Proceedings of the International Congress of Mathematicians
Vancouver, 1974

Levels of Structure in Catastrophe Theory Illustrated by Applications in the Social and Biological Sciences

E. C. Zeeman

Catastrophe theory is a method discovered by Thom [14] of using singularities of smooth maps to model nature.

In such models there are often several levels of structure, just as in a geometry problem there can be several levels of structure, for instance the topological, differential, algebraic, and affine, etc. And, just as in geometry the topological level is generally the deepest and may impose limitations upon the higher levels, so in applied mathematics, if there is a catastrophe level, then it is generally the deepest and likely to impose limitations upon any higher levels, such as the differential equations involved, the asymptotic behaviour, etc. Again, in geometry the complexity of the higher levels may render them inaccessible, so that they can only be handled implicitly rather than explicitly, while at the same time the underlying topological invariants may even be computable. Similarly in applied mathematics the complexity of the differential equations may sometimes render them inaccessible (even to computers), so that they can only be handled implicitly rather than explicitly, while the underlying catastrophe can be modeled, possibly even to the extent of providing quantitative prediction.

Therefore catastrophe theory offers two attractions: On the one hand it sometimes provides the deepest level of insight and lends a simplicity of understanding. On the other hand, in very complex systems such as occur in biology and the social sciences, it can sometimes provide a model where none was previously thought possible. In this paper we discuss various levels of structure that can be superimposed upon an underlying catastrophe and illustrate them with an assortment of examples. For convenience we shall mostly use the familiar cusp catastrophe (see [5], [13], [14], [24]).

Level 1. Singularities.
Level 2. Fast dynamic (homeostasis).
Level 3. Slow dynamic (development).
Level 4. Feedback.
Level 5. Noise.
Level 6. Diffusion.

Thom's classification of elementary catastrophes belongs to Level 1. Levels 2,3,4 refer to ordinary differential equations, and Level 6 refers to partial differential equations.

Level 1. Singulairties. We begin by recalling the main classification theorem. Let C, X be manifolds with $\dim C \leqq 5$, and let $f \in C^\infty(C \times X)$. Suppose that f is generic in the sense that the related map $C \to C^\infty(X)$ is transverse to the orbits of the group $\mathrm{Diff}(X) \times \mathrm{Diff}(R)$ acting on $C^\infty(X)$. (Genericity is open-dense in the Whitney C^∞-topology.) Let $M \subset C \times X$ be given by $\nabla_X f = 0$, and let $\chi : M \to C$ be induced by projection $C \times X \to C$.

THEOREM (THOM). (a) *M is a manifold of the same dimension as C.*
(b) *Any singularity of χ is equivalent to an elementary catastrophe.*
(c) *χ is stable under small perturbations of f.*

The number of elementary catastrophes depends only upon the dimension of C (and not on X):

dim C	1	2	3	4	5	6
elemementary catastrophes	1	2	5	7	11	(∞)

For details of the elementary catastrophes the reader is referred to **[10,]** **[14]**. The first complete proof was given by Mather **[8]**, and other references are **[1]**, **[17]**, **[18]**.

REMARK. The classification of singularities goes infinite for $\dim C \geqq 6$, but the above table can be extended with finite entries, provided the concept of elementary catastrophes is suitably modified, as follows. The singularities correspond to orbits of a group acting on a space of germs (see Level 4 below). In particular the (∞) appears at $\dim C = 6$ because there is a stratum of codimension 6 foliated by orbits of codimension 7. Arnold **[1]** calls the codimension of the foliation, which is 1 in this case, the modality of the stratum. More generally the orbits form a foliated stratification P, which Arnold has shown to be locally finite. The finite numbers of strata of each codimension give the desired extension of the above table.

The reason that catastrophe theory exists is that by a happy accident P is 5-simple, in other words each stratum of P of codim $\leqq 5$ is simple, that is trivially foliated by a single leaf. These strata correspond to the elementary catastrophes of dim $\leqq 5$, and hence the latter are finitely classified differential invariants. For most applications it suffices to have $\dim C \leqq 5$, and so there is no need to worry about the foliation of the higher strata.

Application. Suppose we have some set of objects or events about which we want to test a hypothesis of cause and effect. One of the first things to do is to plot them in cause-effect space, and see if they form a graph. Here C will describe the cause, X the effect, and $f(c, x)$ the probability that cause c will produce effect x. The most likely effects are given by the peaks of probability, where both the gradient vanishes, $\nabla_X f = 0$, and the Hessian is negative definite, $\nabla_X^2 f < 0$. This determines a submanifold G of M (of the same) dimension. Then G will be the desired cause-effect graph in $C \times X$. The events will be represented by a cloud of points clustering near G, with density of clustering depending upon the deviation of the probability distributions.

