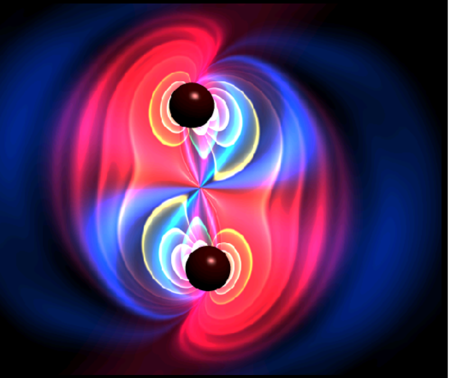

Measuring Gravitational Waves in Numerical Simulations

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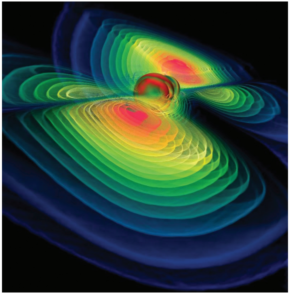
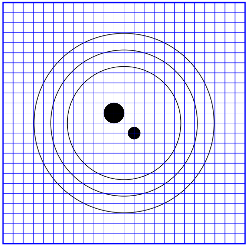
Measuring gravitational waves in a simulation



- Numerical relativity remains the best approximation to allow us to model strong-field gravity.
  - BH-BH, BH-NS, NS-NS mergers.
  - Late inspiral — last dozen orbits.
- Dynamics of the near-zone can be modelled with high accuracy.
- The quantities that we are trying to model, however, are gravitational waves
  - How can we define GWs in the near zone?
  - How can physical effects be disentangled from coordinates?
- Techniques for measuring GWs within a simulation are called **wave extraction**

Quick Review of Numerical Relativity

- Numerical relativity tends to treat spacetime as a succession of snapshots
  - Spacelike slices.
  - Define coordinate $t =$ const. on a slice.
  - Spacing Δt between successive slices.



- Each slice knows the variables:

  g_{ab}: 3-metric in the slice
  K_{ab}: extrinsic curvature of slice
  α, β^a: Lapse, shift (gauge).

- The Einstein equations evolve one snapshot to the next.

- The spatial size of a slice is finite: limited by computational resources (memory, available time):
  - Increasing the domain can be expensive, so we'd like to make measurements as close to the source as possible.
  - <u>But</u>, "gravitational waves" are only defined in the far-zone.

Gravitational waves

- Einstein equations:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = k T_{\alpha\beta}$$

- Linearize around flat space: Assume a metric of the form

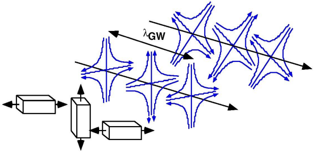
$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \; ,$$

  where $\eta_{\alpha\beta}$ is the flat-space (Minkowski) metric, and $h_{\alpha\beta}$ is a small perturbation.

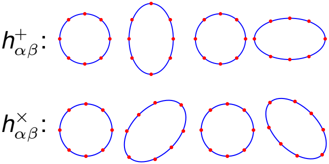
- Substitute into Einstein equations, discard terms nonlinear in $h_{\alpha\beta}$.

$$\Box h_{\alpha\beta} = 16\pi T_{\alpha\beta} \; ,$$

  where $\Box = -\partial_t^2 + \nabla$ is the flat-space d'Alembertian.

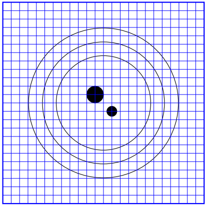


- GWs are quadrupolar.
- Two modes: h_+, $h_\times$.



Gravitational Waves

- To measure GWs locally we need to have some idea of the background we are perturbing.
- To measure h_+ and $h_\times$, we need to be in the transverse-traceless gauge.
- For BH mergers, we have neither in the domain immediately around the sources
  - Need to work in the "wave zone" – somewhat removed from the source.



- Two techniques are standard for finite radius measurement:

Perturbative extraction:	NCSA (1990s), AEI, SXS collab.
Newman-Penrose ψ_4:	Baker et al. 2002, Everybody.

- Alternatively, GWs can be defined asymptotically at $\mathcal{J}^+$
  - The trouble is, how to get there?
- **Characteristic extraction**: Use Einstein equations to transport local data to $\mathcal{J}^+$.

