An introduction to flows on homogeneous spaces

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Homogeneous spaces

G a locally compact group -
(always assumed to be 2nd countable).

Ex.: $\mathbb{R}, \mathbb{R}^n, n \geq 2$, $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$
closed subgroups, quotients by closed normal subgroups,
direct products, covering groups, ...

X a homogeneous space; viz. $X = G/H$, where H is a closed subgroup, consisting of the cosets $gH, g \in G$, equipped with the quotient topology; (this is locally compact).

Equivalently, a topological space with a continuous action
of G which is transitive.

Ex.: $R^n \backslash (0) \approx GL(n, \mathbb{R})/\{g \in GL(n, \mathbb{R}) \mid ge_1 = e_1\}$
(we denote by $\{e_i\}$ the standard basis of $\mathbb{R}^n$).
Similarly $\mathbb{P}^{n-1}$, Grassmannian manifolds

$SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, G/Γ where Γ is a discrete subgroup of G.

Measures on homogeneous spaces

We consider a homogeneous space G/H equipped with the
G-action on the left: $(g, xH) \mapsto (gx)H$ for all $g, x \in G$.

Theorem G/H admits a G-invariant (Borel) measure if and only if $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$.

(Here Δ stands for the modular homomorphism of the group in the suffix.)

$\mathbb{R}^n/\mathbb{Z}^n$ admits an $\mathbb{R}^n$-invariant measure

$\mathbb{R}^n \backslash (0)$ admits a $GL(n, \mathbb{R})$-invariant measure, viz the restriction of the Lebesgue measure.

$\mathbb{P}^{n-1}$ does not admit a $GL(n, \mathbb{R})$-invariant measure.

$SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ admits an $SL(n, \mathbb{R})$-invariant measure.

In the last example the invariant measure is finite. Homogeneous spaces with finite invariant measure are of special interest.

Lattices

Defn.: A closed subgroup Γ of G is called a lattice in G if Γ is discrete and G/Γ admits a finite G-invariant measure.

$\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$.

$SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$.

If Γ is a discrete subgroup of $SL(n, \mathbb{R})$ such that $SL(n, \mathbb{R})/\Gamma$ is compact, then Γ is a lattice.

A lattice for which the corresponding quotient is compact is said to be uniform; otherwise it is said to be nonuniform -

$SL(n, \mathbb{Z})$ is a nonuniform lattice in $SL(n, \mathbb{R})$: Let $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$. Suppose G/Γ is compact. Then there exists a compact subset K of G such that $G = K\Gamma$ $(= \{x\gamma \mid x \in K, \gamma \in \Gamma\})$. Then $G(e_1) = K\Gamma(e_1) \subset K\mathbb{Z}^n$, but this is not possible.

Flows

G a locally compact group and Γ a lattice in G.

For a closed subgroup H of G the H-action on G/Γ is called the flow induced by H on G/Γ. Typically we shall be interested in actions of cyclic subgroups (equivalently of elements of G), or one-parameter flows, namely actions induced by (continous) one-parameter subgroups $\{g_t\}_{t\in\mathbb{R}}$ where $g_t \in G$ for all $t \in \mathbb{R}$.

Ex.:
$G = R^n$, $\Gamma = \mathbb{Z}^n$, $H = \{tv \mid t \in \mathbb{R}\}$, where $v \in \mathbb{R}^n$.
$G = SL(2,\mathbb{R})$, Γ a lattice in G and $H = \{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R}\}$.

This corresponds to the 'geodesic flow' associated with the surfact $\mathbb{H}^2/\Gamma$ where $\mathbb{H}^2$ is the Poincaré upper half-plane. In particular when $\Gamma = SL(n,\mathbb{Z})$ it corresponds to the geodesic flow associated with the 'modular surface'.

'Ergodic' properties

Let (X,μ) be a finite measure space - assume $\mu(X) = 1$.
Let $T : X \to X$ be a measurable transformation preserving μ;
that is, $\mu(T^{-1}(E) = \mu(E)$ for all Borel subsets E.

Defn.: T is said to be *ergodic* if for a measurable subset E, if $\mu(T^{-1}(E)\backslash E) = 0 = \mu(E\backslash T^{-1}(E))$ then $\mu(E) = 0$ or 1 (the latter means $\mu(X\backslash E) = 0$).

Exc.: T is ergodic iff for any measurable set E if $T^{-1}(E) = E$, $\mu(E) = 0$ or 1.