Consider the first two elementary catastrophes, which occur when dim $C = 2$. The fold-catastrophe occurs at the boundary of G, but since there is no dynamic in Level 1 there may not be any catastrophic jump here—all we can say is that the cloud of points appears to terminate.

The cusp-catastrophe occurs when a probability distribution goes bimodal. In this case observers may implicitly recognise the phenomenon, and capture part of it by either naming the two modes, or alternatively framing some form of words, such as a proverb or a belief. However the cusp-catastrophe can often reveal other facets to the phenomenon, and give a new synthesis of understanding. We illustrate the two alternatives by a couple of examples.

EXAMPLE 1. AGGRESSION [22]. According to Konrad Lorenz [7] fear and rage are conflicting drives influencing aggression. Here the two extreme behaviour modes are attack/flight, and X represents a 1-dimensional spectrum of behaviour varying from neutral to the two extremes at either end. The cause C is 2-dimensional, representing the strengths of the fear and rage drives present in the animal at that moment. Lorenz observes that in the case of dogs the coordinates of fear and rage can be read from the facial expression [7, p. 81]. Rage only causes attack, fear only causes flight, and when both are present the effect is one of two extremes but unpredictable. Therefore the probability goes bimodal, and as a first approximation we might expect our cloud of points to cluster around a cause-effect graph equivalent to the cusp-catastrophe as shown in Figure 1. We shall return to this example, and its uses, in Level 2.

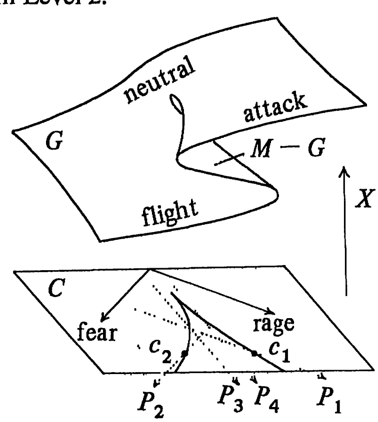

FIGURE 1

Other familiar examples of bimodality which can be modeled by the cusp-catastrophe are (i) liquid/gas [3], [11], [15], (ii) diastole/systole [23], (iii) manic/depressive [25], (iv) dove/hawk [5], or (v) bull/bear ([26], and Example 7 below).

In each case the bimodality is caused either by conflicting factors such as temperature and pressure in (i), or by a splitting factor such as tension in (ii), disease in (iii), cost in (iv) or speculation in (v). Let us now give an example of a proverb.

EXAMPLE 2. MORE HASTE LESS SPEED. This proverb is very familiar in England, although almost unknown in America. Its attraction lies in its brevity and contrariness—it is the opposite of what one would normally expect, especially if the operator is skilled at his task. And this leads to the observation that speed really depends upon two factors, haste and skill, which are conflicting. For, when both factors are present, the probability goes bimodal, because either the operator's skill enables him to increase his speed, or his fumbling haste diminishes it. Therefore again we might expect our cloud of points to cluster around a cusp-catastrophe, as in Figure 2.

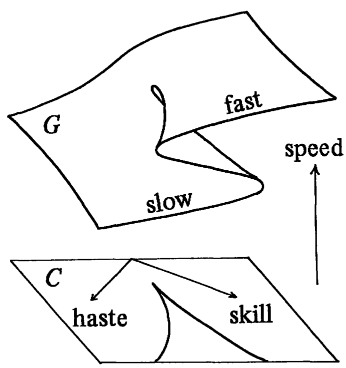

FIGURE 2

We suggest a couple of uses for such a model. Firstly in psychology it might be possible to develop it further into a predictive quantitative model for explicit skills (see Level 2 below). Secondly in sociology it might provide a prototype for reconciling conflicting theories. For, by the theorem, we should expect the cusp-catastrophe to occur in many phenomena, and, although the graph is visually simple, its subtlety is not easy to describe with any brevity in ordinary spoken or written language (see [5, §10]). Therefore although we may often recognise such a phenomenon intuitively, we tend to describe it verbally by an oversimplification, possibly by directing attention only to the unexpected mode. For instance "more haste less speed" directs attention only to the lower front sheet of Figure 2, marked "slow". Similarly two conflicting opinions in a discussion, or two conflicting sociological theories, may in fact each be referring to a single mode of an underlying bimodal phenomenon, and the conflict may sometimes be reconciled by exhibiting the two modes as the two sheets of a cusp-catastrophe growing smoothly out of an area of common agreement.