Newman-Penrose wave extraction

- The Riemann tensor can be invariantly decomposed into trace-free parts:

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - (g_{\alpha[\gamma} R_{\beta]\delta} - g_{\alpha[\delta} R_{\beta]\gamma}) + \frac{1}{6}(g_{\alpha[\gamma} g_{\beta]\delta} - g_{\alpha[\delta} g_{\beta]\gamma}) R$$

  where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor.

- In vacuum, $R_{\alpha\beta} = 0$, $R = 0$, so that

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$$

- Weyl tensor has 10 independent components.

Newman-Penrose wave extraction

- Project the Weyl tensor onto a null tetrad, $\{\ell, n, m, \bar{m}\}$:

$$\ell = \frac{1}{\sqrt{2}}(\hat{t} + \hat{r})\,, \qquad n = \frac{1}{\sqrt{2}}(\hat{t} - \hat{r})\,,$$

$$m = \frac{1}{\sqrt{2}}(\hat{\theta} + i\hat{\phi})\,, \qquad \bar{m} = \frac{1}{\sqrt{2}}(\hat{\theta} - i\hat{\phi})\,.$$

- The 10 independent components are 5 complex-valued scalars:

$$\begin{aligned}
\psi_0 &= C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta\,, \\
\psi_1 &= C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \ell^\gamma \bar{m}^\delta\,, \\
\psi_2 &= C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \bar{m}^\gamma n^\delta\,, \\
\psi_3 &= C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \bar{m}^\gamma n^\delta\,, \\
\psi_4 &= C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta\,.
\end{aligned}$$

- Asymptotically, these fall-off as:

$$C_{\alpha\beta\gamma\delta} \simeq \frac{\psi_4}{r} + \frac{\psi_3}{r^2} + \frac{\psi_2}{r^3} + \frac{\psi_1}{r^4} + \frac{\psi_0}{r^5}$$

- The gravitational radiation measured by distant observers is ψ_4.

$$\ddot{h}_+ - i\,\ddot{h}_\times = \frac{1}{r}\psi_4\,.$$

Perturbative extraction

- Method assumes a background Schwarzschild metric, $g^0_{\alpha\beta}$:

$$g_{\alpha\beta} = g^0_{\alpha\beta} + h_{\alpha\beta},$$

where $g^0_{\alpha\beta}$ corresponds to

$$ds^2 = -(1 - M/r)^{-1}\, dt^2 + (1 - M/r)\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

- The perturbations are expanded in a basis of Regge-Wheeler harmonics:

$$h_{\alpha\beta} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}_{\alpha\beta}\,.$$

- These variables can be used to define first order gauge-invariant variables:

$Q^+_{\ell m}$:	even-parity mass multipoles,
$Q^\times_{\ell m}$:	odd-parity current multipoles

- Related to the GW strain by:

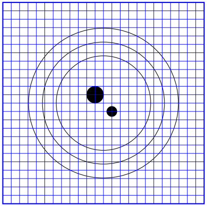
$$h_+ - ih_\times = \frac{1}{\sqrt{2}r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(Q^+_{\ell m} - i \int Q^\times_{\ell m}\, dt \right) {}_{-2}Y^{\ell m}\,.$$

Finite radius extraction

- Both methods require waves to be measured at some distance from the source:
  - Newman-Penrose: Requires the peeling property ($1/r$ fall-off in ψ_4)
  - Perturbative: Requires a known background

- Typically we measure geometric variables on a topological $r = \text{const.}$ sphere around the source

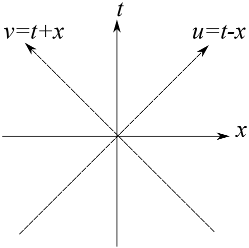
- Several finite radius spheres are chosen, e.g. from $r = 100M$ to $r = 200M$.
  - Inner bound set by above requirements.
  - Outer bound set by available resolution and possible grid boundary effects.

