Accurate & Efficient inspiral templates for spinning compact binaries

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Outline

- Introductory slides for computing time-domain inspiral templates for spinning compact binaries
- New approach to obtain $h_{\times,+}(t)$ for spinning compact binaries
- Benefits of our approach & problems with the traditional approach

In collaboration with **Ms. Anuradha Gupta, TIFR**

GW phasing: I

- The *response function* of a laser-interferometric detector to GWs from ICBs with non-spinning components

$$h(t) \equiv \Delta L/L = \frac{C}{d}\left[\omega(t)\right]^{2/3} \sin 2\phi(t)\,,$$

d: the distance to the binary; $\phi(t)$: orbital phase & $\omega = \dot{\phi}(t)$

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- $h(t)$ requires PN-accurate expressions for $h_{\times,+}(t)$, the two GW polarization states, & how ω varies with time

- The secular phase evolution via the energy balance argument $\frac{dE}{dt} = -\mathcal{L}$

$$\frac{d\phi(t)}{dt} = \frac{c^3}{G\,m}\,x^{3/2}\,; \qquad \frac{d\,x(t)}{dt} = -\mathcal{L}(x)\Big/\frac{d\mathcal{E}}{dx}$$

where $x(t) \equiv \left(\frac{G\,m\,\omega(t)}{c^3}\right)^{2/3}$ is the dimensionless PN expansion parameter

GW phasing: II

- To construct 'ready-to-use' $h_{\times,+}(t)$ for data analysis purposes, we need to tackle two aspects of the dynamics
- Problem of finding equations to describe the dynamics; $\ddot{X}$, $\dot{\mathbf{S}}_1$, $\dot{\mathbf{S}}_2$;
- Problem of computing GW luminosity $\mathcal{L}$, polarization states $h_{\times,+}$

  Blanchet, Buonanno, Damour, Faye, Iyer, Jaranowski, Schäfer, Will, ...

$\ddot{X}$	N	1PN	2PN	2.5PN	3PN	3.5PN	4PN	4.5PN	5PN	5.5PN	6PN
$\dot{\mathbf{S}}_1$, $\dot{\mathbf{S}}_2$	—	N	1PN	—	2PN						
$\mathcal{L}$	—	—	—	N	—	1PN	1.5PN	2PN	2.5PN	3PN	3.5PN
$h_{\times,+}$	—	—	N	0.5N	1PN	1.5PN	2PN	2.5PN	3PN		

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- Linearly project interferometric data against each member of specific *template banks/families*

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  For GWs from non-spinning ICBs, templates should belong to a two-dimensional signal manifold

- Signal manifold's dimensionality is more for spinning compact binaries spiraling along quasi-circular orbits
  - The practice is to invoke 'approximate/phenomenological' template families, characterized by fewer parameters
  - These templates have 'good overlaps' with $h_{\times,+}(t)$ for spinning compact binaries obtained via the 'traditional phasing prescription'

Buonanno, Chen, Vallisneri; Ajith ; Brown *et.al*,.......

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How to search for GWs from ICBs: II

- We are taking a closer look at traditional approach to construct $h_{\times,+}(t)$ for spinning compact binaries inspiralling along quasi-circular/eccentric orbits
- It turns out that the traditional approach inherits few undesirable features
- *Kidder (1995), Buonanno, Chen, Vallisneri (2003), Arun et. al. (2009)* developed the widely used $h_{\times,+}(t)$ for spinning compact binaries inspiralling quasi-circular orbits

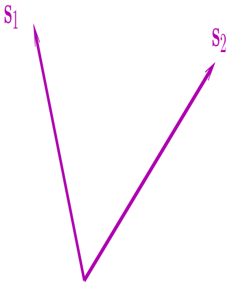
GWs from spinning ICBs

- We want to describe binaries containing two spinning compact objects, characterized by $(m_1, m_2, \chi_1, \chi_2)$

  We will invoke $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$ to describe binary orbits [This is NOT a common practice]

- These vectors precess around the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$

- GW emission shrinks the relative orbit & we want to model resulting GW polarization states during the inspiral phase



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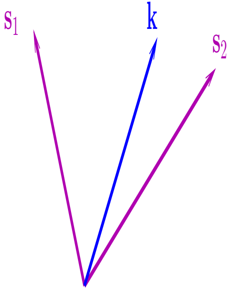

