A STUDY GUIDE FOR MY MINI-COURSE: QUANTITATIVE GEOMETRY OF HYPERBOLIC 3-MANIFOLDS

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My mini-course will be devoted to the techniques that have been developed in my joint work with Culler and others for studying the relationship between classical topological invariants of a 3-manifold and such quantitative invariants as its volume, diameter, minimal injectivity radius, and Margulis number. Much of this work involves the $\log(2k-1)$ theorem, a quantitative result about the geometry of free Kleinian groups, which was proved in a restricted version in [7] (for the case $k = 2$) and in [3] (for all $k \geq 2$). The final version of the theorem depends on the Marden conjecture, which was proved by Agol (unpublished) and independently by Calegari and Gabai in [4]. In my first lecture I will explain the statement and proof of the $\log(2k-1)$ theorem. I will begin with a brief discussion of the predecessor of the $\log 3$ theorem that was proved in [22], and with a discussion of the classical Banach-Tarski paradox, which the proof of the $\log(2k-1)$ theorem parallels in a surprising way. A general reference for the Banach-Tarski paradox is [23]. I will also discuss Patterson-Sullivan measures, which are used in an essential way in the proof. A general reference for the theory of such measures is [18].

In my second lecture, I will discuss a variety of applications of the $\log(2k-1)$ theorem to comparisons between topological and quantitative-geometry invariants. Some of the results that I will touch on were proved in [7]; in [3] (building on techniques from [8]); in [6] (which involves a refined version of the $\log 3$ theorem); in [13] (which builds on results proved in [10] and [11]); in [1] (which builds on techniques from [3] and [9]); and in [12], [19]; [2] and its successor [14]; [5] (which involves a variant of the $\log(2k-1)$ theorem for Kleinian groups that are not free); [15], [16], [17], [20], and [21].

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