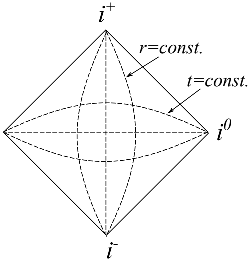


- Results are fit to the expected $1/r$ fall-off, and extrapolated in r to get the result for distant observers

Null compactification

- GWs are difficult to measure locally without a known background.
- At large radii, GWs can be defined unambiguously for asymptotically flat spacetimes.
- 1960s: Bondi, Sachs, Penrose and collaborators:
  - Rigorous description of null infinity, $\mathcal{J}^+$.
  - Definition of mass, radiated energy ("news function") at $\mathcal{J}^+$.
  - Einstein equations in null coordinates.





Compactify u and v, define:

$$U = \arctan(u)\,, \qquad V = artctan(v)\,.$$

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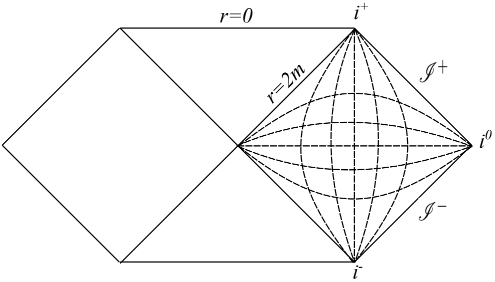

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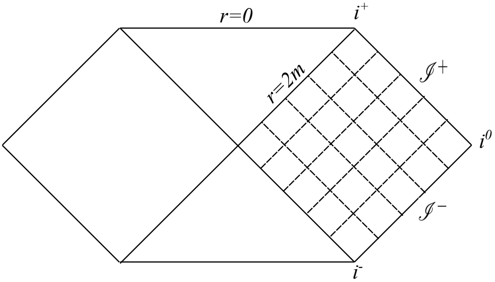

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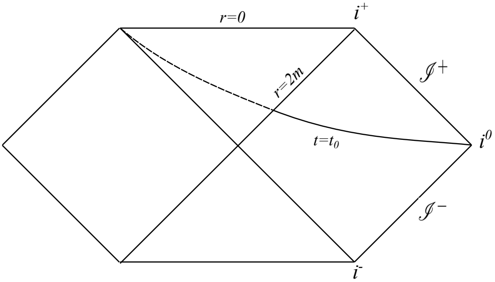

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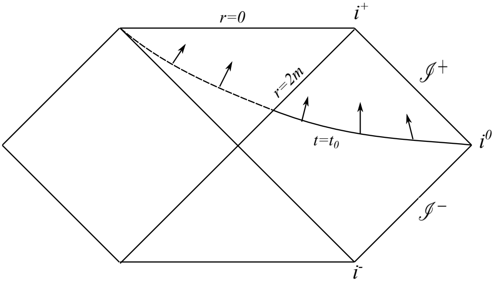

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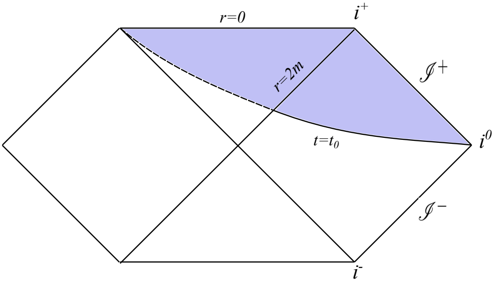

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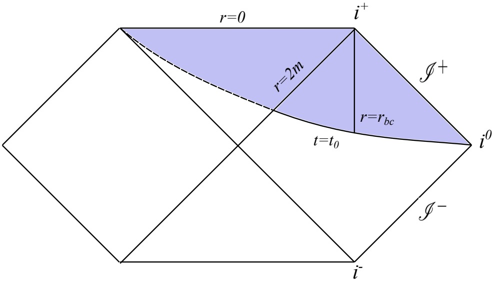

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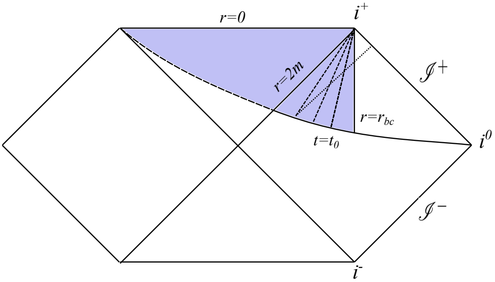
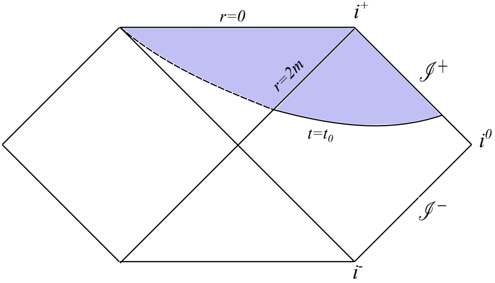