Level 2. Dynamic. In addition to the generic function $f \in C^\infty(C \times X)$ suppose we are given a dynamic D as follows. Denote the associated function $C \to C^\infty(X)$ by $c \mapsto f_c$. Then $D = \{D_c\}$ is a family of differential equations on X, parametrised by C, such that, for each $c \in C$, f_c is a Lyapunov function for D_c. In other words, f_c increases (or decreases) along the orbits of D_c, and so the maxima (or minima) of f_c are the attractors of D_c. Therefore D_c is gradient-like, and this is the restriction that Level 1 imposes on Level 2.

The graph G now represents the attractors of D. In applications we no longer intuitively imagine a cloud of points clustering statically near G, but points flowing dynamically onto G and then staying there. The model is half dynamic and half static. It is sometimes useful to think of the parameter space C as control, and X as behaviour space. If we slowly move the control c then the behaviour x responds by moving continuously on G for as long as possible; in other words it is a theorem of Level 2 that the system obeys the delay rule of Thom **[5]**, **[14]**.

If c now crosses the bifurcation set, then x may cross the boundary ∂G of G. In this case the dynamic will carry x rapidly onto some other sheet of G. The word "rapidly" assumes that the movement of control is slow compared with the dynamic, and it is the sudden jump that occurs at the fold-catastrophes in Level 2 that is responsible for the name "catastrophe theory".

EXAMPLE 3. The *catastrophe machine* described in **[9]**, **[24]** is a simple toy made out of a cardboard disk and two elastic bands, which exhibits the catastrophic jump well, and the uninitiated reader is recommended to make one for himself. Here the function f of Level 1 is the potential energy in the elastic given by Hooke's law of elasticity, and the dynamic D is given by Newton's law of motion, suitably damped so as to minimise f.

EXAMPLE 1. Returning to our first example we see that it can be promoted from Level 1 to Level 2. For we may reinterpret X as the space of states of that part of the brain governing mood (perhaps the hypothalamus), and D as the associated dynamic representing neurological activity. Then the attractors of D represent the attacking/retreating frames of mind, providing the background mood against which behavioural decisions are taken. Although X must necessarily be very high dimensional, and D consequently inaccessible in the sense of being only implicit, nevertheless G will still be 2-dimensional. Therefore the cusp-catastrophe can still provide an explicit model, which for individual animals might be made quantitative and predictive. Moreover since it is a Level 2 model, even though D is only implicit, there will be catastrophic jumps of mood, resulting in sudden attacks or disengagements. For example in Figure 1 the path P_1, representing increasing rage at a fixed level of fear, as for instance in a cornered dog, will lead to a sudden attack at c_1 while path P_2 to a sudden disengagement at c_2. Meanwhile paths P_3, P_4 illustrate how nearby paths can lead to divergent behaviour. Similarly humans, when made angry and frightened, are unpredicatable and are denied access to rational behaviour, and may jump from abuse to apology, even from hysteria to tears.

The interest of this example is that it may provide a general model for control of aggression, valid for different species under varying circumstances, and may give insight into how such controls develop and have evolved. More generally it provides a prototype for relating the neurology to the psychology of moods underlying behaviour.

EXAMPLE 2. Our second example may also be promoted from Level 1 to Level 2, because, if we consider the performance of an individual, his tendency to adjust his speed to x, say, within the limitations of his skill and assuming a given amount of haste, is another way of saying there is an implicit dynamic that moves the speed to x.

A path P_1, such as in Figure 1, here represents an increasing skill at a fixed level of haste, as for instance when learning to ride a bicycle, and at the point c_1 a catastrophe occurs when the individual is suddenly able to ride. Moreover the greater the haste—for instance the swifter reactions that are needed to ride a more unstable machine—then the greater the skill needed before the catastrophe occurs. Meanwhile a path P_2 here represents increasing haste at a fixed skill, as for instance a wireless operator trying to read faster and faster Morse code, and at the point c_2 a catastrophe occurs as the performance drops sharply. Moreover the greater the skill, the greater the haste possible before the catastrophe occurs.

In general Level 2 is much easier to test experimentally than Level 1, because the cloud of points more accurately determines G, and the catastrophes determine ∂G. Whenever a phenomenon exhibits any one of the four qualities of bimodality, divergence, catastrophic jumps or hysterisis delays, then it may be possible to model it by the cusp-catastrophe, in which case it may be possible to predict the other three qualities. Sometimes the cusp-catastrophe can also be useful in applications where the control space C is high dimensional, as shown by the following example.

EXAMPLE 4. ECOMOMIC GROWTH. Let X represent the space of states of an economy, and C the external pressures on that economy together with the controls available to the government. Let D represent the implicit response of the economy. We should expect C to be high dimensional, and so at first sight the theorem is of little use. However the evolution of the economy is in fact only a 1-dimensional path in C lifted to a 1-dimensional path in G, and the corresponding 1-dimensional catastrophes are the slumps, inflation explosions, etc.