Similar definitions for group actions to be ergodic. Also the statement analogous to the Excercise holds in general (more technical).

Defn.: T is said to be *mixing* if for any two measurable subsets
A and B, $\mu(T^{-k}(A) \cap B) \to \mu(A)\mu(B)$, as $k \to \infty$.

Remark: Mixing implies ergodicity. The converse does not hold
(as we shall see).

More about mixing

The action of a noncompact locally compact group, on (X,μ)
as above, is said to be *mixing* if for any divergent sequence
$\{g_k\}$ in G and any two measurable subsets A and B,
$\mu(gk(A) \cap B) \to \mu(A)\mu(B)$, as $k \to \infty$.

Remarks: For the cyclic subgroup the above definition coincides with the previous one. Also, mixing implies ergodicity in the general case as well.

If G-action on (X, μ) is mixing and H is a closed noncompact subgroup of G then H-action is mixing. The analogous statement however does not hold for ergodicity.

There are a variety of weaker and stronger forms of mixing, for cyclic as well as general group actions, that we shall not go into here.

Topological implications

Proposition Let X be a G-space with a G-invariant measure μ, with $\mu(X) = 1$. Suppose that $\mu(\Omega) > 0$ for all nonempty open subset Ω. Then we have the following:

i) if the action is ergodic then almost all G-orbits are dense in X; that is

$$\mu(\{x \in X \mid Gx \text{ not dense in } X\}) = 0.$$

ii) if the action is mixing then for any divergent sequence $\{g_k\}$

$$\mu(\{x \in X \mid \{g_k x\} \text{ not dense in } X\} = 0.$$

Proof of i): Let $\{\Omega_j\}$ be a countable basis for the topology on X. Then each $G\Omega_j$ is an open G-invariant subset and hence by ergodicity $\mu(G\Omega_j) = 1$, and in turn $\mu(\cap_j G\Omega_j) = 1$. The assertion now follows, since for all $x \in \cap_j G\Omega_j$ the G-orbit intersects each Ω_j and hence is dense in X.

Translation flows on tori

Let $v = (\alpha_1, \ldots, \alpha_n)^t$. Then we have the following, for the flows on tori, namely $\mathbb{R}^n/\mathbb{Z}^n$.

Proposition The translation of $\mathbb{R}^n/\mathbb{Z}^n$ by v is ergodic if and only if $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ (that is, no nontrivial linear combination $\sum_1^n q_i\alpha_i$, with $q_i \in \mathbb{Q}$, is rational).

Proof: Let T denote the translation of $\mathbb{R}^n/\mathbb{Z}^n$ by v and let E be a measurable subset such that $T^{-1}(E) = E$. Let f denote the characteristic function of

E. Consider the Fourier expansion of f in $L^2(\mathbb{R}^n/\mathbb{Z}^n)$, say $f = \sum_\chi a_\chi \chi$, where the summation is over the all characters on $\mathbb{R}^n/\mathbb{Z}^n$. The invariance of E under T implies that
$f \circ T$ and f are equal. We see that $f \circ T = \sum_\chi a_\chi \chi(v)\chi$ and hence by uniqueness of the Fourier expansion it follows that $a_\chi \chi(v) = a_\chi$.
The condition as in the hypothesis is seen to then imply that
$a_\chi = 0$ for all nontrivial characters χ, namely that f is constant.
Hence $\mu(E) = 0$ or 1.

More about the translation flows

We note that every orbit of the translation action is a coset of the subgroup of $\mathbb{R}^n/\mathbb{Z}^n$ generated by $v + \mathbb{Z}^n$. When the action is ergodic then there exists a dense coset, and hence the subgroup is dense as well. (On the other hand the latter statement may be proved directly and used to deduce ergodicity.) Conversely, when the subgroup generated by $v+\mathbb{Z}^n$ is dense in $\mathbb{R}^n/\mathbb{Z}^n$ *all* orbits are dense - note that ergodicity assures only *almost all* orbits to be dense, so what we see here is a rather special situation.

When all orbits are dense the action is said to be minimal. Thus for translations of tori ergodicity implies minimality.

The above also means that when the translation action is ergodic the subgroup generated by v and $\mathbb{Z}^n$ is dense in R^n.