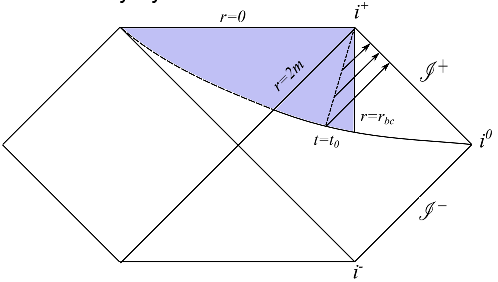
Hyperboloidal Slices

- **Foliate spacetime by spacelike slices that intersect $\mathcal{J}^+$**
- Can be specified by conditions on initial extrinsic curvature, gauge
- Slices are asymptotically null or spacelike at $\mathcal{J}^+$
- Formalisms worked out by Friedrich, Rinne, Zenginoglu
- Still some aspects to be worked out:
  - Initial data
  - Gauges
  - Regularization at $\mathcal{J}^+$



Characteristic extraction

- **Transport data from a standard 3+1 evolution to $\mathcal{J}^+$ using the Einstein equations in null coordinates**
- Use the Bondi null formulation of the Einstein equations
- Inner boundary data given by $3 + 1$ evolution
- Relies on radial null geodesics to define coordinates:
  - Need to be careful of caustics
  - Not suited to very dynamical domain



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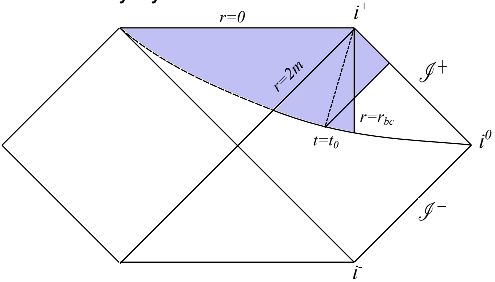

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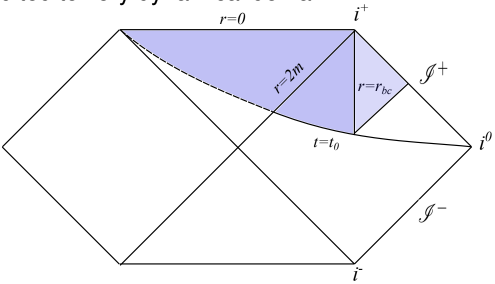
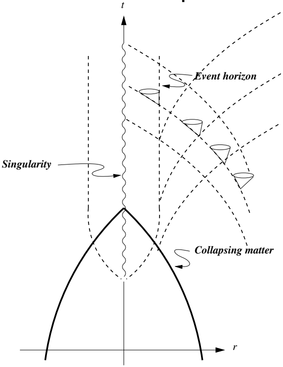


Einstein in a Bondi frame

- Characteristic evolutions make use of a null formulation of the Einstein equations.



- Main idea: Coordinatize spacetime along null geodesics
- This leads to a number of advantages:
  - Spacetime can be compactified.
  - Einstein equations reduce to a simple heirarchy.
  - Miminal number of variables.
  - Asymptotic boundary conditions are purely outgoing.
  - Well defined energy at $\mathscr{I}$.

- General form of the metric [Bondi et al. 1962]:

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right) du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dy^A + r^2 h_{AB} dy^A dy^B$$

Bondi line element

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right) du^2 - 2e^{2\beta}du dr$$
$$- 2r^2 h_{AB}U^B du d\theta^A + r^2 h_{AB} d\theta^A d\theta^B.$$

Coordinates:

- u labels a family of null hypersurfaces: $k_\alpha = -\partial_\alpha u$ is normal to $u =$ constant surfaces, and
$$g^{\alpha\beta}k_\alpha k_\beta = 0.$$
- $\theta^A = (\theta^1, \theta^2)$ are angular coordinates labelling outgoing null geodesics which generate the surfaces.
- r is an areal radius running along each generator.

