Extended bodies in Newtonian gravity II: action principle and binding energy

1. Constructing the action describing a binary system of extended objects

In general, a Lagrangian is constructed from the kinetic and potential energies T and V as $\mathcal{L} = T - V$, and the action is given by $S = \int dt\mathcal{L}$. The total energy receives contributions from each body. As discussed in lecture, each of these can be split into a contribution from the center-of-mass motion of the body and an internal contribution from the dynamics of the multipole moments.

Consider a binary of a point mass B and an object having multipole moments up to the octupole ($\ell = 3$), with all higher moments vanishing.

(a) Consider first each of the bodies in its "buffer" zone, i.e. at distances large compared to the size of the body but small compared to the distance to the companion. Last time we discussed the multipole expansions of the potentials in this zone for $C = A$ or B:

$$U_C^{\text{body}}(x) = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{n_C^{<L>}}{r_C^{\ell+1}} M_C^{<L>}, \qquad U_C^{\text{companion}} = -\sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x - z_C)^L \mathcal{E}_L^C. \tag{1}$$

We also defined the mass multipole moments M_C^L and tidal moments $\mathcal{E}_C^L$ of an object C:

$$M_C^L = \int_C d^3x \rho_C (x - z_A)^L, \qquad \mathcal{E}_C^L = -\left(\partial_L U_C^{\text{companion}}\right)_{z_C}. \tag{2}$$

For the system under consideration, write down the companion's potential for each object.

(b) Write down the tidal moments $\mathcal{E}_L$ of object A for $\ell = 2, 3$ by using the fact that

$$\partial_L r^{-1} = (-1)^\ell (2\ell - 1)!! \frac{n^{<L>}}{r^{\ell+1}} \tag{3}$$

(c) Compute the potential energy V_C of the center-of-mass motion of each body from the definition

$$V_C = -\frac{1}{2}\int_C d^3x \rho_C U_C^{\text{companion}}. \tag{4}$$

(d) Express your result for V_B in terms of $\mathcal{E}_L$ by using your result from (b) and the fact that $n_{BA}^i = -n_{AB}^i$ with $n_{AB}^i \equiv n^i$, where n^i is the unit vector associated with the relative separation $n^i = (z_A^i - z_B^i)/r$ where $r = |\boldsymbol{z}_A - \boldsymbol{z}_B|$.

(e) From your results above, write down the orbital kinetic energy for each object and obtain the total kinetic and potential energies of the two-body system, T and V respectively.
Express your result in terms of μ, M, r, v^2 and the multipole and tidal moments of body A by using the definitions of the reduced mass $\mu = M_A M_B/(M_A + M_B)$, the total mass $M = M_A + M_B$, the relative separation r, and the relation of the velocities to the relative velocity v given by

$$\dot{z}_A^i = \frac{M_B}{M} v^i, \qquad \dot{z}_B^i = -\frac{M_A}{M} v^i. \tag{5}$$

(f) Show that the action $S = \int dt\, [T - V]$ describing the binary in this approximation is given by

$$S = \int dt \left[\frac{\mu}{2} v^2 + \frac{\mu M}{r} - \frac{1}{2} Q_{ij}\mathcal{E}_{ij} - \frac{1}{6} Q_{ijk}\mathcal{E}_{ijk} + L_{\text{internal}}(Q, \dot{Q}\ldots)\right], \tag{6}$$

where L_{internal} describes the dynamics of the object's quadrupole and octupole moments.
From the pattern in the multipolar terms in Eq. (6) one can generalize to a compact result for arbitrary multipole moments

$$S = \int dt \left[\frac{\mu}{2} v^2 + \frac{\mu M}{r} - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} Q_L \mathcal{E}^L + L_{\text{internal}}(Q, \dot{Q}\ldots)\right] \tag{7}$$

This provides an efficient way to compute the binary dynamics and, as discussed in lecture, has a straightforward generalization to general relativity for electric-type tidal interactions in the adiabatic approximation. For two extended bodies, to linear order in the multipole moments, the finite-size contribution from the other object can be added linearly to (6).

2. **Energetics of a binary on circular orbits**

We will continue to consider the system described by Eq. (6) but for simplicity keep only the quadrupole contribution, omitting the octupolar terms. While in exercise 1, the multipole moments of body A were arbitrary, we will now specialize to the case where they are tidally induced and given by the adiabatic approximation

$$Q_{ij}^{\text{adiab}} = -\lambda \mathcal{E}_{ij}, \tag{8}$$

where λ is the tidal deformability parameter. In this approximation the internal Lagrangian describes only the elastic potential energy associated with the tidal deformation

$$L_{\text{internal}}^{\text{adiab}} = -\frac{1}{4\lambda} Q_{ij} Q^{ij} \tag{9}$$

Throughout this exercise, we will assume that tidal effects are small and can be treated as linear perturbations.

(a) Combine Eqs. (6), (9), (8), and your result for $\mathcal{E}_{ij}$ from 1.(b), all specialized to the quadrupole case, to obtain the action explicitly in terms of $(r, \dot{r}, \dot{\phi})$. A useful identity is

$$n_{<L>} n^{<L>} = \frac{\ell!}{(2\ell-1)!!} \tag{10}$$

(b) Use the Euler-Lagrange equations to compute the equations of motion for r and ϕ.

(c) Assume that the orbit is circular so that $\ddot{r} = 0 = \dot{r}$ and define the orbital frequency $\dot{\phi} = \Omega$. Starting from the radial equation of motion specialized to circular orbits, express the radius as $r(\Omega) = M^{1/3}\Omega^{-2/3}(1+\delta r)$ and compute the linear tidal corrections δr.

(d) Calculate the energy of the system from the results that led to (6). Specialize to adiabatic quadrupoles and circular orbits, and express the energy in terms of Ω, to linear order in the tidal effects.
The quantity $E(\Omega)$ is a useful quantity that can also be computed from numerical relativity simulations. Comparisons of these results against theoretical models provide useful tests of the predictions. We will make further use of these results when computing the tidal effects on the GWs.