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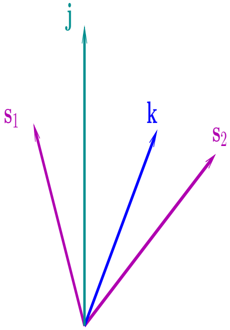
- The precessional dynamics arises due to spin-orbit and spin-spin interactions
  They spiral along quasi-circular orbits



Traditional way of constructing $h_{\times,+}(t)$: I

Kidder (1995), Buonanno, Chen, Vallisneri (2003), Arun et. al. (2009)

- The traditional approach begins by computing PN-accurate expressions for $h_{\times,+}$ in terms of dynamical variables $\Phi', \iota', \alpha', \dot{\Phi}'$ & few constant angles
- Φ' describes how $\mathbf{r}$ varies in an orbital triad, defined by $\mathbf{L}_N = \mu\, \mathbf{r} \times \mathbf{v}$

  $\mathbf{r} = r(\cos\Phi'\, \mathbf{i}' + \sin\Phi'\, \mathbf{j}')$ $[\mathbf{i}', \mathbf{j}', \mathbf{l}]$: an orbital triad based on $\mathbf{L}_N = L_N\, \mathbf{l}$
- (ι', α') specify $\mathbf{L}_N$ in an invariant frame associated with $\mathbf{J}$ at the initial epoch
- The two spins are initially specified by four angles in the orbital triad $[\mathbf{i}', \mathbf{j}', \mathbf{l}]$

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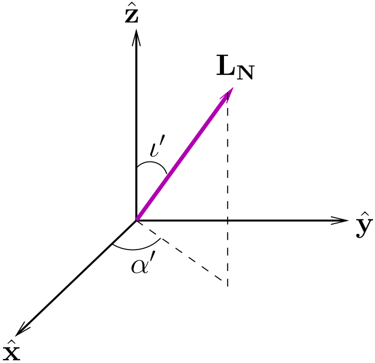
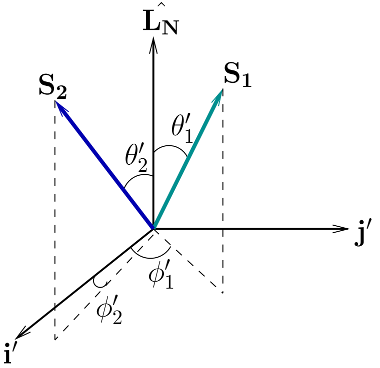
Therefore, TEN parameters are required to specify generic spinning binary $(m_1, m_2, \chi_1, \chi_2)$; $(\iota', \alpha', \theta_1', \phi_1', \theta_2', \phi_2')$



Source frame



$L_{\rm N}$-based triad

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- It is possible to show $\mathbf{v} = r \left(\frac{d\Phi'}{dt} + \frac{d\alpha'}{dt} \cos\iota' \right) \boldsymbol{\lambda}$
- We have $\mathbf{v} \equiv r\, \omega\, \boldsymbol{\lambda}$ for non-spinning Newtonian (& PN-accurate) circular orbits
- Equating these two expressions for $\mathbf{v} \implies \frac{d\Phi'}{dt} = \left(\omega - \frac{d\alpha'}{dt} \cos\iota' \right)$

Traditional way of constructing $h_{\times,+}(t)$: III

Kidder (1995), Buonanno, Chen, Vallisneri (2003), Arun et. al. (2009)

- The effect of GW emission is incorporated via the energy balance arguments

$$\frac{d\,\omega(t)}{dt} = -\mathcal{L} \Bigg/ \frac{d\mathcal{E}}{d\omega}$$

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- Solve numerically differential equations for $\left[\dot{\mathbf{L}}_N, \dot{\mathbf{S}}_1, \dot{\mathbf{S}}_2, \dot{\Phi}(\omega, \iota', \alpha'), \dot{\omega}\right]$ to obtain temporal variations to $\dot{\Phi}', \Phi, \iota'$ and α'

- Numerically implement these variations in PN-accurate expressions for $h_{\times,+}(\Phi', \iota', \alpha', \dot{\Phi}', ...)$

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This is how GW phasing for spinning compact binaries done traditionally