The flow induced by $\{tv \mid t \in \mathbb{R}\}$ is ergodic if and only if $\alpha_1, \dots, \alpha_n$. Equivalently the flow is ergodic if and only if there exists $t \in \mathbb{R}$ such that the translation action of tv is ergodic.

For one-parameter translation flows also ergodicity implies minimality.

Unitary representations

$\mathcal{H}$ a (separable) Hilbert space.

$\mathcal{U}(\mathcal{H})$ the group of all unitary operators on $\mathcal{H}$.

A unitary representation π of G over $\mathcal{H}$ is a homomorphism of G into $\mathcal{U}(\mathcal{H})$ which is continuous with respct to the strong operator topology; that is $g \mapsto \pi(g)\xi$ is continuous for all $\xi \in \mathcal{H}$.

Let (X, μ) be a measure space with $\mu(X) = 1$ equipped with a G-action and let $\mathcal{H} = \mathcal{L}^{\in}(\mathcal{X}, \mu)$. The action induces a unitary representation of G over $\mathcal{H}$, by

$$\pi(g)f(x) = f(g^{-1}x) \text{ for all } g \in G, f \in \mathcal{H} \text{ and } x \in X$$

(following standard abuse of notation we view elements of $\mathcal{H}$ as pointwise functions - while this does involves some technical issues, in the final analysis there is no ambiguity).

Applying the associated unitary representation

The ergodicity and mixing conditions can be translated to the following in terms of the associated unitary representation.

Proposition: i) The G-action on X is ergodic if and only if there is no nonconstant function in $\mathcal{H}$ fixed under the action of $\pi(g)$ for all $g \in G$.

ii) The G-action on X is mixing if and only if for any divergent sequence $\{g_k\}$ in G and all $\phi, \psi \in \mathcal{H}$,

$$\langle \pi(g_k)\phi \rangle \to \langle \phi, 1 \rangle \langle 1, \psi \rangle \text{ as } k \to \infty.$$

It is convenient to consider the restriction of π to the orthocomplement of constants, $\mathcal{H}_0 = \{\phi \in \mathcal{H} \mid \phi \perp 1\}$ (which is an invariant subspace).

Proposition If $\pi(g)(\phi) \to 0$ as $g \to \infty$ for all $\phi \in \mathcal{H}_0$ then the action is mixing.

Mautner phenomenon

Let e denote the identity element in G and for $g \in G$ let

$$H_g^+ = \{x \in G \mid g^k x g^{-k} \to e \text{ as } k \to \infty\}$$

and

$$H_g^- = \{x \in G \mid g^k x g^{-k} \to e \text{ as } k \to -\infty\}$$

There are called the contracting and expanding horospherical subsgroups corresponding to g.

The following simple observation, known as Mautner phenomenon is very useful in proving ergodicity and mixing properties.

Theorem Let π be a unitary representation of G over a Hilbert space $\mathcal{H}$. Let $g \in G$ and $\phi \in \mathcal{H}$ be such that $\pi(g)(\phi) = \phi$. Then $\pi(x)(\phi) = \phi$ for all x in the subgroup generated by $H_g^+ \cup H_g^-$.

Proof of Mautner phenomenon

We may assume $\|\phi\| = 1$.

Let $x \in H_g^+$ be arbitrary. We have

$$\langle \pi(x)\phi, \phi\rangle = \langle \pi(g)\pi(x)\phi, \pi(g)\phi\rangle = \langle \pi(g)\pi(x)\pi(g^{-1})\phi, \phi\rangle,$$

under the given condition. Thus

$$\langle \pi(x)\phi, \phi\rangle = \langle \pi(g)\pi(x)\pi(g^{-1})\phi, \phi\rangle$$

and by successive application of the same it equals $\langle \pi(g^k)\pi(x)\pi(g^{-k})\phi, \phi\rangle$ for all k.

As the latter converges to $\langle \phi, \phi\rangle = 1$ we get that $\langle \pi(x)\phi, \phi\rangle = 1$.

But then $\|\pi(x)\phi - \phi\|^2 = 2 - \Re\langle \pi(x)\phi, \phi\rangle = 0$, which shows that $\pi(x)(\phi) = \phi$.

Similarly the same holds for $x \in H_g^-$ and hence for the subgroup generated by $H_g^+ \cup H_g^-$.