A typical problem facing the government is the realisation that whereas its present policy is now at control point c_0, it may have to change policy in the next few months to c_1, due to external pressures, balance of payments, etc. The government's freedom of action may be limited merely to choosing the path from c_0 to c_1. However such choice may be critical as we now explain. Suppose there is a choice between two paths P_1 or P_2. For simplicity let us assume that neither path involves a catastrophe. The question that must be asked is: Does the circle $P_1 \cup P_2$ link any codimension-2 stratum Σ of the bifurcation set? For if it does, then a

2-dimensional disk E spanning $P_1 \cup P_2$ will pierce Σ, and the section of G over E will contain a cusp-catastrophe, as shown in Figure 3. The lifts Q_1, Q_2 of P_1, P_2 will exhibit divergence, which could radically affect growth, inflation, unemployment, etc.

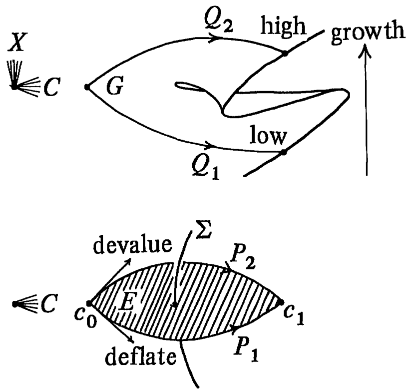

FIGURE 3

For example suppose P_1 represented deflation followed by devaluation (as in the U.K. in 1967), and P_2 the reverse order (as in France in 1968). Then Q_1 could lead to low growth because, with reduced stocks, firms would be unable to exploit the devaluation, whereas Q_2 could lead to high growth, because firms could switch sales of stock from the curtailed home market to the export market, without losing growth momentum. Therefore economists should be concerned not only with the more obvious codimension-1 problems of catastrophe, but also with the more hidden codimension-2 problems of divergence and choice.

Level 3. Development. In addition to $f \in C^\infty(C \times X)$ and the dynamic D suppose that we have time T occurring as one of the axes in the control space C. It is assumed that T is slow compared with the fast time occurring in the dynamic D.

EXAMPLE 5. EMBRYOLOGY. Level 3 occurs in Thom's main application of catastrophe theory to embryology [13], [14], [16], where the control space, $C = S \times T$, represents space-time, and X represents the states of a cell. For instance X may be a bounded open subset of R^n, with several thousand coordinates representing various chemical and physical parameters of the cell. The dynamic D represents the homeostasis of a cell returning it swiftly to equilibrium, and T the slow development of the cells.

An example of a result in this context is the following:

THEOREM [27]. *Whenever a tissue differentiates into two types, the frontier between them first forms to one side and then moves through the tissue before stabilising in its final position.*

The proof uses the cusp catastrophe as illustrated in Figure 4. S is taken to be 1-dimensional perpendicular to the frontier. Development paths of cells are given by lifting time lines to G. The frontier first forms at c_1, then moves as a wave through S along the cusp branch c_1c_2, and then stabilises at c_2, where the cusp touches the time line c_2c_3. Such a wave is often a hidden switching on of genes, and morphogenesis may be caused after some delay by a secondary wave of physical manifestation. For example in [27] detailed models are given for the morphogenesis of gastrulation and neurulation in amphibia, and of culmination in slime-mold.

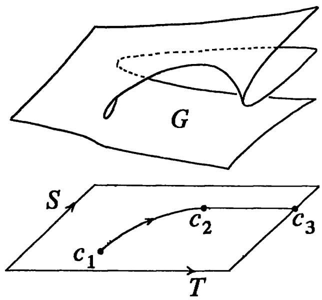

FIGURE 4

Space-catastrophes. The above result depended upon the time-axis not being tangential to the cusp-axis, which can be justified by an appeal to genericity. However to put this type of genericity on a mathematical footing requires a generalisation of the classical theory as follows, which Wassermann [19] calls space-catastrophe theory. (He also studies the dual concept of time-catastrophe theory.)

Let E_n denote the ring of germs at 0 of C^∞-functions $R^n \to R$, m_n the maximal ideal, and G_n the group of germs of C^∞-diffeomorphisms $R^n, 0 \to R^n, 0$. Then G_n acts on m_n, leaving m_n^2 invariant. Classical catastrophe theory [1], [8], [14], [17], [18] consists of analysing the foliated stratification P of m_n^2 by G_n. The elementary catastrophes of dimension s are given by the strata of P of codimension s. Since P is 5-simple, the elementary catastrophes for $s \leqq 5$ are finitely classified differential invariants, independent of n (for $n \geqq 2$).