Bondi line element

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$$- 2r^2 h_{AB} U^B du d\theta^A + r^2 h_{AB} d\theta^A d\theta^B.$$

- In these coordinates, the line element is parameterized by six functions of the coords:
$$\beta, \quad V, \quad U^A, \quad h_{AB}$$
- h_{AB} is the conformal geometry of 2-surfaces defined by constant u, r spheres, and satisfies:
$$\det(h_{AB}) = \det(q_{AB}), \qquad h^{AC} h_{CB} = \delta^A{}_B,$$
with q_{AB} the unit sphere metric.
- The two independent components of h_{AB} represent the radiative degrees of freedom in the spacetime.
  - In later equations, replaced by J — complex-valued scalar, spin-weight 2.

Bondi line element

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right) du^2 - 2e^{2\beta} du dr \\ - 2r^2 h_{AB} U^B du d\theta^A + r^2 h_{AB} d\theta^A d\theta^B.$$

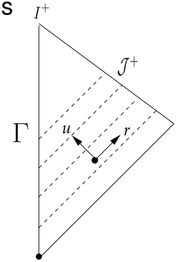
- In these coordinates, the line element is parameterized by six functions of the coords:

$$\beta, \quad V, \quad U^A, \quad h_{AB}$$

- V is an analogue of a Newtonian potential.
- The scalar β measures the expansion of the light cone between the asymptotic frame and the world tube.
- U^A are angular shift components.

Bondi evolution system

- Heirarchy of equations
  - Hypersurface equations: integrated radially along null slices
  - Evolution equations: Evolve data to the next slice



- Hypersurface equations:

$$\beta_{,r} = N_\beta \,,$$

$$(r^2 Q)_{,r} = -r^2(\bar{\eth} J + \eth K)_{,r} + 2r^4 \eth \left(r^{-2}\beta\right)_{,r} + N_Q \,,$$

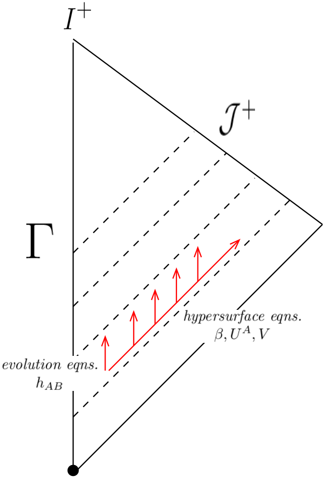
$$U_{,r} = r^{-2} e^{2\beta} Q + N_U \,,$$

$$(r^2 \hat{W})_{,r} = \frac{1}{2} e^{2\beta} \mathcal{R} - 1 - e^{\beta} \eth \bar{\eth} e^{\beta} + \frac{1}{4} r^{-2} \left(r^4 \left(\eth \bar{U} + \bar{\eth} U\right)\right)_{,r} + N_W \,.$$

- Evolution equations:

$$2\,(rJ)_{,ur} = \left(r^{-1} V\,(rJ)_{,r}\right)_{,r} - r^{-1}\left(r^2 \eth U\right)_{,r} + 2r^{-1} e^{\beta} \eth^2 e^{\beta} - \left(r^{-1} W\right)_{,r} J + N_J \,.$$

Evolution Scheme



Boundary data supplied on the initial $u =$ constant slice, and at Γ.

1. Given h_{AB} on a null slice Σ.

2a. On Σ, solve for β:

$$(\beta)_{,r} = \frac{1}{16} r h^{AC} h^{BD} h_{AB,r} h_{CD,r}.$$

2b. Solve for U^A:

$$(r^2 Q_A)_{,r} = F_Q(h_{AB}, \beta),$$
$$(U_A)_{,r} = r^{-2} e^{2\beta} Q_A.$$

2c. Solve for V:

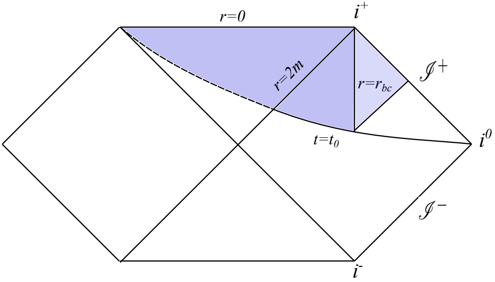
$$(V)_{,r} = F_V(h_{AB}, \beta, U_A).$$

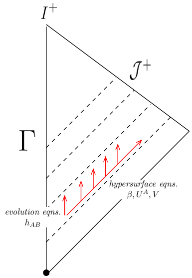
3. Evolve h_{AB} to the next slice using:

$$(r h_{AB})_{,ur} = F_H(h_{AB}, \beta, U_A, V).$$

4. Repeat.

Coupling 3+1 to Null Evolutions





- Metric data from a standard 3+1 evolution is stored on an $r =$ const. world-tube Γ.
  - Stored in file as time series of spherical harmonic coefficients
  - Store ADM variables $(g_{ab}, K_{ab}, \alpha, \beta^i)$.
- Null evolution code using data at Γ as inner boundary data.
  - Change of variables.
  - Locate Γ in Bondi coordinates.
- Null evolution currently a post-processing step:
  - One-way transfer of information $\rightarrow$ *extraction*

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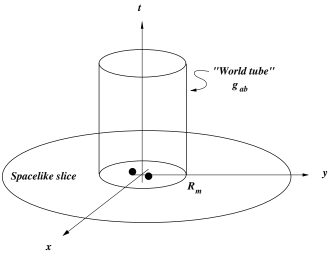

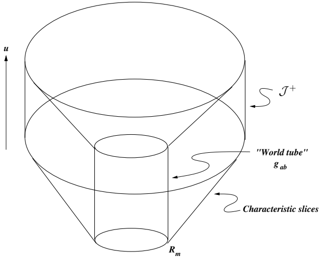

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Measurements at $\mathcal{J}^+$

- As $r \to \infty$, the Bondi variables go to zero with known fall-offs.
- In particular,

$$J = 0 + \frac{J_1}{r} + O(\frac{1}{r^2})\,.$$

- We define the gravitational "news function" by

$$\mathcal{N} \;=\; \frac{1}{2}(J_1)_{,u} \;=\; -\lim_{r\to\infty}\frac{1}{2}r^2 J_{,ur}\,.$$

- The Bondi mass-loss formula is:

$$\frac{dm}{du} = \int_{\mathcal{J}^+} |\mathcal{N}|^2$$

- The Newman-Penrose quantity ψ_4 is related to the news by:

$$\mathcal{N}_{,u} = -\frac{1}{2}\bar{\psi}_4$$

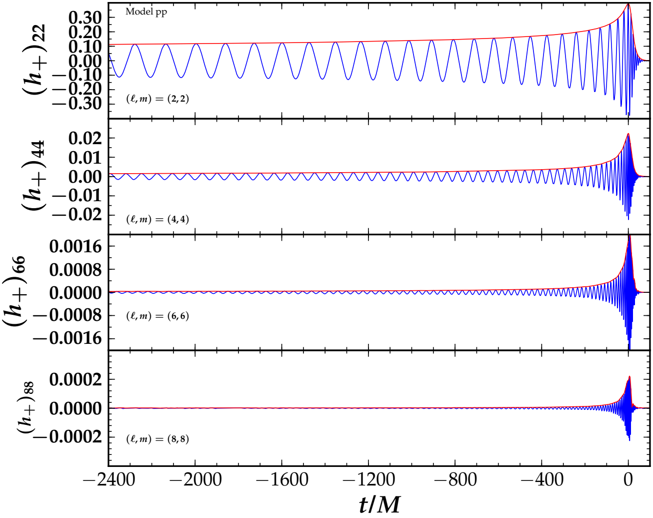
Summary of numerical GW measures

	Variables	Relation to h
Perturbative	Q^+, $Q^\times$	$Q^+ + \int Q^\times \, dt$
Newman-Penrose	ψ_4	$\int \int \psi_4 \, dt \, dt$
Bondi news	$\mathcal{N}$	$\int \mathcal{N} \, du$

- In each case, at least one integration is required to get the strain.
- Integration of noisy time-series can be problematic.
  - Results in spurious drifts that need to be removed.

