = \pi_u z_0, \quad e_u(0) = \pi_u \mu_0.$$

Thus we can obtain the exponential stabilization of the system coupling the feedback control law and the estimator that we have determined.

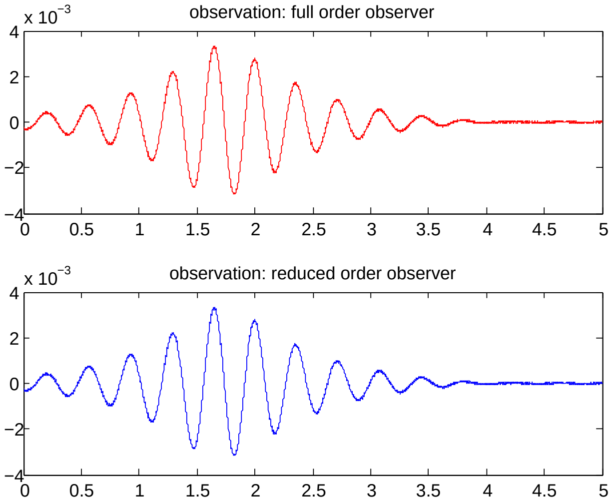
Tests - Cylinder - Boundary Measure of the vorticity

Time interval needed for the convolution w. r. to time

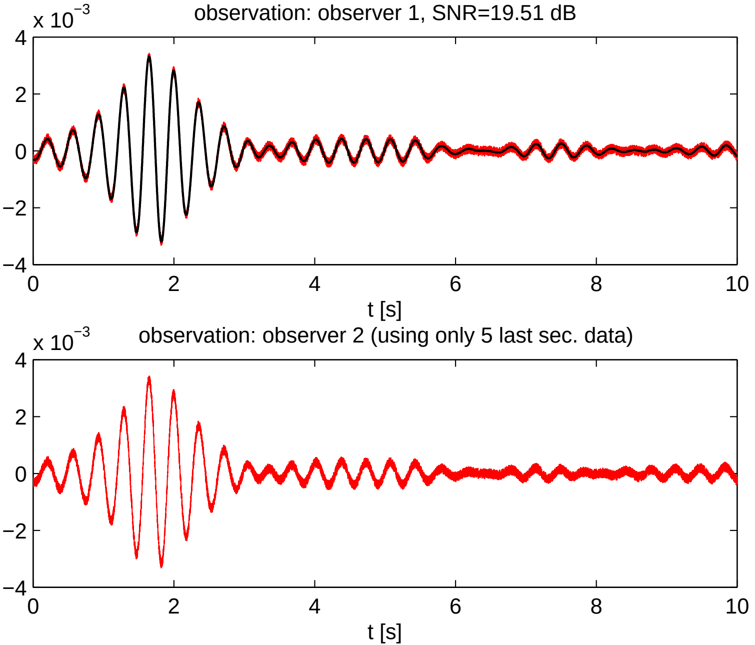
Representation of $P_e H_u^* R^{-1} H_s z_{s,e}^1(t)$ and $P_e H_u^* R^{-1} H_s z_{s,e}^2(t)$.



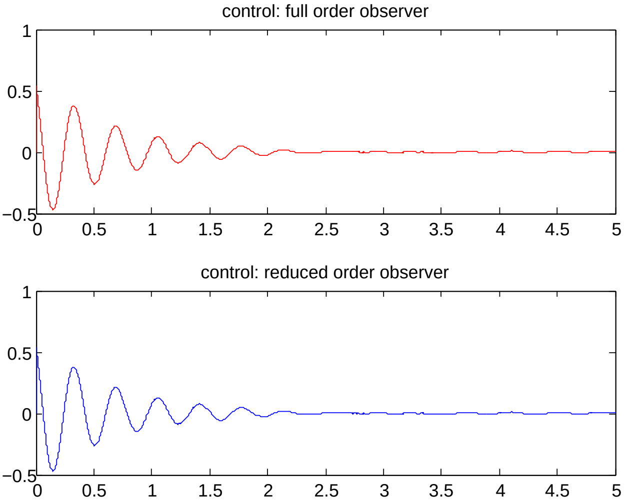
Comparison between the reduced observer and the exact one for the L.N.S.E.



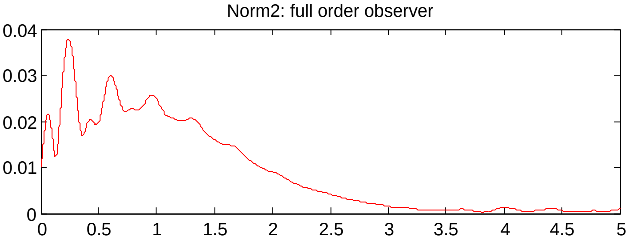
Comparison with a higher level of noise and pressure measurements

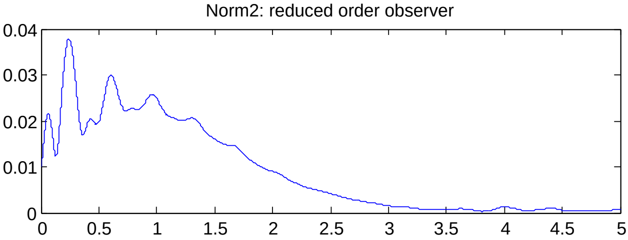


Comparison of control norms determined with the reduced and exact observers



Comparison of solution norms determined with the reduced and exact observers





References

J.M. Schumacher, Schumacher, J. M., A direct approach to compensator design for distributed parameter systems,, SIAM J. Control and Optimization, 21 (1983), 823-836.

J.M. Schumacher, Dynamic feedback in finite and infinite dimensional linear systems. Mathematical Center Tracts, No. 143, Mathematisch Centrum, Amsterdam 1981.

R.F. Curtain, Finite-dimensional compensators design for parabolic distributed systems with point sensor and boundary input. IEEE Trans. Automat. Control, AC-26, pp 98 - 104, 1982.

R.F. Curtain, Stabilization of boundary control distributed systems via integral dynamic output feedback of a finite-dimensional compensator. 5th International Conference on Analysis and Optimization of Systems, INRIA, Versaille, France, December 1982.

N. Fujii, Feedback stabilization of distributed parameters systems by a functional observer. SIAM J. Control and Optimization,vol. 18, No. 2, pp. 108 - 120, 1980.

Banks, H. T., Smith, R. C., Wang, Y., Smart Material Structures, modeling, estimation and control Masson/John Wiley, Paris/Chichester, 1996.

Burns, J. A., Marrekchi, H. , Optimal fixed-finite-dimensional compensator for Burgers' equation with unbounded input/output operators, Computation and Control III, (Bozeman, MT, 1992), pp. 83-104.

Curtain, R., Salamon, D., Finite dimensional compensators for infinite dimensional systems with unbounded input operators, SIAM J. Control and Optimization, 24 (1986), pp. 797-816.

Ji, G. and Lasiecka, I., Partially observed analityc systems with fully unbounded actuators and sensors-FEM algorithms, Comp. Optim. and Applic., 11 (1998), p. 111-136.

Lasiecka, I., Galerkin approximations of infinite dimensional compensators for flexible structures with unbounded control action, Acta Appl. Math., 28 (1992), pp. 101-133.

Lasiecka, I., Finite element approximations of compensator design for analytic generators with fully unbound controls/observations, SIAM J. Control and Optimization, 33 (1995), pp. 67-88.