For the generalisation we need some more definitions. We say $\alpha \in G_{n+r}$ *covers* $\beta \in G_r$ if $\pi\alpha = \beta\pi$, where $\pi : R^{n+r} \to R^r$ is the projection. Define

$$G_n^r = \{(\alpha, \beta) \in G_{n+r} \times G_{1+r};\ \exists\, \gamma \in G_r \text{ such that } \alpha, \beta \text{ cover } \gamma\}.$$

Then G_n^r acts on m_{n+r}, leaving $m_n^2 + m_r E_{n+r}$ invariant. For space-catastrophe theory we choose $r = 1$ (representing time), and analyse the foliated stratification Q of $m_n^2 + m_1 E_{n+1}$ by G_n^1. The space-catastrophes of dimension s are given by the strata of Q of codimension $s + 1$.

Wassermann [19] has shown that Q is 2-simple, and hence the 1-space-catastrophes are finitely classified differential invariants, independent of $n \geqq 1$. There are exactly four, namely the beginning c_1, the middle, and the end c_2, of the wave in Figure 4, and the "silent" dual of c_2. Therefore the above theorem is valid, and exhibits them all.

However Q is not 3-simple, for Wassermann has shown that the P-strata of swallowtails and umbilics are not only substratified by Q but also foliated. Therefore the number of singularities in 2-space goes infinite, and although the 2-space-catastrophes will still be finitely classifiable, they will no longer be differential invariants. Some will be—for example the cusp-projection of a fold-surface into 2-space (analogous to the fold-projection of a fold-curve into 1-space at c_2 in Figure 4). This example has been used to model the pattern formation of somites in amphibia [27].

However Thom [14], [16] uses the swallowtail, butterfly and umbilics extensively in embryology, and hence it is important to classify the 2- and 3-space-catastrophes. Therefore mathematically we need to analyse the strata of Q up to codimension 4, and to understand the nature of the loss in differentiability implied by their foliation.

Level 4. Feedback. Here we assume that the slow flow is not as simple as merely taking a coordinate in the control space, but may go in different directions on different sheets of G. In fact it may be conceived as a form of feedback:

$$C \underset{\text{slow feedback } F}{\overset{\text{fast dynamic } D}{\rightleftarrows}} X.$$

More precisely, in addition to f and D, suppose we are given a C^∞-map F: $C \times X \to TC$, where TC denotes the tangent bundle of C, and $F(c, x)$ is a tangent at c, for each $c \in C$, $x \in X$. Therefore D and F together form an ordinary differential equation on $C \times X$ (with the proviso that D is fast and F slow).

EXAMPLE 6. HEARTBEAT AND NERVE IMPULSE [23]. Explicit examples of differential equations in form of feedbacks on the cusp-catastrophe were taken as models. In each case the flow possessed a stable equilibrium, which if suitably disturbed by an "external agent", triggered a catastrophe via D, and a return to equilibrium via F. In the heartbeat the return involved a second catastrophe (relaxation after contraction), whereas in the nerve impulse the return was smooth (repolarisation). These models possess two interesting features. Firstly the feedback does not give a flow precisely on G, but only near G, the order of nearness depending upon the ratio K of fast/slow. If $K \to \infty$ we obtain an idealised flow on G, with instantaneous catastrophes, generalising the relaxation oscillations of electrical engineering. Secondly the words "external agent" above reveal the inadequacies of the models, in being only ordinary differential equations describing the local behaviour of heart muscle and nerve fragment; what is needed is to embed the latter in a larger partial differential equation that describes the global behaviour as waves. We return to this problem in Level 6.

EXAMPLE 7. STOCK EXCHANGES [26]. The cusp-catastrophe is used to model the behaviour of stock exchanges, as follows. The excess demand is the normal factor controlling the rate of change of index, and the speculative content of the market a splitting factor. The dynamic D represents the immediate response of index to investors, and F the somewhat slower feedback. Plausible economic hypotheses lead to a flow that exhibits periodic bull market, recession, bear market and recovery. However, to make this model realistic, we should promote it to Level 5 by including noise.

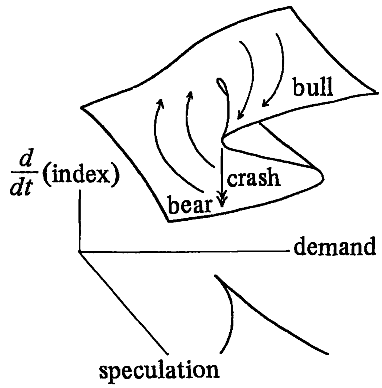

FIGURE 5

EXAMPLE 8. FUNNEL. In classifying the generic low-dimensional feedback-catastrophes, Takens [12] has recently discovered an interesting new type, the simplest of which he calls the funnel. In the associated idealised flow a 2-dimensional piece of G is funneled through a single fold-point P. Figure 6 illustrates the following explicit example:

Fast dynamic D: $\dot{x} = -K(x^2 + 2b)$, K large constant.
Slow feedback F: $\dot{a} = 1$, $\dot{b} = 3a + 4x$.