Ergodicity of the geodesic flow

Let $G = SL(2, \mathbb{R})$ and $g = \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, with $0 < \lambda < 1$. Then it can be seen that

$$H_g^+ = \{\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}\}$$

and

$$H_g^- = \{\{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R}\}.$$

We note also that the subgroups H_g^+ and H_g^- as above generate the whole of $SL(2,\mathbb{R})$. Hence we have

Corollary Let ϕ be a unitary representation of $SL(2,\mathbb{R})$ over a Hilbert space $\mathcal{H}$. If $\phi \in \mathcal{H}$ is fixed by g as above then it is fixed by the $SL(2,\mathbb{R})$ action.

Corollary Let Γ be a lattice in $SL(2,\mathbb{R})$. Then the action of g on G/Γ is ergodic.

Ergodicity of the Horocycle flow

The action of the subgroup $\{\{h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (which played a role in the proof of ergodicity of the geodesic flow) is called the *horocycle flow.*

(The other (lower trianglular) subgroup is conjugate to this one, and need not be considered separately, with regard to its properties - only in dealing with the geodesic flow we need both, in which case we talk of the contracting and expanding horocycle flows.)

Theorem The horocycle flow on G/Γ is ergodic.

Proof. Let $\mathcal{H} = L^2(G/\Gamma)$, π the associated representation, and $\phi \in \mathcal{H}$ be fixed by $\pi(h_t)$ for all t. We assume $\|\phi\| = 1$.

Let F be the function on G defined by $F(g) = \langle \pi(g)\phi, \phi \rangle$. It is a continuous function and $F(h_s g h_t) = F(g)$ for all $s, t \in \mathbb{R}$.

Now let f be the function on $\mathbb{R}^2 \backslash (0)$ defined by $f(ge_1) = F(g)$ for all $g \in G$. Then f is a well-defined continuous function and $f(h_s v) = f(v)$ for all $v \in \mathbb{R}^2 \backslash (0)$.

Proof contd.

For all $v \in \mathbb{R}^2$ which are not on the x-axis the orbits of $\{h_s \mid x \in \mathbb{R}\}$ consist of horizontal lines; the points on the x-axis are fixed points of the flow. Thus the invariance property of f as above implies that it is constant along horizontal lines, except perhaps the x-axis. But then by continuity it must be constant also along the x-axis (that is, $f(te_1) = f(e_1)$ for all $t \neq 0$.

We thus get that for $g = \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix}$, $F(g) = f(ge_1) = f(\lambda e_1) = f(e_1) = F(e) = 1$. Arguing as before we
see that this implies that ϕ is fixed by $\pi(g)$.

Since the action of g is ergodic it follows that ϕ is a constant function. Hence the action of $\{h_t\}$ on G/Γ is ergodic.

Flows on $SL(n, \mathbb{R})/\Gamma$

Let me now mention the stronger results with regard to ergodicity and mixing on homogeneous spaces of $SL(n, \mathbb{R})$), modulo a lattice.

Theorem Let H be a closed noncompact subgroup of $SL(n, \mathbb{R})$. Then its action on G/Γ is mixing. In particular it is ergodic.

The action on $SL(n, \mathbb{R})/\Gamma$ by a compact subgroup of $SL(n, \mathbb{R})$ can not be ergodic; by the result on density of orbits such an action would have to be transitive, which is not possible for a proper subgroup. Thus the action of a closed subgroup H on $SL(n, \mathbb{R})/\Gamma$ is ergodic if and only if the subgroup is noncompact.

In particular, for $G = SL(2, \mathbb{R})$ for almost all $x \in G/\Gamma$ the trajectories $\{g^j x \mid j \in \mathbb{Z}\}$ are dense in G/Γ, if $g = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, for any $t \neq 0$.

Orbits of individual points

It is however a nontrivial matter in general to determine for which particular x the trajectory is dense under a flow. The horocycle flows are well behaved in this respect.

Theorem: Let $G = SL(2, \mathbb{R})$ and Γ be a lattice in G. Let $\{h_t\}$ be
a horocycle flow. Then we have the following:

i) if G/Γ is compact then for any $t \neq 0$ all trajectories of h_t are
dense in $\Gamma \backslash G$; that is, the horocycle flow is minimal.

ii) for any x, either $h_t x = x$ for some $t \neq 0$ or the trajectory of x under h_t is dense for all $t \neq 0$.