$h_{\times,+}(t)$ for spinning ICBs: I

Gupta & Gopakumar, *Submitted to Phys. Rev. D*

- An accurate & efficient prescription to compute time-domain $h_{\times,+}(t)$ for spinning compact binaries spiraling along quasi-circular orbits

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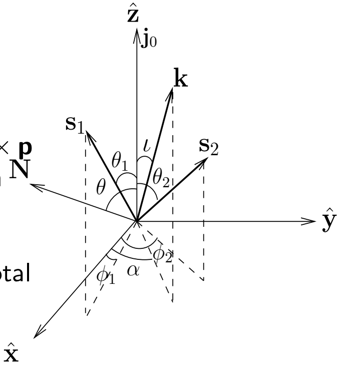
  We need to specify only *EIGHT* independent parameters

- We invoke the orbital angular momentum $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$ to describe the orbit (NOT its Newtonian version $\mathbf{L}_{\rm N}$)

- We construct an invariant frame such that the total angular momentum vector at the initial epoch is along the z-axis (A Standard practice)

- Further, we specify $\mathbf{L}$ as well as $\mathbf{S}_1$ and $\mathbf{S}_2$ in such an invariant (source) frame (NO orbital triad is invoked)



$h_{\times,+}(t)$ for spinning ICBs :II

- The expression for h_+ at the dominant (quadrupolar order):

$$
\begin{aligned}
h_+|_{Q}(t) \quad &= \quad \frac{2\,G\,\mu\,v^2}{c^4\,R'}\Big\{\left(\frac{3}{2}\cos^2\iota-\frac{3}{2}\right)(1-C_\theta^2)\cos 2\Phi-(1+\cos\iota)\,S_\theta\,C_\theta\sin\iota\cos(2\Phi+\alpha)\\
&-\frac{1}{4}(\cos^2\iota+2\,\cos\iota+1)(1+C_\theta^2)\cos(2\alpha+2\Phi)-\frac{1}{4}(\cos^2\iota-2\cos\iota+1)(1+C_\theta^2)\cos(2\alpha-2\Phi)\\
&-S_\theta\,C_\theta\sin\iota\cos\iota\cos(\alpha-2\Phi)+S_\theta\,C_\theta\sin\iota\cos(\alpha-2\Phi)\Big\}
\end{aligned}
$$

$$v^2/c^2=(G\,m\,\dot{\Phi}/c^3)^{2/3}\sim x; \qquad \cos\theta=\mathbf{N}\cdot\mathbf{j}_0\ ..$$

Note that $h_{\times,+}$ NOT $\propto \sin 2\,\Phi$ or $\cos 2\,\Phi$

- (ι,α) specify $\mathbf{k}$, the unit vector along $\mathbf{L}$, in the invariant frame

  Φ via $\mathbf{r}=r(\cos\Phi\,\mathbf{i}+\sin\Phi\,\mathbf{j})$ $\qquad [\mathbf{i},\mathbf{j},\mathbf{k}]$ $\mathbf{L}$-based orbital triad

- We need to specify how these angles vary

$h_{\times,+}(t)$ for spinning ICBs: III

- ι & α evolutions via

$$\dot{\mathbf{k}} = \frac{c^3}{Gm} x^3 \left\{ \delta_1 \, q \, \chi_1 \, (\mathbf{s}_1 \times \mathbf{k}) + \frac{\delta_2}{q} \chi_2 \, (\mathbf{s}_2 \times \mathbf{k}) \right\}$$

$$\dot{\mathbf{s}}_1 = \frac{c^3}{Gm} x^{5/2} \, \delta_1 \, (\mathbf{k} \times \mathbf{s}_1)$$

$$\dot{\mathbf{s}}_2 = \frac{c^3}{Gm} x^{5/2} \, \delta_2 \, (\mathbf{k} \times \mathbf{s}_2)$$

- Variations in Φ & ω are via

$$\dot{\Phi} = \frac{x^{3/2}}{(G\,m/c^3)} - \cos\iota \, \dot{\alpha}$$

$$\dot{x} = \frac{64}{5} \frac{c^3}{Gm} \eta \, x^5 \left\{ 1 + x(..) + x^{1.5}(..) + x^2(..) \right\}$$