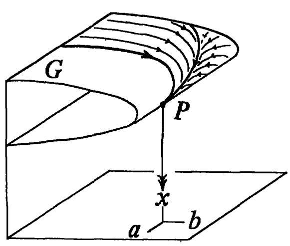

FIGURE 6

Funnels may occur in biological regulation, for instance choosing x, b to model the internal self-regulation of a cell, and a the production by the cell of some hormone for use outside the cell, whose production-rate needs to be funneled precisely.

Level 5. Noise. We may superimpose on $\{f, D, F\}$ stochastic noise in the form of random small displacements of control and behaviour. For most noise the dynamic D carries the state rapidly back onto G, and the slow flow F proceeds as before, and so the noise can be ignored. However in two cases noise can cause catastrophes, firstly if control-noise crosses the bifurcation set, and secondly more interestingly if the behaviour-noise crosses a separatrix.

EXAMPLE 7. In the stock exchange example noise represents external events and consequent jumpiness in the market, and may cause recessions to occur before the bifurcation set is reached.

EXAMPLE 9. RIOTS [4]. This model reports joint work in progress with prison psychologists P. Shapland, C. Hall and H. Marriage, and statistician J. Harrison. We start with a truism: The more tension in an institution the more disorder. This applies not only to institutions such as prisons, universities, firms, or countries but also to individuals. In the case of prisons, an analysis of data suggests that the tension (or distress or frustration) can be measured by the numbers reporting sick, suitably smoothed, and the disorder can be measured by correlating independent assessments of the seriousness of incidents. Alienation (or lack of communication) seems to be a splitting factor, producing the two modes that we have labelled quiet and disturbed in Figure 7, and the data suggest that this may be measured by the numbers of disciplinary reports. The feedback flow represents the increase in tension during quiet (over months) and the release during disturbance (over days). Noise describes incidents, and if the noise level crosses the separatrix AA' at B then the incidents will escalate and spark a riot causing a catastrophe. Some types of prison population (e.g., young long-term) have a higher noise level, and are therefore more susceptible to riots. When the tension has subsided after a few days an incident may cause the reverse catastrophe at B'. The same incident might not have done so earlier, which explains the advantage of playing it cool.

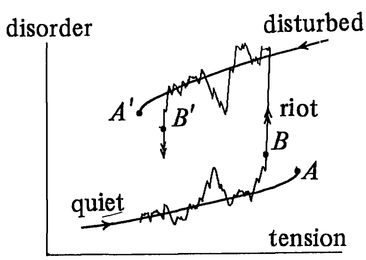

FIGURE 7

EXAMPLE 10. PHASE TRANSITION [3], [11], [15]. If the noise is frequent, and the noise-level high, the state will, averaged over time, seek the absolute maximum (or minimum) of f. This explains why Van der Waals' equation for liquid/gas phase transition has to be supplemented with Maxwell's rule [3], [5], [14], instead of obeying the delay rule. On the other hand, if the noise level is kept low then partial delays can be induced, such as in the supersaturated and superevaporated states of the cloud and bubble chambers. The usual proof of Maxwell's rule in statistical mechanics involves integration by steepest descent, but since this method breaks down near the critical point, it would be interesting if a new abstract proof could be devised, parallel to the proof of Thom's theorem, in order to enhance critical point analysis.

Level 6. Diffusion. The following arises out of joint work in progress with Sharon

Hintze, stimulated by papers of Winfree [20], [21] and Kopell and Howard [6] on the Zhabotinsky reaction. First the mathematics.

Let Y be a manifold and g be a C^∞-vector field on Y. The associated ordinary differential equation is

$$\dot{y} = g(y). \tag{1}$$

In particular we shall be concerned with the type of differential equation given by Level 4, namely $Y = C \times X$ and $g = \{D, F\}$. Suppose now that Y represents the space of local states of some medium in space-time $S \times T$, and that g represents the reaction of that medium. Suppose further that the medium not only reacts but also diffuses. Then, following [6], the global state $y\colon S \times T \to Y$ of the medium satisfies the reaction-diffusion partial differential equation:

$$\partial y/\partial t = g(y) + k\nabla^2 y \tag{2}$$

where k is a constant (more precisely a vector bundle map $k\colon TY \to TY$) representing the different rates of diffusion of the various components of Y. We are particularly interested in whether or not the medium can sustain stable periodic wave trains, or stable pulses (isolated waves). If it can, and θ is the speed, then the global state y can be factored $S \times T \to R \to Y$ such that $\partial y/\partial t = \theta \dot{y}$ and $\nabla^2 y = \ddot{y}$, where the dot denotes differentiation with respect to R. Therefore the partial differential equation (2) reduces to the ordinary differential equation

$$\theta\dot{y} = g(y) + k\ddot{y}. \tag{3}$$

This equation (3) is the central interest of Level 6; compare it with equation (1) above. If k is small then (3) can be regarded as a singular perturbation of (1), but in important applications k is large, and so new methods are needed.

