Computational Fluid Dynamics in Relativity

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Introduction

Table of Contents

1. Conservation Laws
2. Scalar equations
3. Systems of Equations
4. Finite Volume Numerical Methods for Scalar Equations

Conservation Laws

- Consider a fluid flowing through a region of space in one dimension. Assume that the quantity U in the fluid is conserved, and that $u(x,t) = U/V$ is a local density. Consider the fluid in a small cell at $x \in [x_1, x_2]$.
- The conserved quantity $U(t)$ in the cell at time t is

$$U(t) = \int_{x_1}^{x_2} \mathrm{d}x\, u(x,t). \tag{1}$$

- A conservation law can be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} \mathrm{d}x\, u(x,t) = f\left(u(x_1,t)\right) - f\left(u(x_2,t)\right) \tag{2}$$

where $f(u)$ is a flux function.

Conservation Laws

- A formal solution can be written

$$\begin{aligned}\int_{x_1}^{x_2} \mathrm{d}x\, u(x,t_2) &= \int_{x_1}^{x_2} \mathrm{d}x\, u(x,t_1) \\ &\quad + \int_{t_1}^{t_2} \mathrm{d}t\, f\left(u(x_1,t)\right) - \int_{t_1}^{t_2} \mathrm{d}t\, f\left(u(x_2,t)\right)\end{aligned} \tag{3}$$

- If $u(x,t)$ and $f(u)$ are differentiable, then we can exchange the order of integration and differentiation, to derive the differential form

$$\int_{t_1}^{t_2} \mathrm{d}t\, \mathrm{d}x \left\{ \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} f\left(u(x,t)\right) \right\} = 0. \tag{4}$$

XXX

- Define the average value of u in a cell on the interval $I_i = \left[x_{i-1/2}, x_{i+1/2}\right]$.

$$\bar{u}_i = \frac{1}{x_{i+1/2} - x_{i-1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{d}x \, u(x) \tag{5}$$

- For uniform cells of width $\triangle x = x_{i+1} - x_i$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_i = -\frac{1}{\triangle x}\left\{f\left(u(x_{i+1/2}, t)\right) - f\left(u(x_{i-1/2}, t)\right)\right\} \tag{6}$$

Key Ideas

Key Ideas

- Conservation laws (integral formulation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \mathrm{d}x\, u(x,t) = f\left(u(x_1,t)\right) - f\left(u(x_2,t)\right) \tag{7}$$

- The differential form

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x} f\left(u(x,t)\right) = 0 \tag{8}$$

- This is the key equation for developing numerical methods

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_i = -\frac{1}{\triangle x}\left\{f\left(u(x_{i+1/2},t)\right) - f\left(u(x_{i-1/2},t)\right)\right\} \tag{9}$$

Table of Contents

Scalar Conservation Laws

- The linear advection equation is the simplest hyperbolic equation

$$u_t + a u_x = 0 \tag{10}$$

  where a is a real constant.

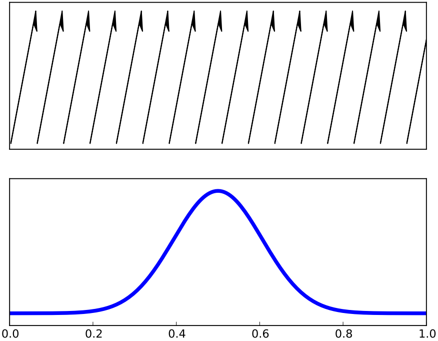
- The Cauchy problem is on the domain $-\infty < x < \infty$ and $t \geq 0$ with the initial conditions

$$u(x,0) = u_0(x). \tag{11}$$

- The solution is

$$u(x,t) = u_0(x - at) \tag{12}$$

Characteristics



- The lines $x'(t) = a$ and $x(0) = x_0$ are characteristics, and the solution is constant along these lines
- Differentiate $u(x,t)$ along the characterstics

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}u(x(t),t) &= \frac{\partial}{\partial t}u(x(t),t) + \frac{\partial}{\partial x}u(x(t),t)x'(t) \\ &= u_t + au_x \\ &= 0 \end{aligned} \tag{13}$$

Characteristics

A more general equation has the form

$$u_t + \left(a(x)u\right)_x = 0 \tag{14}$$

$$u_t + a(x)u_x = -a'(x)u \tag{15}$$

The characteristics are

$$\begin{aligned} x'(t) &= a(x(t)) \\ x(0) &= x_0 \end{aligned} \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t),t) = -a'(x(t))u(x(t),t) \tag{17}$$

Burgers' Equation

This simplest model equation for fluids with nonlinear and viscous effects is Burger's equation

$$u_t + uu_x = \epsilon u_{xx} \tag{18}$$

If we set the viscous coefficent $\epsilon = 0$, then we have the *inviscid* Burger's equation.
In conservation form, the inviscid Burger's equation is

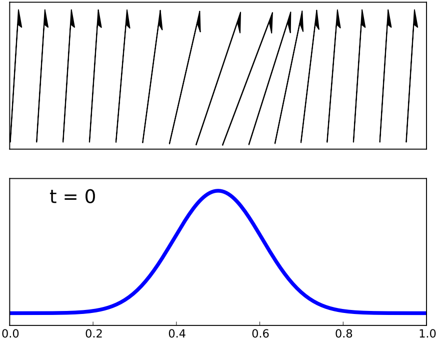
$$u_t + f(u)_x = 0 \tag{19}$$

with the flux function

$$f(u) = \frac{1}{2}u^2. \tag{20}$$

This equation is genuinely nonlinear, $f''(u) > 0$ for all u.