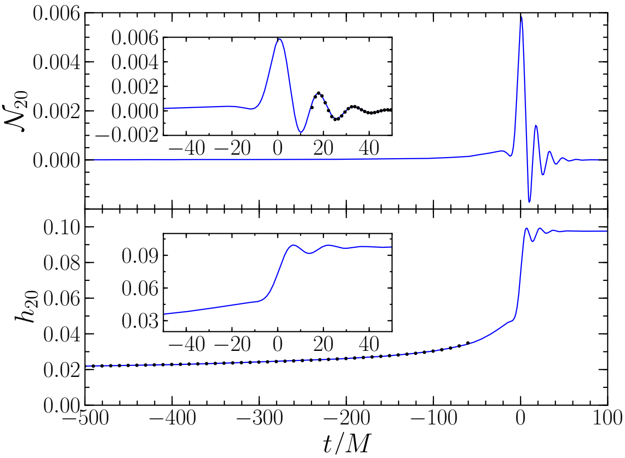
Some Characteristic Extraction Results

Binary BH: $m_1 = m_2$, $s_1 = +0.8$ $s_2 = +0.4$



Gravitational memory $(\ell, m) = (2, 0)$ mode

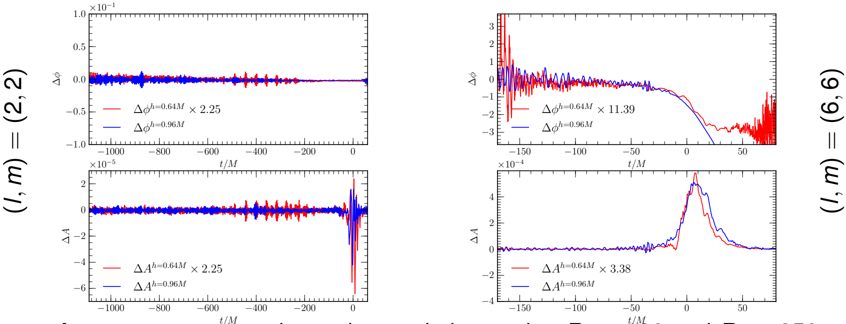
Non-spinning, equal-mass binary:



- Integration constant for *h* determined by fit to PN estimate.

GWs independent of world-tube radius

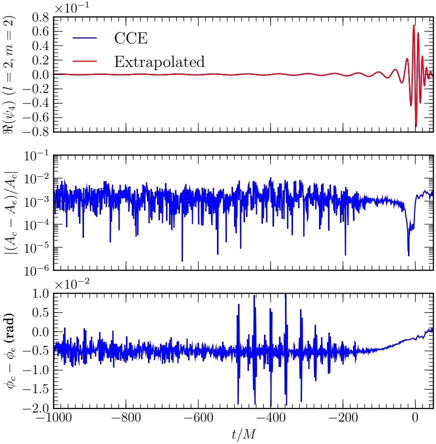
- A key feature of characteristic extraction is that the results should be indepdendent of the world-tube radius:
  - The full Einstein equations are used over the entire domain (no linearized assumption)
  - Bondi coordinates at $\mathscr{I}$ are invariantly defined



- A convergence test shows that evolutions using $R_\Gamma = 100$ and $R_\Gamma = 250$ produce identical results up to numerical truncation error.

Comparison with finite radius measurements

- We can estimate the error of finite radius ψ_4 measurements by comparing with CCE results.



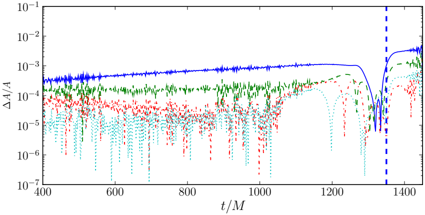
- Extrapolation using radii: $r = 300M - 1000M$ from source
- Max. amplitude diff: 1.08%
- Max. phase diff: 0.019rad
- This is good news for numrel $\longrightarrow$ standard extrapolation techniques are quite accurate
- Note: Finite radius measurements usually carried out within $r = 200M$
  - This can increase error by order of magnitude.
- Observed errors are larger than the discretization errors for this resolution.