$x = (G\,m\omega/c^3)^{2/3}$: the usual dimensionless PN expansion parameter

$h_{\times,+}(t)$ for spinning ICBs: IV

- We numerically solve these PN-accurate d-equations for $[\dot{\mathbf{k}}, \dot{\mathbf{s}}_1, \dot{\mathbf{s}}_2]$ and $[\dot{\Phi}, \dot{x}]$
  - We invoke Cartesian components of $[\dot{\mathbf{k}}, \dot{\mathbf{s}}_1, \dot{\mathbf{s}}_2]$
  - $\implies$ we have eleven Eqs to solve (The same as the # of Eqs in the traditional approach)

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  $\Longrightarrow$ freely specify the initial Cartesian components of the two spin vectors

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- At the initial epoch, we freely choose $(\theta_1,\ \phi_1)$ and $(\theta_2,\ \phi_2)$ $\Longrightarrow$ freely specify the initial Cartesian components of the two spin vectors
- We DO NOT freely specify initial orientation of $\mathbf{k}$ (or $\mathbf{L}$)
  - Recall that at the initial epoch $\mathbf{J}$ points along the z-axis of the source frame
  - $\Longrightarrow$ $\mathbf{J}$ can not have components along the x- & y-axes of the invariant frame at $t = 0$
  - This fixes the initial orientation of $\mathbf{k}$

Initial Conditions

- $J_x = 0; J_y = 0 \implies$

  $$k_{\rm x,0} = -\frac{G\,m^2}{c\,L_{\rm 2PN}|_{x=x_0}} \left\{X_1^2\,\chi_1\,\sin\theta_{10}\,\cos\phi_{10} + X_2^2\,\chi_2\,\sin\theta_{20}\,\cos\phi_{20}\right\}$$
  $$k_{\rm y,0} = -\frac{G\,m^2}{c\,L_{\rm 2PN}|_{x=x_0}} \left\{X_1^2\,\chi_1\,\sin\theta_{10}\,\sin\phi_{10} + X_2^2\,\chi_2\,\sin\theta_{20}\,\sin\phi_{20}\right\}$$

  $L_{\rm 2PN}$ provides 2PN-accurate orbital angular momentum in terms $(x, ..)$

- IC for x : $x_0 \sim 2.9 \times 10^{-4} (m\,\omega_0)^{2/3}$
  ω_0 is the initial frequency of aLIGO (a slight subtlety exists)

- Presently, we terminate the numerical integration when $x \sim 1/6$ (further refinements are possible)

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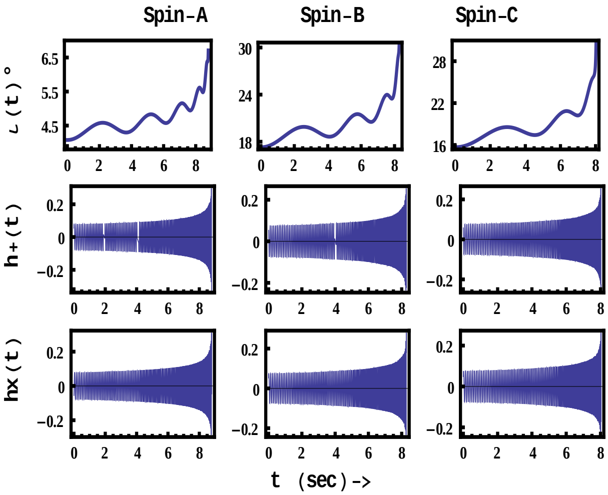
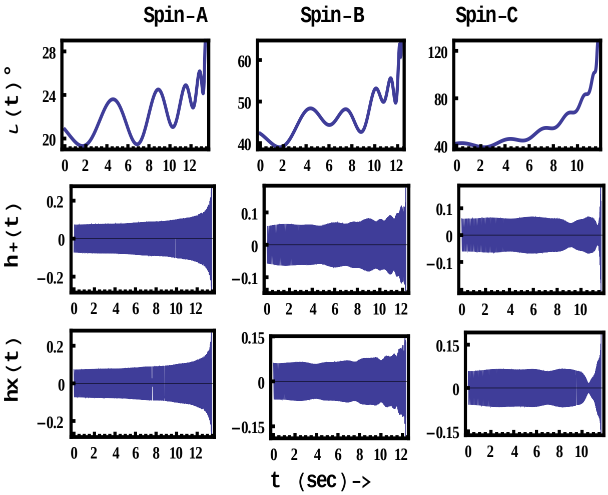
This is how we obtain $h_{\times,+}(t)$ in our approach