Burger's Equation



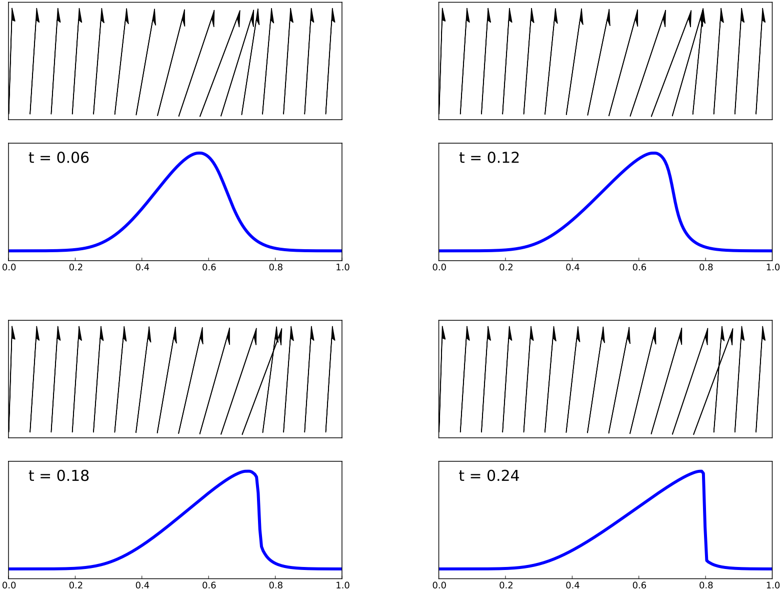
The characteristics for the inviscid Burger's equation satisfy

$$x'(t) = u(x(t), t) \tag{21}$$

and along each characteristic u is constant, as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u(x(t), t) &= \frac{\partial}{\partial t} u(x(t), t) + \frac{\partial}{\partial x} u(x(t), t) x' t() \\ &= u_t + u u_x \\ &= 0 \end{aligned} \tag{22}$$

Shock Formation



Weak Solutions

When the characteristics cross, the solution becomes discontinuous. At this point a classical solution to the PDE no longer exists. We generalize the possible solutions by introducing *weak solutions.*

Multiply $u_t + f(u)_x = 0$ by $\phi(x,t)$ and integrate over space and time

$$\int_0^\infty \int_{-\infty}^\infty \mathrm{d}x\mathrm{d}t\, \phi(x,t)\left[u_t + f(u)_x\right] \tag{23}$$

Integrate by parts

$$\int_0^\infty \int_{-\infty}^\infty \mathrm{d}x\mathrm{d}t\, \left[\phi_t u + \phi_x f(u)\right] = -\int_{-\infty}^\infty \mathrm{d}x\, \phi(x,0)u(x,0) \tag{24}$$

$u(x,t)$ is a *weak solution* of the conservation law of this equation holds for all $\mathcal{C}^1$ functions $\phi(x)$.

Weak Solutions

Let $u(x,t)$ be a piecewise smooth function. The it is a solution of the integral form of the conservation law if and only if

1. $u(x,t)$ is a classical solution in the domains where it is continuous.
2. The Rankine-Hugoniot jump conditions are satisfied across discontinuities

$$s(u_R - u_L) = f(u_R) - f(u_L) \tag{25}$$

where s is the propagation speed of the discontinuity.

Riemann Problem

The Riemann problem is a conservation problem with piecewise constant data with a single discontinuity. Consider the IVP for Burgers' equation

$$u(x,0) = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0. \end{cases} \tag{26}$$

ADD MORE HERE
The solutions to the Riemann problem are self-similar.

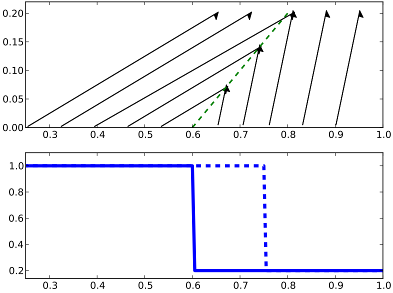
Riemann Problem: Case I

If $u_\ell > u_r$, there is a unique weak solution

$$u(x,t) = \begin{cases} u_\ell & x < st \\ u_r & x > st. \end{cases} \qquad (27)$$

where the shock speed s is

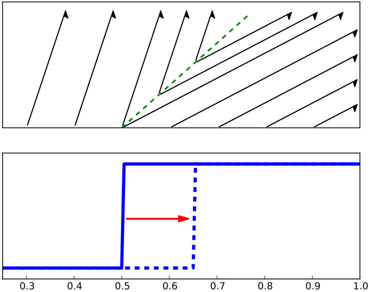
$$s = \frac{1}{2}\left(u_\ell + u_r\right). \qquad (28)$$



Note that characteristics cross at the shock. In the characteristic diagram, characteristics go into the shock.

Riemann Problem: Case II

If $u_\ell < u_r$, there are infinitely many weak solutions, including the previous solution

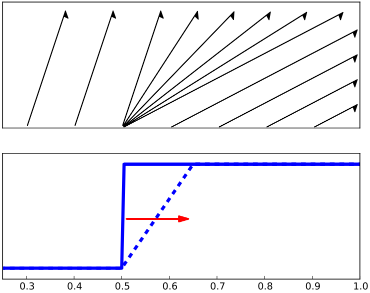


In this case, however, characteristics come out of the shock. Moreover, the solution is unstable to small perturbations. This is not the physical weak solution to Burger's equation.

Riemann Problem: Case II Rarefaction Wave

A stable solution for the case $u_\ell < u_r$ is the *rarefaction wave*

$$u(x,t) = \begin{cases} u_\ell & x < u_\ell t \\ x/t & u_\ell t \le x < u_r t \\ u_r & x > u_r t. \end{cases} \tag{29}$$



This solution is stable to perturbations, and it is the vanishing viscosity solution. This is then the physical solution.

Shock Speed

The shock speed can be determined by conservation. For the conservation equation $u_t + f(u)_x = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-M}^{M} \mathrm{d}x\, u(x,t) = f(u_\ell) - f(u_r). \tag{30}$$

We can evaluate the integral directly

$$\int_{-M}^{M} \mathrm{d}x\, u(x,t) = (M+st)u_\ell + (M-st)u_r \tag{31}$$

and differentiate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-M}^{M} \mathrm{d}x\, u(x,t) = s(u_\ell - u_r) \tag{32}$$

Combining, we have the *Rankine-Hugoniot Jump Condition*

$$s = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} = \frac{[f]}{[u]} \tag{33}$$

Entropy Condition

Weak solutions are not unique. The physical solution can be determined using either

- the vanishing viscosity procedure
- entropy conditions

Going into entropy conditions takes us too far afield, and they are often used when deriving new methods. The important message for now is that numerical methods can calculate solutions that are entropy violating and therefore unphysical. This usually is not a problem with central methods, but can be for other methods.

