Extended bodies in Newtonian gravity III: computation of Love numbers

I. RELEVANT BACKGROUND AND SETUP

A. Basic equations

We consider a Newtonian star described by solutions to the Poisson equation for the gravitational potential

$$\nabla^2 U = -4\pi G\rho, \tag{1a}$$

the continuity equation (which is a statement of conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1b}$$

and Euler's equation (which is a statement of the conservation of momentum)

$$\frac{\partial v^i}{\partial t} + (\boldsymbol{v} \cdot \nabla) v^i = -\frac{\partial^i p}{\rho} + \partial_i U + a^i_{\rm ext}, \tag{1c}$$

where $a^i_{\rm ext}$ is the acceleration due to external forces.

B. Equilibrium configuration

The unperturbed object we consider is a static spherical star in isolation, with density profile $\rho_0(r)$, mass M and radius R. In this case the system (1) simplifies to equation of hydrostatic equilibrium and the mass integral (in units with $G = c = 1$)

$$\frac{dp_0}{dr} = \frac{\rho_0 m_0}{r^2}, \qquad m_0(r) = 4\pi \int_0^r r^2 \rho_0 dr, \tag{2}$$

where p_0 is the pressure and the subscript 0 denotes the unperturbed quantities. The mass-radius relation $M = M(R)$ is determined by the equation of state $p_0 = p_0(\rho_0)$.

C. Tidal perturbations

Suppose now that the star is placed in an external static tidal gravitational field. For simplicity we will consider only a quadrupolar perturbation, however, a similar calculation can be applied for any multipole moments. The external potential is

$$U_{\rm tidal} = -\frac{1}{2}\mathcal{E}_{ij} x^i x^j. \tag{3}$$

The star will deform in response to the tidal field and settle down to a new static configuration which has a nonzero quadrupole moment Q_{ij}. This quadrupole moment can be defined in terms of an expansion of the gravitational potential outside the star

$$U_{\rm total}(x) = \frac{M}{r} + \frac{3}{2} Q_{ij} \frac{n^{<ij>}}{r^3} - \frac{1}{2}\mathcal{E}_{ij} r^2 n^{<ij>} \tag{4}$$

in the sense that the quadrupole moment is associated with the piece of the exterior potential that falls off as $1/r^3$. Similarly, the tidal moment is related to the coefficient of the piece of $U_{\rm total}$ that grows as r^2. To linear order in the external tidal perturbation and in the adiabatic limit, the quadrupole distortion will be a linear response of the form

$$Q_{ij} = -\lambda \mathcal{E}_{ij} \tag{5}$$

where λ is the tidal deformability constant of the star.

Our goal in this exercise is to derive all the inputs needed to compute λ. This will require computing $U_{\rm total}$ using information about the perturbed interior of the star. Matching this interior description to the asymptotic behavior at large distances in (4) and will enable us determine λ.

II. EXERCISES

A. 1. Units of Love number

The generalized linear, adiabatic response of the star to a multipolar tidal perturbation is

$$M^L = -\lambda_\ell \mathcal{E}_L, \qquad \text{where} \qquad M^L = \int \delta\rho(x') x'^{<L>} d^3x', \qquad \mathcal{E}_L = -GM_B \partial_L r^{-1}, \tag{6}$$

where $\delta\rho$ is the density perturbation. Note that G has units [length]^3[mass]$^{-1}$[time]$^{-2}$. Determine the units of λ_ℓ.

B. 2. Linear Perturbations

The perturbed NS is still described by (1) but with $p = p_0 + \delta p$, $\rho = \rho_0 + \delta\rho$, and $U = U_0 + \delta U$ being the perturbed pressure, density, and gravitational potential. The external acceleration is due to the tidal potential $\boldsymbol{a}_{\rm ext} = \nabla U_{\rm tidal}$.

We can represent the fluid perturbation by a Lagrangian displacement $\xi(x,t)$, which is defined so that the fluid element at position x in the unperturbed star is at position $x + \xi(x,t)$ in the perturbed star. Then, to linear order, the Eulerian perturbations in velocity are

$$v = \delta v = \dot{\xi}, \tag{7}$$

Substitute the perturbed quantities into Euler's equation (1c), working only to linear order in the deviations from the background star, and write the result as an equation for $\ddot{\xi}$.

This equation is the foundation for the usual theory of stellar perturbations in terms of modes where one assumes a solution of the form $\xi(x,t) = e^{-i\omega t}\xi(x)$. This is needed to compute for example the oscillation mode frequencies of the neutron star [among the topics for next week]. However, for our problem of computing the Love number, it is not actually necessary to use a mode decomposition. Instead, we can specialize to static perturbations where $\dot{\xi} = \ddot{\xi} = 0$.

C. 3. Specialization to a relation $p = p(\rho)$

Specialize to an equation of state of the form $p = p(\rho)$ to relate $\partial_i p$ and δp to $\partial_i \rho$ and $\delta\rho$. Show that using these relations the terms involving δp and $\delta\rho$ in your result from 2. can be combined into the form

$$-\partial_i \left(\frac{1}{\rho_0} \frac{dp_0}{d\rho_0} \delta\rho \right). \tag{8}$$

D. 4. Algebraic relation between $\delta\rho$ and $\delta U_{\rm tot}$

Specialize your result from 2. using 3. to static perturbations where $\ddot{\xi} = 0$, substitute the external acceleration due to tidal forces $a^i_{\rm ext} = \partial_i U_{\rm tidal}$, and perform the integration to obtain an algebraic relation between $\delta\rho$ and $\delta U_{\rm tot} = \delta U + U_{\rm tidal}$.