Our signal manifold is essentially EIGHT dimensional $(m_1, m_2, \chi_1, \chi_2)$; $(\theta_1, \phi_1, \theta_2, \phi_2)$

Temporal evolution of $h_{+,\times}$ and ι : $q = 1$



Temporal evolution of $h_{+,\times}$ and ι : $q = 4$



Why our approach ?: I

- The initial values of the angular variables, (α, ι), that explicitly appear in $h_{\times,+}(t)$ are dependent variables in our approach
  These two angles are also uniquely estimated
- There are a number of undesirable features present in the traditional way of obtaining $h_{\times,+}(t)$ spinning ICBs

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  - It is customary to invoke precessional equation appropriate for $\mathbf{L}$ to describe $\mathbf{L}_{\mathrm{N}}$
  - Detailed computations show that it leads to a feature that $\mathbf{v}$ will have components along $\mathbf{L}_{\mathrm{N}} \equiv \mu\, \mathbf{r} \times \mathbf{v}$ at 1.5PN order
    This is easily observed while expressing $\mathbf{v}$ in the co-moving triad $(\mathbf{n}, \mathbf{n} \times \mathbf{L}_N, \mathbf{L}N)$
  - Non-vanishing components of $\mathbf{v}$ along $\mathbf{L}_{\mathrm{N}}$ lead to unphysical 3PN order terms in the evolution equation for Φ'

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  - Non-vanishing components of **v** along $\mathbf{L}_{\rm N}$ lead to unphysical 3PN order terms in the evolution equation for Φ'
  - Φ' evolution should be 3.5PN-accurate for aLIGO templates

Why our approach ?: II

It is impossible to constrain initial orientation of $\mathbf{L}_N$ or $\mathbf{L}$ in the traditional approach by demanding that $\mathbf{J}$ at the initial epoch should point along the z-axis

- This is mainly because of specifying freely the two spins in orbital triad $[\mathbf{i}', \mathbf{j}', \mathbf{l}]$, at the initial epoch

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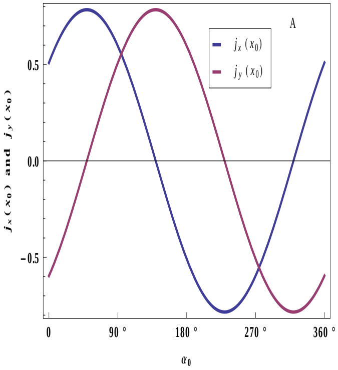
- This is mainly because of specifying freely the two spins in orbital triad $[\mathbf{i}', \mathbf{j}', \mathbf{l}]$, at the initial epoch
- In the traditional approach $J_x = 0; J_y = 0$ at initial epoch $\Longrightarrow$

$$
\begin{aligned}
k_{\rm x}(x_0) = \sin\iota' \cos\alpha' \quad &= \quad -\frac{Gm^2}{cL_0}\{X_1^2\chi_1(\sin\tilde\theta_1\ \cos\tilde\phi_1 \sin\alpha' + \sin\tilde\theta_1\ \sin\tilde\phi_1 \cos\alpha' \cos\iota' + \cos\tilde\theta_1 \sin\iota' \cos\alpha') \\
&\quad +X_2^2\chi_2(\sin\tilde\theta_2 \cos\tilde\phi_2 \sin\alpha' + \sin\tilde\theta_2 \sin\tilde\phi_2 \cos\alpha' \cos\iota' + \cos\tilde\theta_2 \sin\iota' \cos\alpha')\} \\
k_{\rm y}(x_0) = \sin\iota' \sin\alpha' \quad &= \quad -\frac{Gm^2}{cL_0}\{X_1^2\chi_1(-\sin\tilde\theta_1 \cos\tilde\phi_1 \cos\alpha' + \sin\tilde\theta_1 \sin\tilde\phi_1 \sin\alpha' \cos\iota' + \cos\tilde\theta_1 \sin\iota' \sin\alpha') \\
&\quad +X_2^2\chi_2(-\sin\tilde\theta_2 \cos\tilde\phi_2 \cos\alpha' + \sin\tilde\theta_2 \sin\tilde\phi_2 \sin\alpha' \cos\iota' + \cos\tilde\theta_2 \sin\iota' \sin\alpha')\}
\end{aligned}
$$