Table of Contents

1. Conservation Laws

2. Scalar equations

3. Systems of Equations

4. Finite Volume Numerical Methods for Scalar Equations

Linear Hyperbolic Systems

Consider the linear system in one dimension

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = 0 \tag{34}$$

$$\mathbf{u}(x,0) = u_0(x). \tag{35}$$

where $\mathbf{A}$ is a constant matrix. The system is hyperbolic if $\mathbf{A}$ is diagonalizable with real eigenvalues $\{\lambda_1, \lambda 2, \ldots, \lambda_N\}$

$$\mathbf{A} = \mathbf{R\Lambda R^{-1}} \tag{36}$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \tag{37}$$

Characteristic Variables for Linear Systems

We define the characteristic variables

$$\mathbf{v} = \mathbf{R}^{-1}\mathbf{u} \tag{38}$$

where $\mathbf{R}$ is a matrix of eigenvectors $\mathbf{r}_i$ of $\mathbf{A}$

$$\mathbf{R} = \left(\ \mathbf{r}_1 \,\middle|\, \mathbf{r}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{r}_N\ \right). \tag{39}$$

As $\mathbf{A}$ is a constant matrix, $\mathbf{R}$ is also a constant matrix. Multiply the linear equation by $\mathbf{R}^{-1}$

$$\mathbf{R}^{-1}\partial_t\mathbf{u} + \mathbf{R}^{-1}\mathbf{A}\partial_x\mathbf{u} = 0 \tag{40}$$

and then writing the equation in terms of characteristic variables decouples the equations into a set of scalar advection equations

$$\partial_t\mathbf{v} + \mathbf{\Lambda}\partial_x\mathbf{v} = 0 \tag{41}$$

with velocities λ_i.

Solution of the Linear System

Each decoupled characteristic variable has the solution

$$v_i(x,t) = v_i(x - \lambda_i t, 0) \tag{42}$$

where $\mathbf{v}(x,0) = \mathbf{R}^{-1}\mathbf{u}_0(x)$. The solution in terms of the original variables is

$$\mathbf{u}(x,t) = \mathbf{R}\mathbf{v}(x,t) \tag{43}$$

$$\mathbf{u}(x,t) = \sum_{i=1}^{N} v_i(x,t)\mathbf{r}_i \tag{44}$$

$$\mathbf{u}(x,t) = \sum_{i=1}^{N} v_i(x - \lambda_i t, 0)\mathbf{r}_i \tag{45}$$

The solution is a superposition of N waves, each advected independently with no change in shape. The ith wave has shape $v_i(x,0)\mathbf{r}_i$ and propagates with speed λ_i.

The Riemann Problem for Linear Systems

The Riemann problem is

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = 0 \tag{46}$$

with piece-wise constant initial data

$$\mathbf{u}(x,0) = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0. \end{cases} \tag{47}$$

For simplicity assume that the system is strictly hyperbolic, which means that the eigenvalues of $\mathbf{A}$ are real and distinct

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N. \tag{48}$$

We decompose the initial data as

$$\mathbf{u}_\ell = \sum_i \alpha_i \mathbf{r}_i, \qquad \mathbf{u}_r = \sum_i \beta_i \mathbf{r}_i \tag{49}$$

then

$$v_i(x,0) = \begin{cases} \alpha_i & x < 0 \\ \beta_i & x > 0. \end{cases} \tag{50}$$

Solution of the Riemann Problem

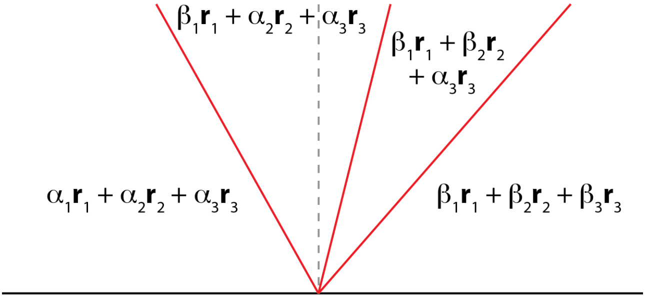
The solution of the Riemann problem is then

$$v_i(x,t) = \begin{cases} \alpha_i & x - \lambda_i t < 0 \\ \beta_i & x - \lambda_i t > 0 \end{cases} \tag{51}$$

At this point we can simply write the solution in terms of $\mathbf{u}(x,t) = \mathbf{R}\mathbf{v}(x,t)$. Let's develop a different form of the solution that is useful in developing HRSC numerical methods.

The diagram on the next slide shows a characteristics for a representative Riemann problem. It may be helpful to refer to this diagram in the following discussion.

Solution of the Riemann Problem II


$\beta_1\mathbf{r}_1 + \alpha_2\mathbf{r}_2 + \alpha_3\mathbf{r}_3$
$\beta_1\mathbf{r}_1 + \beta_2\mathbf{r}_2 + \alpha_3\mathbf{r}_3$
$\alpha_1\mathbf{r}_1 + \alpha_2\mathbf{r}_2 + \alpha_3\mathbf{r}_3$
$\beta_1\mathbf{r}_1 + \beta_2\mathbf{r}_2 + \beta_3\mathbf{r}_3$

The solution for $\mathbf{u}$ can be written in terms of α and β. Let $P(x,t)$ be the maximum value of p for which $x - \lambda_p t > 0$, then

$$\mathbf{u}(x,t) = \sum_{p=1}^{P(x,t)} \beta_p \mathbf{r}_p + \sum_{p=P(x,t)}^{N} \alpha_p \mathbf{r}_p. \tag{52}$$

The Wave Solution of the Riemann Problem

The jump across the ith-wave can be written

$$[\mathbf{u}] = (\beta_i - \alpha_i)\mathbf{r}_i. \tag{53}$$

These jumps satisfy the Rankine-Hugoniot conditions

$$[\mathbf{f}] = \mathbf{A}[\mathbf{u}] = (\beta_i - \alpha_i)\mathbf{A}\mathbf{r}_i = \lambda_i[\mathbf{u}]. \tag{54}$$

This jump, traveling with constant speed λ_i, is called the ith-wave. Finally, the solution for $\mathbf{u}$ can be written in terms of these jumps. This form of the solution is useful in developing HRSC methods.

$$\begin{aligned}\mathbf{u}(x,t) &= \mathbf{u}_\ell + \sum_{\lambda_i < x/t} (\beta_i - \alpha_i)\mathbf{r}_i & (55)\\ &= \mathbf{u}_r - \sum_{\lambda_i \geq x/t} (\beta_i - \alpha_i)\mathbf{r}_i & (56)\end{aligned}$$

Table of Contents

1. Conservation Laws

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Classical Finite Difference Schemes

To solve equations they must be discretized

Many different ways to discretize the equations can be used. Some of the simplest methods are based on the substitution of a finite difference approximation for the derivatives

Some classical FD schemes (two-level, explicit schemes) for the advection equation

$$\partial_t u + \partial_x u = 0 \tag{57}$$

are on the following slides.