For instance (3) can be regarded as a flow on TY, with the same fixed points as (1), on the zero section Y. An attractor of (1) may be a saddle point of (3), and a homoclinic orbit of this saddle will represent a pulse solution of (2). Meanwhile a closed orbit of (3) represents a wave train solution of (2). Therefore we seek homoclinic and closed orbits of (3), that are stable with respect to (2). As yet relatively little is known, even when g represents a canonical elementary catastrophe with the simplest form of feedback.

Example 6. In the heartbeat and nerve impulse dynamics [23] Conley and Carpenter [2] have shown the existence of homoclinic and closed orbits, and the next problems are to prove stability and fit data.

Example 11. Zhabotinsky Reagents. Belousov discovered a mixture of chemicals that oscillates in colour at about twice a minute, and later Zhabotinsky and Zaiken observed that circular wave trains would propagate through this reagent, entraining the oscillation. Winfree [21] then modified Belousov's reagent by adding a little more bromide and a little less acid, so as to stop the oscillation. He called his mixture the Z-reagent, after Zhabotinsky and Zaiken, and showed that it could sustain both pulses and rotating scroll-shaped wave trains. In [21] Winfree offers equations which beautifully explain the geometry of the patterns, but which can be mildly criticised on four counts. Firstly his dynamic is discontinuous, and the obvious way to make the model differentiable is to approximate it by a catastrophe model. Indeed as Kopell and Howard [6] point out there are both fast (fractions of a second) and slow (minutes) reactions, as well as a very slow (hours) loss of energy. Therefore one would normally expect the reaction dynamic to belong to Level 4. Secondly Winfree's equations do not illustrate the modification Belousov $\to Z$. However this can be illustrated naturally in catastrophe theory, by modifying one constant, causing a Hopf bifurcation, as we show below. Thirdly his equations exhibit a jump return, like the heartbeat, whereas his photographs illustrate a smooth return, blue $\to$ red, as opposed to the catastrophic hard edge, red $\to$ blue, more like the repolarisation of the nerve impulse. This feature can be accommodated by using the cusp-catastrophe [20], [23]. Fourthly he does not offer mathematical proof of existence and stability.

As Winfree has pointed out [20], the first two criticisms are answered by a 2-dimensional fold-catastrophe model as follows (cf. [23, Figures 7, 9]). Let $Y = R^2$, and let g be given by

$$\begin{aligned} &D\text{, fast dynamic:} \quad \dot{x} = -(x^3 - 3x + a), \\ &F\text{, slow feedback:} \quad \dot{a} = \varepsilon(x - \lambda), \end{aligned}$$

where ε, λ are constants and ε is small. For the Belousov reagent choose $\lambda < 1$, and for the Z-reagent $\lambda > 1$. Then by [23] the decrease of the parameter λ past the value $\lambda = 1$ gives the Hopf bifurcation. The resulting flows are illustrated in Figure 8, with the catastrophe slow manifold shown dotted. For the Belousov reagent a theorem of Kopell and Howard [6] ensures the existence and stability of closed orbits for equation (3) near the Van der Pol attractor, but only provided diffusion is sufficiently small. For the Z-reagent Conley and Carpenter [2] have proved the existence, but not yet the stability, of homoclinic and closed orbits, provided ε is sufficiently small, for the case of a large diffusion of x. What is needed is to handle both cases together and prove stability for large diffusion, giving an estimate on ε, the ratio of slow/fast. Then extend the results to the cusp-catastrophe [23, Example 8]. Finally identity the equations with the explicit chemical reactions, make a quantitative model, and predict the speeds of the various waves.