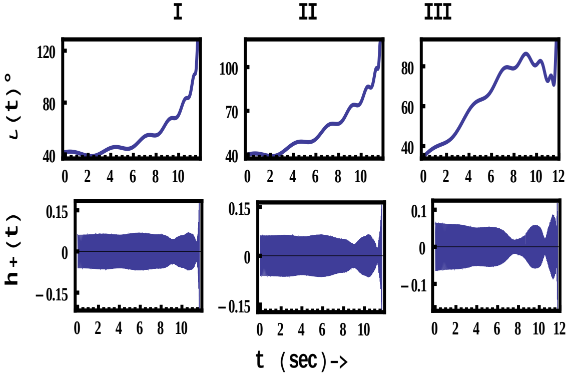
- ι' and α' are present on both sides of the above two equations $\Longrightarrow$ impossible to find a solution for ι' and α' at the initial epoch

Why our approach ?: III

- We may obtain the initial estimate for ι' via $\iota' = \cos^{-1}\left(\mathbf{j}(x_0)\cdot\mathbf{l}(x_0)\right)$
  These unit vectors are along $\mathbf{J}$ and $\mathbf{L}_N$ at the initial epoch
- We can plot the x and y components of $\mathbf{j}(x_0)$ at the initial epoch as function of α
  These plots do NOT cross each other (together) at zero !!
- This leads to an undesirable inconsistency that $\mathbf{J}$ will not point along the z-axis of the invariant frame at $t = 0$



Slight changes in the initial ι or α values can lead to substantially different looking $h_{\times,+}(t)$ for unequal mass binaries



- ι : Correct ι 2° off 7° off

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- Additionally, the traditional approach contains another inconsistency
  Recall that the two spins are specified in an orbital triad $[\mathbf{i}', \mathbf{j}', \mathbf{l}]$

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    - The definition of these units vectors are $\mathbf{i}' = (\mathbf{l} \times \mathbf{j}_0)/|\mathbf{l} \times \mathbf{j}_0|$ & $\mathbf{j}' = \mathbf{l} \times \mathbf{i}'$
      Note that by construction, $\mathbf{j}_0 \cdot \mathbf{i}' \equiv 0$
    - In the literature, spins are freely specified at the initial epoch by FOUR angles ($\mathbf{S}_1 = S_1\,\mathbf{s}_1, \mathbf{S}_2 = S_2\,\mathbf{s}_2$)

$$\mathbf{s}_1 = \sin\theta_1' \, \cos\phi_1' \, \mathbf{i}' + \sin\theta_1' \, \sin\phi_1' \, \mathbf{j}' + \cos\theta_1' \, \mathbf{l} \,,$$
$$\mathbf{s}_2 = \sin\theta_2' \, \cos\phi_2' \, \mathbf{i}' + \sin\theta_2' \, \sin\phi_2' \, \mathbf{j}' + \cos\theta_2' \, \mathbf{l}$$

    - We observe that $\mathbf{J}_0 \cdot \mathbf{i}' \neq 0$ while evaluating $\mathbf{J} = L\,\mathbf{k} + S_1\mathbf{s}_1 + S_2\mathbf{s}_2$
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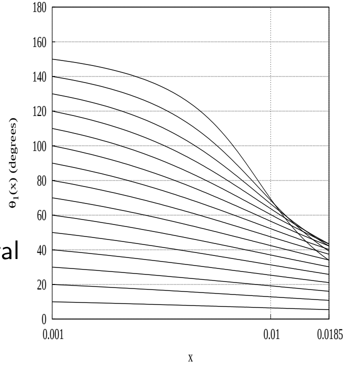
- $\mathbf{J}_0 \cdot \mathbf{i}' \equiv 0$ is a necessary but NOT a sufficient condition to extract the initial estimate for α by equating the x and y components of $\mathbf{J}_0$ to zero

Why our approach ?: V

- The dominant spin orientation at the initial aLIGO frequency should be $\leq \pi/4$ for spin-dominated binaries in our approach
- The above statement requires that the astrophysically produced spin-orbit misalignment should be $\leq 160^\circ$
  This is a very reasonable assumption
- In our approach, we can uniquely compute inspiral templates for binaries experiencing spin-orbit resonances in the aLIGO frequency window
- This is because of our ability to uniquely fix the initial value for α