Classical FD Schemes: FTBS

- **Forward Time-Backward Space (FTBS)**
  In this scheme a forward time discretization is used for the time derivative
  $$\partial_t f \approx (f(t+\triangle t) - f(t))/\triangle t,$$
  and a backward discretization is used for the spatial derivative
  $$\partial_x f \approx (f(x+\triangle x) - f(x))/\triangle x,$$
  The discrete equation is of order $\mathrm{O}(\triangle t, \triangle x)$, with the discrete equation

FTBS

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{\triangle x}(U_i^n - U_{i-1}^n). \qquad (58)$$

Classical FD Schemes: Lax-Friedrichs

- **Lax-Friedrichs (LxF)**
  A natural way to improve the accuracy of the FTBS method is to use a centered spatial derivative, giving the FTCS method

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{2\triangle x}(U_{i+1}^n - U_{i-1}^n). \tag{59}$$

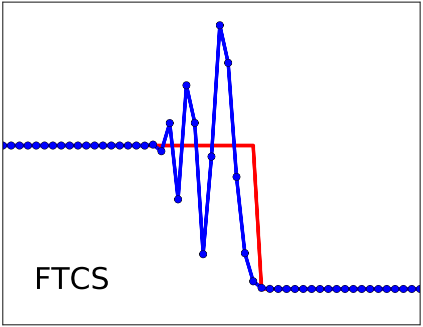
However, this method is unstable. The Lax-Friedrichs method replaces U_i^n on the RHS with the average value $\frac{1}{2}(U_{i-1}^n + U_{i+1}^n)$. The dissipation in this method stabilizes the discretization. While a centered spatial derivative is used, this method is also $\mathrm{O}(\triangle t, \triangle x)$.

Lax-Friedrichs

$$U_i^{n+1} = \frac{1}{2}(U_{i-1}^n + U_{i+1}^n) - \frac{\triangle t}{2\triangle x}(U_{i+1}^n - U_{i-1}^n) \tag{60}$$

FTCS Evolution

While the FTCS method appears to be a reasonable guess for the numerical flux function, unfortunately it is unstable. Here is a numerical solution plotted after only a few steps.



The numerical stencil includes points outside of the physical domain of dependence.

Classical FD Schemes: Lax-Wendroff

- **Lax-Wendroff**

  To get a more accurate scheme, we turn to the Lax-Wendroff method, which is based on the Taylor series expansion

$$u(x,t+\triangle t) = u(x,t) + \left.\frac{\partial u}{\partial t}\right|_{t} \triangle t + \frac{1}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t} (\triangle t)^2 + \cdots \qquad (61)$$

  Using the equation $\partial_t u = -\partial_x u$, we have $\partial_t^2 u = \partial_x^2 u$, giving

$$u(x,t+\triangle t) = u(x,t) + \left.\frac{\partial u}{\partial x}\right|_{t} \triangle t + \frac{1}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,t} (\triangle t)^2 + \cdots \qquad (62)$$

  Using centered derivatives, the discretization is $\mathrm{O}\left(\triangle t, (\triangle x)^2\right)$

Lax-Wendroff

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{2\triangle x}(U_{i+1}^n - U_{i-1}^n) + \frac{\triangle t^2}{2\triangle x^2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) \qquad (63)$$

Classical FD Schemes: Beam-Warming

- **Beam-Warming**
  The Beam-Warming method is obtained in the same way as the Lax-Wendroff method, except that second-order one-sided discretizations are used instead of centered derivatives.

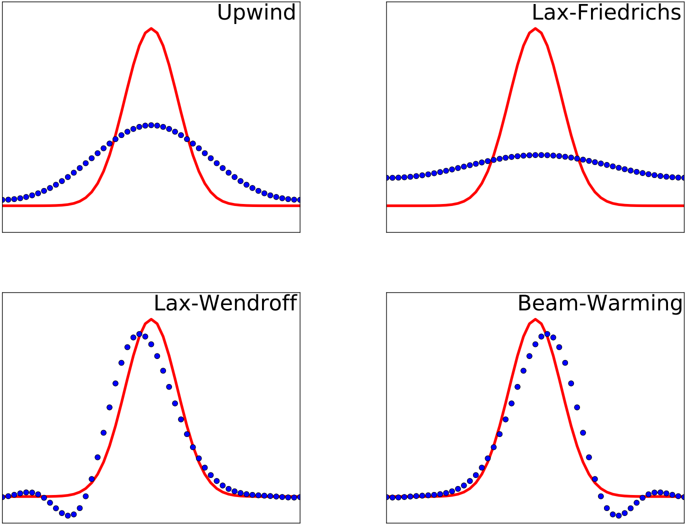
Beam-Warming

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{2\triangle x}(3U_i^n - 4U_{i-1}^n + U_{i-2}^n) + \frac{\triangle t^2}{2\triangle x^2}(U_i^n - 2U_{i-1}^n + U_{i-2}^n) \tag{64}$$

Performance of Classical Methods

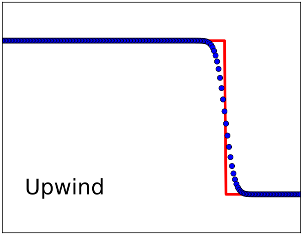
Classical methods work well for smooth solutions. They do not work well for discontinuous methods. Next two slides show examples.

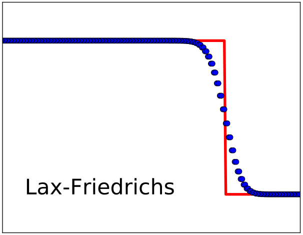
Classical Methods and Smooth Solutions

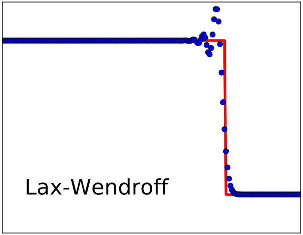


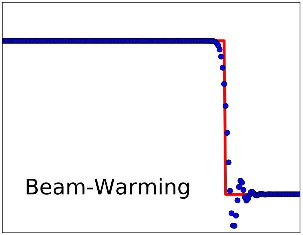
The exact and numerical solutions after 5 crossing times.