If g is any element normalising $\{h_t\}$ (which applies to the elements "contracting" or "expanding" the subgroup as seen before) and x is a periodic point of $\{h_t\}$ then gx is also a periodic point of $\{h_t\}$. The periodic points thus form "cylindars". The number of such cylindars is the same as the number of "cusps" of the associated surface.

Uniformly distributed orbits

Apart from being dense the non-periodic orbits are also "uniformly distributed": Let x be a point with a non-periodic orbit and let $t \neq 0$. Then we have the following:

if Ω is an open subset of G/Γ such that $\mu(\partial\Omega) = 0$ then as $k \to \infty$

$$\frac{|\{0 \le j \le k-1 \mid h_t^j x \in \Omega\}|}{k} \longrightarrow \frac{\mu(\Omega)}{\mu(X)}$$

This means that the trajectory visits "good" set with frequency equal to their proportion in the space (with respect to μ).

For $\Gamma = SL(2,\mathbb{Z})$, $\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma \mid n \in \mathbb{Z}_+\}$ is dense if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P\Gamma$, where P is the subgroup consisting of all upper triangular matrices.

Diophantine approximation

Here is a consequence of the above to Diophantine approximation.

Corollary: Let $\Gamma = SL(2,\mathbb{Z})$ and consider the natural Γ-action on $\mathbb{R}^2$. Then for $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\Gamma(v)$ is dense in $\mathbb{R}$ if and only if $v_1 \neq 0$ and v_2/v_1 is irrational.

Thus given an irrational α, given any $w_1, w_2 \in \mathbb{R}$, and $\epsilon > 0$ there exist $p, q, r, s \in \mathbb{Z}$, with $ps - qr = 1$ such that

$$|p\alpha + q - w_1| < \epsilon \text{ and } |r\alpha + s - w_2| < \epsilon.$$

Note in particular that we can get solution (p, q) to the first inequality consisting of a primitive pair (having gcd 1).

A similar question arose for quadratic forms, in Oppenheim conjecture, which was settled by Margulis in mid 1980's.

Oppenheim conjecture

Theorem: Let $n \geq 3$ and $Q(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$, such that $a_{ij} = a_{ji}$ for all i, j, $\det(a_{ij}) \neq 0$. Suppose that

i) there exists a nonzero $(v_1, \ldots, v_n)$ such that $Q(v_1, \ldots, v_n) = 0$ and

ii) a_{ij}/a_{kl} is irrational for some i, j, k, l.

Then given $a \in \mathbb{R}$ and $\epsilon > 0$ there exist $x_1, \ldots x_n \in \mathbb{Z}$ such that

$$|Q(x_1, \ldots x_n) - a| < \epsilon.$$

Moreover the n-tuple $(x_1, \ldots x_n)$ can be chosen to be primitive.

One can be reduce in a routine way to the case of $n = 3$.

Let H be the subgroup of $SL(3, \mathbb{R})$ consisting of the elements leaving the quadratic form Q fixed, that is $\{g \in G \mid Q(gv) = Q(v) \ \forall v \in \mathbb{R}^3\}$. Then $Q(\mathbb{Z}^3) = Q(H\Gamma\mathbb{Z}^3)$ and hence if $H\Gamma$ is dense then it follows that $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$. One shows that under condition (ii) $H\Gamma$ is not closed. Thus the task is to show that every H-orbit on G/Γ which is not closed is dense. This is analogous to what we saw for the horocycle flows, but more intricate.

Quantitative version

While existence of solutions can be dealt with via consideration of density of orbits, uniform distribution of orbits enables to get asymptotic results on the number of solutions in large balls.

Uniform distribution in the general case is studied via classification of invariant measures.

Using some results of Marina Ratner analogous on classificaiton of invariant measures of unipotent flows, quantitative results are obtained// (D. & Margulis)

In particular, for any $a, b \in \mathbb{R}$, there exists a $c > 0$ such that

$$\#\{x \in \mathbb{Z}^n \mid a < Q(x) < b, \|x\| \leq r\} \approx cr^{n-2}.$$

This is achieved by comparing the number on the left hand side with the volume of the region $\{v \in \mathbb{R}^n \mid a < Q(v) < b, \|v\| \leq r\}$.

For $n \geq 5$, the two sequences of numbers turn out to be asymptotic to each other. For $n = 3, 4$ there are some interesting situations when the solutions can be more than that "expected value".

Geodesic flows

The orbit structure of the geodesic flow is much more complicated. To describe the situation in this respect we recal the geometric form.