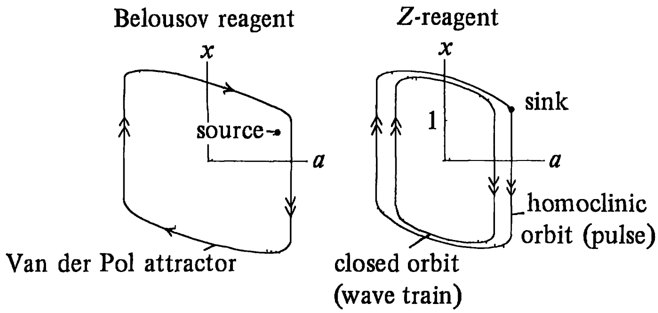

FIGURE 8

References

1. V. I. Arnold, *Singularities of differentiable functions*, these PROCEEDINGS.

2. G. A. Carpenter, *Travelling wave solutions of nerve impulse equations*, Thesis, University of Wisconsin, Madison, Wis., 1974.

3. D. H. Fowler, *The Riemann-Hugoniot catastrophe and Van der Waal's equation*, Towards a Theoretical Biology 4, edited by C. H. Waddington, Edinburgh Univ. Press, 1972, pp. 1–7.

4. C. Hall, P. J. Harrison, H. Marriage, P. Shapland and E. C. Zeeman, *A model for prison disturbances*, Jour. Math. and Stat. Psychology (to appear).

5. C. A. Isnard and E. C. Zeeman, *Some models for catastrophe theory in the social sciences*, Use of Models in the Social Sciences, edited by L. Collins, Tavistock, London, 1974.

6. N. Kopell and L. N. Howard, *Pattern formation in the Belousov reaction* (A.A.A.S., 1974, Some Math. Questions in Biology, VIII), Lectures on Math. in the Life Sci., vol. 7, Amer. Math. Soc., Providence, R. I., 1974, pp. 201–216.

7. K. Lorenz, *On aggression*, 1963; English transl., Methuen, London, 1967.

8. J. N. Mather, *Right equivalence*, Warwick University, 1969 (preprint).

9. T. Poston and A. E. R. Woodcock, *Zeeman's catastrophe machine*, Proc. Cambridge Philos. Soc. **74** (1973), 217–226.

10. ———, *A geometrical study of the elementary catastrophes*, Lecture Notes in Math., vol. 373, Springer-Verlag, Berlin, 1974.

11. L. S. Shulman and M. Revzen, *Phase transitions as catastrophes*, Collective Phenomena **1** (1972), 43–47.

12. F. Takens, *Constrained differential equations*, Dynamical Systems, Warwick, 1974, edited by A. K. Manning, Lecture Notes in Math., vol. 468, Springer-Verlag, Berlin, pp. 80–82.

13. R. Thom, *Topological models in biology*, Topology **8** (1969), 313–335. MR **39** #6629.

14. ———, *Stabilité structurelle et morphogénèse*, Benjamin, New York, 1972.

15. ———, *Phase-transitions as catastrophes*, Conf. on Stat. Mechanics, Chicago, Ill., 1971.

16. ———, *A global dynamical scheme for vertebrate embryology* (A.A.A.S., 1971, Some Math. Questions in Biology, IV), Lectures on Math. in the Life Sci., vol. 5, Amer. Math. Soc., Providence, R. I., 1973, pp. 1–45.

17. D. J. A. Trotman and E. C. Zeeman, *The classification of elementary catastrophes of codimension* ≦ 5, Lecture Notes, Warwick University, 1974.

18. G. Wassermann, *Stability of unfoldings*, Lecture Notes in Math., vol. 393, Springer-Verlag, Berlin, 1974.

19. ———, *(r, s)-stability of unfoldings*, Regensburg Universität, 1974 (preprint).

20. A. T. Winfree, *Spacial and temporal organisation in the Zhabotinsky reaction*, Aakron Katchalsky Memorial Sympos., Berkeley, Calif., 1973.

21. ———, *Rotating chemical reactions*, Scientific American **230** (1974), 82–95.

22. E. C. Zeeman, *Geometry of catastrophes*, Times Literary Supplement, 1971, pp. 1556–1557.

23. ———, *Differential equations for heartbeat and nerve impulse*, Dynamical Systems, edited by M. M. Peixoto, Academic Press, New York, 1973, pp. 683–741.

24. ———, *A catastrophe machine*, Towards a Theoretical Biology 4, edited by C. H. Waddington, Edinburgh Univ. Press, 1972, pp. 276–282.

25. ———, *Applications of catastrophe theory*, Manifolds, Tokyo, 1973, Univ. Tokyo Press, 1975, pp. 11–23.

26. ———, *On the unstable behaviour of stock exchanges*, J. Math. Economics **1** (1974), 39–49.

27. ———, *Primary and secondary waves in developmental biology* (A.A.A.S., 1974, Some Math. Questions in Biology, VIII), Lectures on Math. in the Life Sci., vol. 7, Amer. Math. Soc., Providence, R. I., 1974, pp. 69–161.

UNIVERSITY OF WARWICK
COVENTRY, ENGLAND CV4 7AL

Section 20

History and Education