Classical Methods and Discontinuous Solutions









Performance of Classical Methods

- The first order classical methods capture the discontinuity, but have significant **diffusion.** The diffusion keeps the solution stable at the discontinuity, however it is everywhere. This degrades the solution in long runs, especially in smooth regions. So we look to higher-order methods.
- The second order methods, however, do not handle the discontinuity very well, showing **dispersion.** This is a type of Gibb's phenomenon, and the over- or under-shoots do not converge away. These methods are not suitable for nonlinear equations, as they generate higher harmonics, making the problem worse.
- Godunov introduced a new approach based on the conservation properties of the equations.

Godunov Methods

Godunov introduced a general method for using characteristic information in numerical algorithms to properly upwind the discrete derivative operators. Godunov's method uses a finite volume discretization that is ideally suited for conservation equations. Godunov's method is the basis of many modern numerical methods in fluid dynamics.

We define grid cells centered on the points x_i on the numerical grid

$$\mathcal{C}_i = (x_{i-1/2}, x_{i+1/2}). \tag{65}$$

In each grid cell, we compute the average value of the solution at time t_n, and approximate this average with the discrete value U_i^n

$$U_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{d}x\, u(x, t_n) \equiv \frac{1}{\Delta x} \int_{\mathcal{C}_i} \mathrm{d}x\, u(x, t_n) \tag{66}$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$.

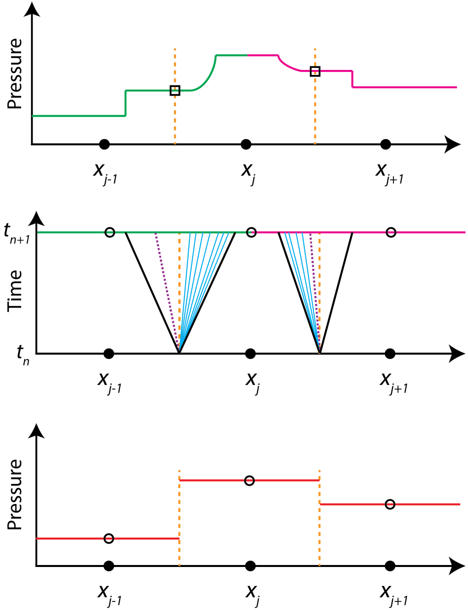
Godunov Algorithm

Godunov's method has 3 primary steps, abbreviated *REA:*

1. **Reconstruct:** Define a piece-wise consant function $\tilde{u}(x, t_n)$ with the value U_i^n in every cell $\mathcal{C}_i$.
2. **Evolve:** Use the functions $\tilde{u}(x, t_n)$ as initial data to solve the hyperbolic system forward in time $t_n < t \leq t_{n+1}$. The functions $\tilde{u}(x, t_n)$ are piecewise-constant functions, defining a Riemann problem at each cell interface $x_{i+1/2}$.
3. **Average:** Average the solution $\tilde{u}(x, t_{n+1})$ in each $\mathcal{C}_i$ to obtain the new approximate solution

$$U_i^{n+1} = \frac{1}{\triangle x} \int_{\mathcal{C}_i} \mathrm{d}x \, \tilde{u}(x, t_{n+1}). \tag{67}$$

Diagram for Godunov's Method



Godunov's method

In practice, the integral form of the conservation equation considerably simplies the implementation of Godunov's method.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_i} \mathrm{d}x\, u(x,t_n) = f(u(x_{i-1/2},t)) - f(u(x_{i+1/2},t)) \tag{68}$$

Integrate from time t_n to t_{n+1}, where $t_{n+1} = t_n + \triangle t$.

$$\begin{aligned}\int_{\mathcal{C}_i} \mathrm{d}x\, u(x,t_{n+1}) - \int_{\mathcal{C}_i} \mathrm{d}x\, u(x,t_n)&\\ = \int_{t_n}^{t_{n+1}} \mathrm{d}t\, f(u(x_{i-1/2},t)) &- \int_{t_n}^{t_{n+1}} \mathrm{d}t\, f(u(x_{i+1/2},t))\end{aligned} \tag{69}$$

Using the definitions of the cell averages we have

$$U_i^{n+1} = U_i^n - \int_{t_n}^{t_{n+1}} \mathrm{d}t\, f(u(x_{i-1/2},t)) - \int_{t_n}^{t_{n+1}} \mathrm{d}t\, f(u(x_{i+1/2},t)) \tag{70}$$

We next define the integrals on the RHS as numerical flux functions.

The Numerical Flux

Let $F^n_{i+1/2}$ be approximate the time integrals on the RHS

$$F^n_{i+1/2} \approx \frac{1}{\triangle t}\int_{t_n}^{t_{n+1}} \mathrm{d}t\, f(u(x_{i+1/2}, t)) \tag{71}$$

We assume that $F^n_{i+1/2}$ can be written in terms of the average values U^n_i and U^n_{i+1}. This defines the *numerical flux function*

$$F^n_{i+1/2} = \mathcal{F}(U^n_i, U^n_{i+1}) \tag{72}$$

This gives a general form for Godunov methods.