E. 5. Spherical harmonic expansion and linear perturbations to Poisson's equation

Using the fact that $\mathcal{E}_L n^{<L>} = \sum_m \mathcal{E}_m Y_{\ell m}(\theta, \phi)$ we can write the external quadrupolar tidal potential as

$$U_{\rm tidal} = -\frac{1}{2} \sum_{m=-2}^{2} \mathcal{E}_m r^2 Y_{2m} \tag{9}$$

Similarly, we decompose the quadrupole as

$$Q_{ij} = \sum_{m=-2}^{2} Q_m \mathcal{Y}^{*2m}_{ij} \tag{10}$$

Note that sometimes the normalization is chosen as $Q_{ij} = \frac{8\pi}{15}\sum_m Q_m \mathcal{Y}^{*2m}_{ij}$ but we will absorb the prefactor into a re-definition of Q_m here for convenience. Using the property that $Y_{\ell m} = \mathcal{Y}^{*\ell m}_L n^L$ we see that when contracting the relation (I C) with n^{ij} it can be written as

$$Q_m = -\lambda \mathcal{E}_m, \tag{11}$$

where it is sufficient to consider a single value of m to compute λ.

It follows that the perturbations to all the quantities for a single value of m can be expanded as

$$\delta\rho = h(r) Y_{2m}(\theta, \phi), \qquad \delta U_{\rm tot} = g(r) Y_{2m}(\theta, \phi). \tag{12}$$

Consider Poisson's equation (1a) in spherical polar coordinates. Note that the Laplace operator is

$$\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2 \sin\theta}\partial_\theta \sin\theta \partial_\theta + \frac{1}{r^2\sin^2\theta}\partial^2_\phi \tag{13}$$

and that the spherical harmonics have the property that

$$\left(\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial^2_\phi\right) Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \tag{14}$$

Write down Poisson's equation to linear order in the perturbations and substitute the decomposition (12). Use Eqs. (13) and (14) to derive a differential equation for $g''(r)$.

F. 6. Master equation

Substitute the decomposition (12) and the expansion (9) in your result from 4. and combine this relation with your result from 5. to obtain a single equation for $g(r)$ in the region $r \leq R$ given by

$$g'' + \frac{2}{r}g' - \frac{6}{r^2}g = -\frac{4\pi}{q(r)}g, \tag{15}$$

where

$$q(r) = \frac{1}{\rho_0}\frac{dp_0}{d\rho_0} \tag{16}$$

In general, except for special choices of the equation of state, this has to be integrated numerically.

Since the tidal contribution to $g(r)$ is a homogeneous solution and given, we can write the equation in terms of the radial part $G(r)$ of δU instead of using g that corresponds to $\delta U_{\rm tot}$, and derive the boundary conditions on G.

Show that regularity at the centre of the star requires that $G(0) = G'(0) = 0$, and that the boundary condition at the star's surface is $G'(R) = -3G(R)/R$, where the derivative is taken just inside the surface. This follows from matching to the external potential $G^{\rm ext} = 3Q_m/(2r^3)$ at the star's surface. Here we assume that the density is continuous (and thus goes to zero at the surface of the star), and that G and G' are also continuous.

G. 7. Love number

For $r > R$ verify that the exterior solution

$$g(r) = \frac{3Q_m}{2r^3} - \frac{1}{2}\mathcal{E}_m r^2 \tag{17}$$

is indeed a solution to (15).

Use the definition of λ from Eq. (11) and write λ as

$$\lambda = \frac{2}{3}k_2 R^5, \tag{18}$$

to eliminate Q_m from Eq. (17).

Next, eliminate $\mathcal{E}_m$ by computing the quantity

$$y(r) = \frac{rg'(r)}{g(r)}. \tag{19}$$

Finally, evaluate Eq. (19) at the surface of the object $r = R$, and solve for k_2 to show that

$$k_2 = \frac{2 - y(R)}{2\left[3 + y(R)\right]|} \tag{20}$$

The strategy for practical computations of the Love number is thus the following: (i) obtain a solution for the background configuration, (ii) compute the perturbed interior described by (15), (iii) evaluate from this $y(R)$, and (iv) finally use it in Eq. (20) to obtain the Love number.

H. Bonus: explicit computation for a constant density star

Now, consider the simple case of a uniform density star. While this means that we do not have to worry about integrating to find the stellar structure, we do have to worry about the discontinuity in the density at the surface of the star in the boundary conditions. Here, one must be careful in matching the interior and exterior solutions because of the density discontinuity. Specifically, the density profile is a step function that vanishes outside the star, so the density can be written as $\rho(r) = \rho_c\left[1 - \Theta(R - r)\right]$, where ρ_c is the constant value and Θ is the Heaviside step function. Therefore, the term $q(r)$ will contribute a delta function discontinuity at the surface that must be taken into account. In computing the density discontinuity's contribution to the outer boundary condition it is convenient to use the full potential. Read Sec. VII B in Damour and Nagar, Phys. Rev. D 80, 084035 (2009) where the procedure is explained. Show that the boundary condition of matching the interior and exterior solutions is $RG'(R) = -3G(R) + 2g(R)$, where $G'(R)$ denotes the value of the derivative taken just inside the star's surface. Solve for G using the boundary conditions and matching the potential at the surface, show that

$$k_2^{\text{incompressible}} = \frac{3}{4} \tag{21}$$