Relativistic Fluids Tutorial # 4

To improve the accuracy of Godunov's method, we can use a higher-order reconstruction. In this tutorial, we will add a piecewise-linear reconstruction to our code that evolves Burger's equation.

A computational grid has points labeled with integer indices, x_{i-1}, x_i, x_{i+1}, ..., which are the centers of computational cells. The boundaries of the computational cell centered at x_i are at $x_{i-1/2}$ and $x_{i+1/2}$. The reconstruction of the conserved variables $\mathbf{u}$ in each cell has the form

$$\boldsymbol{u}^{\ell}_{i+1/2} = \boldsymbol{u}_i + \boldsymbol{\sigma}_i(r_{i+1/2} - r_i), \tag{1}$$

$$\boldsymbol{u}^{r}_{i+1/2} = \boldsymbol{u}_{i+1} + \boldsymbol{\sigma}_{i+1}(r_{i+1/2} - r_{i+1}), \tag{2}$$

where $\boldsymbol{\sigma}_i$ is a "limited" slope that satisfies the TVD property. Different algorithms for calculating the "limited slopes" are available, and four common methods are listed below. First we define slopes at the cell boundary and center

$$\mathbf{s}_{i+1/2} = \frac{\boldsymbol{u}_{i+1} - \boldsymbol{u}_i}{x_{i+1} - x_i}, \qquad \mathbf{s}_i = \frac{\boldsymbol{u}_{i+1} - \boldsymbol{u}_{i-1}}{x_{i+1} - x_{i-1}}. \tag{3}$$

Some TVD slope limiters are

1. **minmod**

$$\boldsymbol{\sigma}_i = \text{minmod}\,(\mathbf{s}_{i-1/2}, \mathbf{s}_{i+1/2}), \tag{4}$$

where

$$\text{minmod}\,(a,b) = \begin{cases} \text{sgn}\,(a)\,\min(\,|a|, |b|\,) & \text{if sgn}\,(a) = \text{sgn}\,(b), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The sign function, sgn (a), is

$$\text{sgn}\,(a) \equiv \begin{cases} +1 & a > 0 \\ -1 & a < 0. \end{cases} \tag{6}$$

2. **The monotonized central-difference limiter (MC)**

$$\boldsymbol{\sigma}_i = \text{minmod}\,\left(\mathbf{s}_i,\, 2\mathbf{s}_{i-1/2},\, 2\mathbf{s}_{i+1/2}\right), \tag{7}$$

where

$$\text{minmod}\,(a,b,c) = \begin{cases} \text{sgn}\,(a)\,\min\,(\,|a|, |b|, |c|\,) & \text{if sgn}\,(a) = \text{sgn}\,(b) = \text{sgn}\,(c) \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

3. **van Leer limiter**

$$\boldsymbol{\sigma}_i = \frac{|\mathbf{s}_{i+1/2}| - |\mathbf{s}_{i-1/2}|}{|\mathbf{s}_{i+1/2}| + |\mathbf{s}_{i-1/2}|}. \tag{9}$$

If the denominator is zero, then $\boldsymbol{\sigma}_i = 0$.

4. **Superbee limiter**

$$\boldsymbol{\sigma}_i = \text{maxmod}\,(\boldsymbol{\sigma}_i^{(1)}, \boldsymbol{\sigma}_i^{(2)}), \tag{10}$$

where

$$\boldsymbol{\sigma}_i^{(1)} = \text{minmod}\,(\mathbf{s}_{i+1/2},\, 2\mathbf{s}_{i-1/2}), \tag{11}$$

$$\boldsymbol{\sigma}_i^{(2)} = \text{minmod}\,(2\mathbf{s}_{i+1/2},\, \mathbf{s}_{i-1/2}), \tag{12}$$

and

$$\text{maxmod}\,(a,b) = \begin{cases} \text{sgn}\,(a)\,\max(\,|a|,|b|\,) & \text{if sgn}\,(a) = \text{sgn}\,(b), \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

To integrate the equations we use an integrator that preserves the TVD property. Huen's method is the optimal 2nd order method

$$\mathbf{q}^{(1)} = \mathbf{q}^n + \triangle t\, L(\mathbf{q}^n), \tag{14}$$

$$\mathbf{q}^{n+1} = \frac{1}{2}(\mathbf{q}^n + \mathbf{q}^{(1)}) + \frac{1}{2}\triangle t\, L(\mathbf{q}^{(1)}). \tag{15}$$

$\triangle t$ is limited by a Courant (CFL) condition.