Finite Volume Discretization

$$U^{n+1}_i = U^n_i - \frac{\triangle t}{\triangle x}\left[\mathcal{F}(U^n_i, U^n_{i+1}) - \mathcal{F}((U^n_{i-1}, U^n_i)\right] \tag{73}$$

Numerical Fluxes in Godunov's Method

In Godunov's method the numerical fluxes are easily evaluated. $\tilde{u}(x_{i+1/2}, t)$ is a solution of the Riemann problem. As the Riemann problem solution is self-similar—the solution is constant along lines $x/t =$ constant—the numerical flux is independent of time. For Godunov's method we write the numerical flux as

$$\mathcal{F}(U_i^n, U_{i+1}^n) = f\left(u^\star(U_i^n, U_{i+1}^n)\right) \tag{74}$$

where $u^\star(U_i^n, U_{i+1}^n)$ indicates the solution of the Riemann problem at the cell interface with initial states U_i^n and U_{i+1}^n

Godunov's Method for a Scalar Conservation Law

Consider the nonlinear scalar equation

$$\partial_t u + \partial_x f(u) = 0 \tag{75}$$

The Riemann problem for one weak solution propagating with speed $s = [f]/[u]$

$$u^\star(u_\ell, u_r) = \begin{cases} u_\ell & \text{if} s > 0 \\ u_r & \text{if} s < 0. \end{cases} \tag{76}$$

The numerical flux is then

$$\begin{aligned} \mathcal{F}(U_i^n, U_{i+1}^n) &= f(u^\star(u_\ell, u_r)) \\ &= \begin{cases} f(u_\ell) & \text{if} s \geq 0 \\ f(u_r) & \text{if} s < 0. \end{cases} \end{aligned} \tag{77}$$

Godunov's Method for a Linear Systems

Godunov's method for a scalar equation is identical to the upwind method discussed above.

Consider the linear system of equations

$$\partial_t \mathbf{u} + \mathbf{A} \partial_x \mathbf{u} = 0, \tag{78}$$

where $\mathbf{A}$ is a matrix with constant coefficients. The solution of the Riemann problem with initial states $\mathbf{U}_i^n$ and $\mathbf{U}^{n+1})i$ is $\mathbf{u}^\star(U_i^n, U_{i+1}^n)$. We expand the difference

$$\boldsymbol{\alpha} = \mathbf{R}^{-1}\left(\mathbf{U}_{i+1}^n - \mathbf{U}_i^n\right) \tag{79}$$

where $\mathbf{R}$ is the matrix whose columns are the (right) eigenvectors $\mathbf{r}_i$ of $\mathbf{A}$.

$$\mathbf{u}^\star(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n) = \mathbf{U}_i^n + \sum_{\lambda_i < 0} \alpha_i \mathbf{r}_i = \mathbf{U}_{i+1}^n - \sum_{\lambda_i > 0} \alpha_i \mathbf{r}_i \tag{80}$$

Godunov's Method for Linear Systems

We can simplify the equations by introducing the notation

$$\begin{aligned} \lambda_p^+ &= \max(\lambda_p, 0), & \mathbf{\Lambda}^+ &= \text{diag}\left(\lambda_1^+, \lambda_2^+, \ldots, \lambda_N^+\right) \\ \lambda_p^- &= \min(\lambda_p, 0), & \mathbf{\Lambda}^- &= \text{diag}\left(\lambda_1^-, \lambda_2^-, \ldots, \lambda_N^-\right) \end{aligned} \tag{81}$$

Note that $\mathbf{\Lambda}^+ + \mathbf{\Lambda}^- = \mathbf{\Lambda}$.

Then the numerical flux is

$$\begin{aligned} \mathcal{F}(U_i^n, U_{i+1}^n) &= A\mathbf{u}^\star(U_i^n, U_{i+1}^n) \\ &= AU_j^n + \sum_{\lambda_p<0} \alpha_p \lambda_p \mathbf{r}_p \\ &= AU_j^n + \mathbf{A}^-(\mathbf{U}_{j+1}^n - \mathbf{U}_j^n) \end{aligned} \tag{82}$$

An equivalent form of the numerical flux is

$$\begin{aligned} \mathcal{F}(U_i^n, U_{i+1}^n) &= AU_{j+1}^n - \sum_{\lambda_p>0} \alpha_p \lambda_p \mathbf{r}_p \\ &= AU_{j+1}^n - \mathbf{A}^+(\mathbf{U}_{j+1}^n - \mathbf{U}_j^n) \end{aligned} \tag{83}$$

Godunov's Method for Linear Systems

Now average these two expressions to get

$$\begin{aligned}\mathcal{F}(U^n_i, U^n_{i+1}) &= \frac{1}{2}\mathbf{A}(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}) + \frac{1}{2}(\mathbf{A}^- - \mathbf{A}^+)(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j) \\ &= \frac{1}{2}\mathbf{A}(\mathbf{U}^n_j + \mathbf{U}^n_{j+1}) - \frac{1}{2}|\mathbf{A}|(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j) \qquad (84)\end{aligned}$$

where

$$|\mathbf{A}| = \mathbf{A}^+ - \mathbf{A}^- = \mathbf{R}|\mathbf{\Lambda}|\mathbf{R}^{-1} \qquad (85)$$

and

$$|\mathbf{\Lambda}| = \text{diag}\,(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|) \qquad (86)$$

Combining everything . . .

Godunov's method for a linear system

$$\mathbf{U}^{n+1}_j = \mathbf{U}^n_j - \frac{\triangle t}{2\triangle x}\mathbf{A}(U^n_{j+1} - \mathbf{U}^n_{j-1}) + \frac{\triangle t}{2\triangle x}|\mathbf{A}|(U^n_{j+1} - 2\mathbf{U}^n_j + \mathbf{U}^n_{j-1}) \qquad (87)$$

Improving on Godunov

Godunov's method gives a general way to use characteristic information in numerical algorithms for nonlinear equations. There are a couple of issues to consider:

- Godunov's original method used the full solution of the Riemann problem, which can be difficult to find and comes with a high computational cost. (The solution typically requires solving nonlinear transcendental equations using iterative methods.)
- Much of the structure in the Riemann problem's solution is lost in the final averaging process.
- The overall method is just first order, $\mathrm{O}(\triangle t, \triangle x)$.

To address these issues, we will consider two strategies

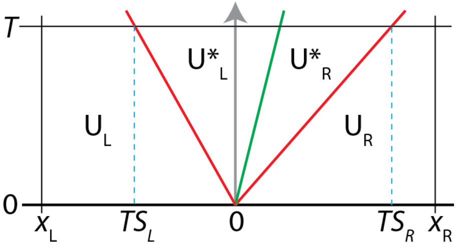
- An approximate solution to the Riemann problem could be used in the method. Since the details of the solution are lost in the averaging process, an approximate solution that can be found with low computational cost would be preferable.
- Higher order reconstructions of the initial states can improve the accuracy.