- There are few prescriptions to estimate α at x_0

  Let $\mathbf{N} = (\sin\theta, 0, \cos\theta)$ & $\mathbf{j}$ at x_0 to be $(0, 0, 1)$

  Invoke these identities

  $\cos\alpha' = ((\mathbf{j}_0 \times \mathbf{N}) \cdot (\mathbf{j}_0 \times \mathbf{k}))/(|\mathbf{j}_0 \times \mathbf{N}||\mathbf{j}_0 \times \mathbf{k}|)$

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- Does there exist a prescription to compute $h_{\times,+}(t)$ where $\mathbf{j}_0$ will be along the z-axis of the invariant frame ?

- Can you force $\mathbf{j}_0$ to be along z-axis & obtain $h_{\times,+}(t)$ while specifying the two spins in an orbital triad ?

Ajith's SpinTaylorT5: I

- Ajith implemented *SpinTaylorT5* in LAL. It provides $h_{\times,+}(t)$ for inspiraling spinning compact binaries

- The $\mathbf{L}_N$-based orbital triad is invoked to freely specify the two spins at the initial epoch
- Let me denote such an orbital triad by $(\mathbf{a}, \mathbf{b}, \mathbf{k})$ [ignore that $\mathbf{l} \neq \mathbf{k}$]

  $\mathbf{a} = (\mathbf{k} \times \mathbf{j}_0)/|\mathbf{k} \times \mathbf{j}_0|$ and $\mathbf{b} = \mathbf{k} \times \mathbf{a}$

- He computed θ_J and ϕ_J from the Cartesian components of $\mathbf{J}$ in the $(\mathbf{a}, \mathbf{b}, \mathbf{k})$ frame

  $\rightarrow$ $\mathbf{j}_0$ can take the form $(\sin\theta_J \cos\phi_J\,, \sin\theta_J \sin\phi_J\,, \cos\theta_J)$ in the $(\mathbf{a}, \mathbf{b}, \mathbf{k})$ orbital triad
- Note that $\theta_J = \iota$ by the definition

Ajith's SpinTaylorT5: II

- All the four vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{k}$ and $\mathbf{j}_0$ were rotated by a proper rotational matrix that involves θ_J and ϕ_J
- It is easy to numerically verify that in the rotated frame $\mathbf{j}_0 = (0, 0, 1)$
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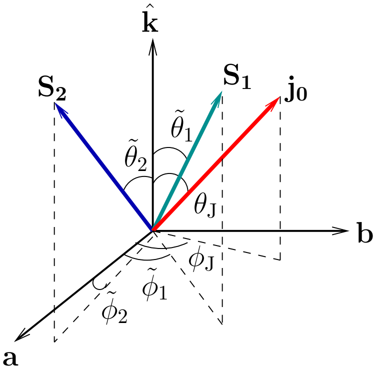
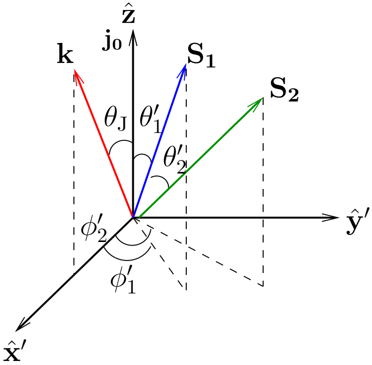
- Note that the expressions for $h_{\times,+}(t)$ require that $\mathbf{k} = (\sin\iota\,\cos\alpha\,,\,\sin\iota\,\sin\alpha\,,\,\cos\iota)$ in the invariant frame associated with $\mathbf{j}_0$ & $\mathbf{N}$
- We know that $\theta_J = \iota$. However, $\phi_J \neq \alpha$
- What are its implications ?
  Is it possible that the expressions for $h_{\times,+}(t)(\iota, \alpha)$ are written in a frame & the orbital evolution is done in a slightly different frame ?

Ajith's SpinTaylorT5: III



An orbital triad with **k** for **l**



Note that **k** has no projection onto the x-y plane

???

- We can show that $h_{\times,+}(t)$ via our approach is unique & no internal inconsistency exists at the initial epoch..
- Can we develop a way to compare $\Phi + \alpha$ in our & the traditional approaches to compute $h_{\times,+}(t)$?

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- Can we compare with secular orbital evolutions arising from full general relativity ?