The Poincaré upper half plane is

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\},$$

equipped with the Riemannian metric, called the Poincare metric,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

The distance between any two points z_1, z_2 is given by

$$d(z_1, z_2) = \inf \int y(t)^{-1} \sqrt{(\frac{dx}{dt}(t))^2 + (\frac{dy}{dt}(t))^2} dt,$$

with inf taken over piecewise C^1 curves $(x(t), y(t)$ joining z_1 and z_2.

The geodesics in this metric are vertical lines over points of the x-axis, or semicircles orthogonal to the x-axis; the x-axis is not in $\mathbb{H}$ but at its "boundary". We denote by $S(\mathbb{H})$ the "unit tangent bundle", viz. the set of pairs (z, ξ) where $z \in \mathbb{H}$ and ξ is tangent direction at the point z.

The "geodesic flow" corresponding to $\mathbb{H}$ is the flow $\{\varphi_t\}_{t \in \mathbb{R}}$ defined on $S(\mathbb{H})$ as follows: let $(z, \xi) \in S(\mathbb{H})$ be given and let $\gamma(t)$ be the geodesic (parametrized by the length parameter) starting at z and pointing in the direction ξ; then we choose $\varphi_t(z,\xi) = (z_t, \xi_t)$, where $z_t = \gamma(t)$ and $\xi = \gamma'(t)$.

The group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ has an action on $\mathbb{H}$, where $g \approx \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$ acts by

$$g(z) = \frac{az+b}{cz+d} \text{ for all } z \in \mathbb{H}.$$

This action leaves invariant the Poincaré metric, and hence the action of each $g \in PSL(2,\mathbb{R})$ is an isometry. (These isometries form a subgroup of index 2 in the group of all isometries with respect to the Poincaré metric.

The action on $\mathbb{H}$ induces also, canonically, an action of $PSL(2,\mathbb{R})$ on $S(\mathbb{H})$.

Using the action $PSL(2,\mathbb{R})$-action we can identify $S(\mathbb{H})$ with $PSL(2,\mathbb{R})$, via the correspondence

$$g \in PSL(2,\mathbb{R}) \leftrightarrow g(i,v),$$

where v is the direction at i pointing upward.

Under the identification the geodesic flow corresponds to $\psi_t : PSL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ given, for all $t \in \mathbb{R}$, by

$$\psi_t(g) = g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

(the product of matrices, viewed modulo $\{\pm I\}$).

Geodesic flow of the modular surface

We now consider images of the geodesics in $\mathbb{H}$ in the quotient $PSL(2,\mathbb{Z})\backslash\mathbb{H}$.

Recall that a geodesic in $\mathbb{H}$ is determined by two (distinct) points in $\mathbb{R} \cup \{\infty\}$, which we call its "endpoints"; the end points constitute an ordered pair, as the geodesics are considered oriented.

For $\alpha, \beta \in \mathbb{R}$ let $g(\alpha, \beta)$ be the geodesic in $\mathbb{H}$ with endpoints (α, β), and $\overline{g}(\alpha, \beta)$ be its image in $PSL(2, \mathbb{Z})\backslash\mathbb{H}$.

i) $\overline{g}(\alpha, \beta)$ is a closed subset if and only if α and β are rational.

ii) $\overline{g}(\alpha, \beta)$ is periodic if and only if α and β are conjugate quadratic numbers.

iii) $\overline{g}(\alpha, \beta)$ is contained in a compact subset of $PSL(2, \mathbb{Z})\backslash\mathbb{H}$ if and only if α and β are badly approximable.

A real number θ is said to be "badly approximable" if there exists a $\delta > 0$ such that for all $p, q \in \mathbb{Z}$, $q \neq 0$, $|\theta - \frac{p}{q}| > \delta/q^2$. **Continued fractions**

Every real number θ has a "continued fraction" expansion as

$$\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}},$$

where $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for all $k \in \mathbb{N}$.

The expansion terminates if θ is rational and is infinite if θ is irrational.

The a_k's are called the partial quotients of the continued fraction expansion. The above properties have the following analogues:

θ is a quadratic number if and only if the partial quotients are eventually periodic; i.e. there exist m and l such that $a_{k+l} = a_k$ for all $k \geq m$.

θ is badly approximable if and only if the partial quotients are bounded; i.e. there exists M such that $a_k \leq M$ for all k.