Approximate Riemann Solvers

A solution to the Riemann problem requires solving nonlinear transcendental methods iteratively for millions or billions of points. Most of the structure of the Riemann problem is lost in the averaging process.
There are two types of approximate solvers. Those that approximate the Riemann problem solution, and those that solve a related Riemann problem exactly.
Two very popular approximate Riemann solvers are the HLL family of solvers and the Roe Riemann solver. HLL is an example of the first type of approximate Riemann solver, and Roe is an example of the second.

Consistency Relation for Approximate Riemann Solvers

An approximate Riemann solver may not satisfy the conservation law for the original equation. To ensure that conservation is satisfied, a consistency relation must be satisified.



The sructure of the exact Riemann problem is shown schematically in the diagram. Here S_L and S_R are the minimum and maximum signal speeds respectively. The conservation equation is

$$\int_{x_L}^{x_R} \mathrm{d}x\, \mathbf{u}(T,x) = \int_{x_L}^{x_R} \mathrm{d}x\, \mathbf{u}(0,x) + \int_0^T \mathrm{d}t\, \left[\mathbf{f}(\mathbf{u}(t,x_L)) - \mathbf{f}(\mathbf{u}(t,x_R))\right]$$

Consistency Relation for Approximate Riemann Solvers

The RHS can be evaluated to give the *consistency equation* for the Riemann solution

$$\int_{x_L}^{x_R} \mathrm{d}x\, \mathbf{u}(T,x) = x_R\mathbf{u}_R - x_L\mathbf{u}_L + T\left(\mathbf{f}(\mathbf{u}_L) - \mathbf{f}(\mathbf{u}_R)\right). \tag{88}$$

The integral on the LHS can be written

$$\int_{x_L}^{x_R} \mathrm{d}x\, \mathbf{u}(T,x) = \int_{TS_L}^{TS_R} \mathrm{d}x\, \mathbf{u}(T,x) + (TS_L - x_L)\mathbf{u}_L + (x_R - TS_R)\mathbf{u}_R \tag{89}$$

Combining these equations, and dividing by $T(S_R - S_L)$ gives

$$\frac{1}{T(S_R - S_L)}\int_{x_L}^{x_R} \mathrm{d}x\, \mathbf{u}(T,x) = \frac{S_R\mathbf{u}_R - S_L\mathbf{u}_L + \mathbf{f}(\mathbf{u}_L) - \mathbf{f}(\mathbf{u}_R)}{S_R - S_L} \tag{90}$$

Consistency for Approximate Riemann Solvers

The integral average of the state $U^\star$ in the Riemann problem is a constant,

$$\bar{\mathbf{u}} = \frac{S_R\mathbf{u}_R - S_L\mathbf{u}_L + \mathbf{f}(\mathbf{u}_L) - \mathbf{f}(\mathbf{u}_R)}{S_R - S_L} \tag{91}$$

We can perform a similar analysis, integrating the conservation equation from $[x_L, 0] \times [0, T]$. We get

$$\mathbf{f}_L(0) = \mathbf{f}(\mathbf{u}_L) - S_L\mathbf{u}_L - \frac{1}{T}\int_{TS_L}^{0} \mathrm{d}x\,\mathbf{u}(T,x) \tag{92}$$

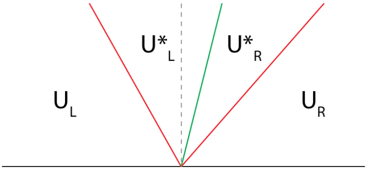
Interating over the interval $[0, x_R] \times [0, T]$ gives

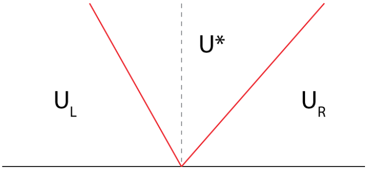
$$\mathbf{f}_R(0) = \mathbf{f}(\mathbf{u}_R) - S_R\mathbf{u}_R + \frac{1}{T}\int_{0}^{TS_R} \mathrm{d}x\,\mathbf{u}(T,x) \tag{93}$$

Here $\mathbf{f}_L(0)$ and $\mathbf{f}_R(0)$ indicate the flux evaluated at $x = 0$, using integral equations from either the left or right. The consistency equation shows that $\mathbf{f}_L(0) = \mathbf{f}_R(0)$.

The HLL Approximate Riemann Solver

Harten, Lax, and van Leer introduced this an approximate Riemann solver in 1983 that simplifies the Riemann problem by using only a single intermediate state and only the minimum and maximum signal speeds.





The HLL approximate solution of the Riemann problem is $\xi \equiv x/t$

$$\mathbf{u}^{\star}(\xi) = \begin{cases} \mathbf{u}_L & \text{if} \xi \leq S_L \\ \bar{\mathbf{u}} & \text{if} S_L < \xi < S_R \\ \mathbf{u}_R & \text{if} \xi \geq S_R \end{cases} \tag{94}$$

The HLL Flux

We defined the numerical flux above as $\mathcal{F}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) = \mathbf{f}(\mathbf{u}^\star)$ However,

$$\mathcal{F}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) \neq \mathbf{f}(\bar{\mathbf{u}}), \tag{95}$$

as this solution does not satisfy the consistency relation. Using either (92) or (93) and $\bar{\mathbf{u}}$, we find that the numerical flux is

$$\mathcal{F}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n) = \frac{S_R \mathbf{f}(\mathbf{u}_L) - S_L \mathbf{f}(\mathbf{u}_R) + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L}. \tag{96}$$

The HLL Family of Approximate Riemann Solvers

Advantages of the HLL Methods

- Very simple Riemann solver, and currently the most commonly used approximate solver for relativistic fluids.
- More dissipation than some other common methods, but also tends to prevent entropy-violating solutions.

Variants of the HLL method

- HLLE:
- HLLEM:
- HLLC:

A First Guess—FTCS

Godunov's method is expensive because it requires the solution of the Riemann problem. We can speed this up by finding approximate solutions to the Riemann problem.

The choice of the numerical flux function must be done carefully. To examine some numerical issues with nonlinear equations, we first look at some simple choices motivated by finite difference equations.

As a first guess we might choose an *average* numerical flux

$$\mathcal{F}(U_i^n, U_{i+1}^n) = \frac{1}{2}\left[f(U_i^n) + f(U_{i+1}^n)\right] \tag{97}$$

The update

$$U_i^{n+1} = U_i^n - \frac{\triangle t}{2\triangle x}\Big[f(U_{i+1}^n) - F((U_{i-1}^n)\Big] \tag{98}$$

This is the Forward-Time Central-Space (FTCS) method, as the time derivative is a one-sided forward derivative, while the spatial derivative is centered.