Extrapolation error from small radii

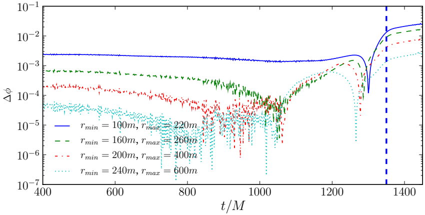
Experiment:

- Measure ψ_4 at $r = 1000M$ from the source
  - (on a *large* grid where this radius is not influenced by outer boundary)
- Compare with smaller radius extrapolations to estimate the wave at $r = 1000M$

Amplitude exrapolation error



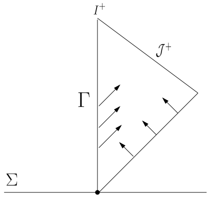
Phase extrapolation error



Initial data for characteristic evolutions

- A remaining potential inconsistency is the specification of data on the initial null cone

- Information travelling *inwards* from the past can influence the measured signal



- For 3+1 codes, initial data corresponds to the solution of elliptic constraint equations, typically under assumption of conformal flatness
- In the characteristic domain, conformal flatness corresponds to a simple prescription:

$$J = 0$$

- Perhaps we can do better by imposing an outgoing wave solution on the initial characteristic slice

A linearized characteristic solution

- Bishop (2005) developed a linearized characteristic solution, representing purely outgoing waves:

$$\beta_{2,\nu}(r) = b_1 \quad \text{(constant)}$$

$$j_{2,\nu}(r) = (12b_1 + 6i\nu c_1 + i\nu^3 c_2)\frac{\sqrt{6}}{9} + \frac{2\sqrt{6}c_1}{r} + \frac{\sqrt{6}c_2}{3r^3}$$

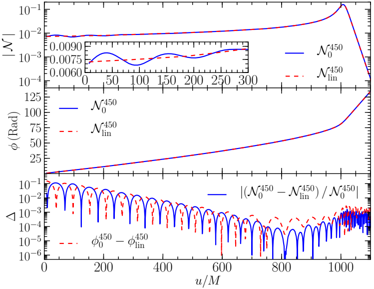
$$u_{2,\nu}(r) = \sqrt{6}\left(\frac{\nu^4 c_2 + 6\nu^2 c_1 - 12i\nu b_1}{18} + \frac{2b_1}{r} + \frac{2c_1}{r^2} - \frac{2i\nu c_2}{3r^3} - \frac{c_2}{2r^4}\right)$$

$$w_{2,\nu}(r) = r^2\frac{12i\nu b_1 - 6\nu^2 c_1 - \nu^4 c_2}{3} + r\frac{-6b_1 + 12i\nu c_1 + 2i\nu^3 c_2}{3} + 2\nu^2 c_2 - \frac{2i\nu c_2}{r} - \frac{c_2}{r^2}$$

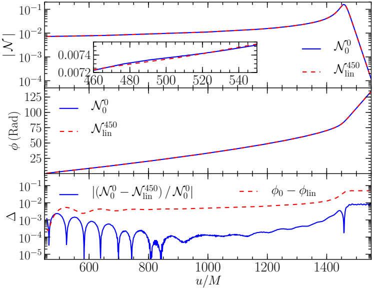
- The free constants are fixed by the known 3+1 data at the world-tube
- Purely outgoing solution can be matched with the GW signal at the world-tube, Γ [Bishop, DP, Reisswig 2011]:
  - Evaluate “junk” radiation in null initial data
  - Diagnose radiation content of initial 3+1 slice

Initial data on the world tube

- Model problem: Equal mass non-spinning binary



Linearized vs. Conf. flat data — initialized after junk radiation



Conf. flat (before junk) vs. Linearized (after junk)

- Initial data on the null side can have an influence.
- But consistent data seem to agree pretty well, i.e.:
  - 3+1 conformally flat? $\rightarrow$ use $J = 0$.
  - 3+1 wavey (after junk)? $\rightarrow$ use linearized wave J.
- Comparison with linearized solution suggests a component of incoming radiation in 3+1 initial data — lasts up to $800M$.

Summary

- Numerical relativity evolutions involve a number of systematic errors beyond discretization.
- One of these is the extrapolation error due to finite radius measurements.
- For extrapolation radii $< 200M$, this error may be significantly larger than achievable discretization accuracies.
- Evaluating waves at $\mathscr{I}^+$ is one way to reduce this ambiguity.