The last statement in particular tells us that badly approximable numbers exist. They form a set of Lebesgue measure 0, but nevertheless constitute a large set in other ways: the Hausdorff dimension of the set is 1, the maximal possible for a subset of $\mathbb{R}$.

Generic numbers

A real number θ to be generic if in the continued fraction expansion $(a_0, a_1, \ldots, a_k, \ldots)$ every finite block of positive integers occurs;
that is given $(b_1, \ldots, b_l)$, $b_k \in \mathbb{N}$, there exists m such that $a_m = b_1, \ldots a_{m+l-1} = b_l$. With this we can to the above list,

iv) $\overline{g}(\alpha, \beta)$ is dense in $PSL(2, \mathbb{Z})\backslash\mathbb{H}$ if and only if at least one of α and β is generic.

The set of generic numbers is a set of full Lesbegue measure. The ergodicity result mentioned earlier tells us that the set of pairs for which the conclusion as in (iv) holds must correspond to the endpoints being from a set of full measure. The point of (iv) however is that it gives a specific set of numbers, in terms of their continued fraction expansions, for which it holds. Also historically this result was proved, by E. Artin, before the ergodicity result came up. Recently in a paper with Nogueira a strengthening of the result was obtained with a different method.

Binary quadratic forms

For binary quadratic forms $Q(x, y) = (x - \alpha y)(x - \beta y)$, where $\alpha, \beta \in \mathbb{R}$ we have the following:

If α, β are badly approximable then there exists $\delta > 0$ such that $Q(\mathbb{Z}^2) \cap (-\delta, \delta) = \{0\}$, so in particular $Q(\mathbb{Z}^2)$ is not dense in $\mathbb{R}$.
(this may contrasted with the Oppenheim "conjecture" for $n \geq 3$).

If one of α and β is generic then $Q(\mathbb{Z}^2)$ is dense in $\mathbb{R}$.

With our alternative method we proved also the following strengthening.

If one of α and β is a positive generic number then $Q(\mathbb{N}^2)$ is dense in $\mathbb{R}$.

Recently we (D. & Nogueira) studied continued fractions for complex numbers, in terms of the Gaussian integers, and proved analogues of the density result for complex binary forms. In this case also this contrasts the analogue of Oppenheim conjecture, which is known (A. Borel & Gopal Prasad).

Higher dimensional situations

Theorem (Ratner): Let $G = SL(n, \mathbb{R})$ (or more generally a Lie group) and Γ be a lattice in G. Let $\{u_t\}$ be a unipotent one-parameter subgroup of G. Then for any $g \in G$ there exists a closed subgroup F of G such that the following holds:

i) $\Gamma g F$ is closed, $\Gamma \backslash \Gamma g F$ admits a F-invariant probability measure μ, and

ii) $\{\Gamma g u_t \mid t \geq 0\}$ is dense and uniformly distributed with respect to μ.

Subsets as seen above are called "homogeneous subsets". For the horocycle flow, where $G = SL(2,\mathbb{R})$, and $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$,
only G and $\{g^{-1}u_t g \mid t \in \mathbb{R}\}$ turn out to be the possible candidates for the subgroup F as above; correspondingly the $\{u_t\}$-orbits are either uniformly distributed in $\Gamma\backslash G$ or periodic.

This theorem has found a variety of applications in diophantine approximation and geometry that we shall not go into. **Actions of diagonal subgroups**

Even though the orbit structure of the geodesic flows is rather intricate as discussed, for $n \geq 3$ the orbit structure of the corresponding subgroup D_n on $\Gamma\backslash SL(n,\mathbb{R})$, Γ a lattice in $SL(n,\mathbb{R})$ is expected to be good, in a way similar to the unipotent case.

In particular, Margulis has conjectured that all orbits of D_n having compact closure are homogeneous.

This Margulis conjecture implies a well-known conjecture in diophantine approximation called Littlewood conjecture:

Let α, β be irrational real numbers. Then

$$\liminf_n \; n\{n\alpha\}\{n\beta\} > 0$$

where for any $\theta \in \mathbb{R}$, $\{\theta\}$ denotes the "fractional part" of θ.

Landmark work is done by Einsiedler, Katok and Lindenstrauss on these questions, but in its full form the question remains open. It has been shown in particular that the set of pairs (α, β) for which the Littlewood conjecture does not hold has Hausdorff dimension 0.

Thanks for your interest.