The Lax-Friedrichs Method

Since the FTCS method is unstable, let's look at a light modification known as the Lax-Friedrichs method

$$U_i^{n+1} = \frac{1}{2}(U_{i-1}^n + U_{i+1}^n) - \frac{\triangle t}{2\triangle x}\left[f(U_{i+1}^n) - F((U_{i-1}^n).\right] \qquad (99)$$

This method is very similar to the FTCS method, except U_i^n on the RHS has been replaced by the average $\frac{1}{2}(U_{i-1}^n + U_{i+1}^n)$. This can be written as a Finite Volume method with the numerical flux

$$\mathcal{F}(U_i^n, U_{i+1}^n) = \frac{1}{2}\left[f(U_i^n) + f(U_{i+1}^n)\right] - \frac{\triangle x}{2\triangle t}(U_{i+1}^n - U_i^n) \qquad (100)$$

The second term acts as a dissipation term in an advection-diffusion equation

$$u_t + f(u)_x = \beta u_{xx} \qquad (101)$$

with $\beta = \frac{1}{2}(\triangle x)^2/\triangle t$.

Local Lax-Friedrichs

The Roe Approximate Riemann Solver

(102)

High Resolution Methods

The upwind and Godunov schemes that we have seen so far are only first order accuate in space and time, $\mathrm{O}(\triangle t, \triangle x)$. We desire higher accuracy. The CFL condition requires $\triangle t = \mathrm{O}(\triangle x)$, so we need to improve the accuracy in both time and space.
High-Resolution Shock-Capturing methods are at least 2nd order in space *and* produce well-resolved, non-oscillatory discontinuities.

The strategy is

- Improve temporal accuracy using a semi-discrete method.
- Improve spatial accuracy using
  - Flux-limiter methods
  - Higher-order reconstructions

Semi-discrete Methods

The semi-descrete approach simply means that the discretization in time is done independently of the discretization in space. This approach is very useful for our purposes for two reasons

- It naturally incorporates source terms in the balance equations. (And in general relativity there are many source terms!)
- It allows us to independently choose the level of temporal and spatial accuracy.

For the general balance law

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u}) \tag{103}$$

we discretize first in space to get the semi-discrete equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{\mathcal{F}(\mathbf{u}_{i+1/2}) - \mathcal{F}(\mathbf{u}_{i-1/2})}{\triangle x} + \mathbf{s}(\mathbf{u}). \tag{104}$$

This discretization method is also called the *Method of Lines.*

ODE Solvers That Preserve the TVD Property

In the semi-discrete balance law

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{\mathcal{F}(\mathbf{u}_{i+1/2}) - \mathcal{F}(\mathbf{u}_{i-1/2})}{\triangle x} + \mathbf{s}(\mathbf{u}), \tag{105}$$

the LHS is simply an ODE. These (coupled) ODEs can integrated with an appropriate numerical scheme. What do I mean by appropriate?

The RHS of the balance law was differenced by techniques that satisfy the TVD property. The integration in time should preserve the TVD property. Without proof or derivation, the following slides show some integration methods that preserve the TVD property. For more information see

- C.-W. Shu and S. Osher, "Efficient implementation of essentially non-oscillatory shock capturing schemes," *Journal of Computational Physics,* **77** 439–471 (1988).
- S. Gottlieb and C.-W. Shu, "Total variation dimenishing Runge-Kutta schemes," *Mathematics of Computation* **FIXME** XXX–XXX (XXXX).

2nd Order TVD Runge-Kutta Scheme

Let $L(\mathbf{u})$ represent the RHS of the balance law

$$L(\mathbf{u}) = -\partial_x \mathbf{f}(\mathbf{u}) + \mathbf{s}(\mathbf{u}). \tag{106}$$

The optimal second order TVD Runge-Kutta method is Huen's method with $\mathrm{CFL} = 1$.

Huen's method

$$\begin{aligned} \mathbf{u}^{(1)} &= \mathbf{u}^n + \triangle t L(\mathbf{u}^n) \\ \mathbf{u}^{n+1} &= \frac{1}{2}(\mathbf{u}^n + \mathbf{u}^{(1)}) + \frac{1}{2}\triangle t L(\mathbf{u}^{(1)}) \end{aligned} \tag{107}$$

3rd Order TVD Runge-Kutta Schemes

The optimal 3rd order scheme with $\mathrm{CFL} = 1$ is

3rd Order TVD Runge-Kutta

$$\begin{aligned} \mathbf{u}^{(1)} &= \mathbf{u}^n + \triangle t L(\mathbf{u}^n) \\ \mathbf{u}^{(2)} &= \frac{3}{4}\mathbf{u}^n + \frac{1}{4}\mathbf{u}^{(1)} + \frac{1}{4}\triangle t L(\mathbf{u}^{(1)}) \\ \mathbf{u}^{n+1} &= \frac{1}{3}\mathbf{u}^n + \frac{2}{3}\mathbf{u}^{(2)} + \frac{2}{3}\triangle t L(\mathbf{u}^{(2)}) \end{aligned} \tag{108}$$

Memory is always a concern for fluids in general relativity. This method can be written such that only the solution at the initial and advanced time needs to be stored, with one set of work arrays.

Flux-limiter Methods

In examples with Burgers equation we found that

1. First order methods (Lax-Friedrichs, Godunov) produce monotonic solutions at discontinuities. These schemes are also diffusive.
2. Second order methods (Lax-Wendroff, MacCormack) work well when the flow is smooth, but produce unphysical oscillations at discontinuities (dispersion).

With flux-limiting we attempt to have the best of both worlds. Let the function Φ be a limiter function that is 0 for discontinuous functions and 1 for smooth functions, and varies in between $[0,1]$ depending on how smooth the numerical solution is. We define a numerical flux function $\mathcal{F}$ by combining a high-order numerical flux $\mathcal{F}_H$ and a low-order flux $\mathcal{F}_L$

$$\mathcal{F} = \mathcal{F}_L + \Phi\left[\mathcal{F}_H - \mathcal{F}_L\right] \tag{109}$$

For highly relativistic fluids, experience has shown that flux-limiter methods do not work as well as using higher-order reconstructions.