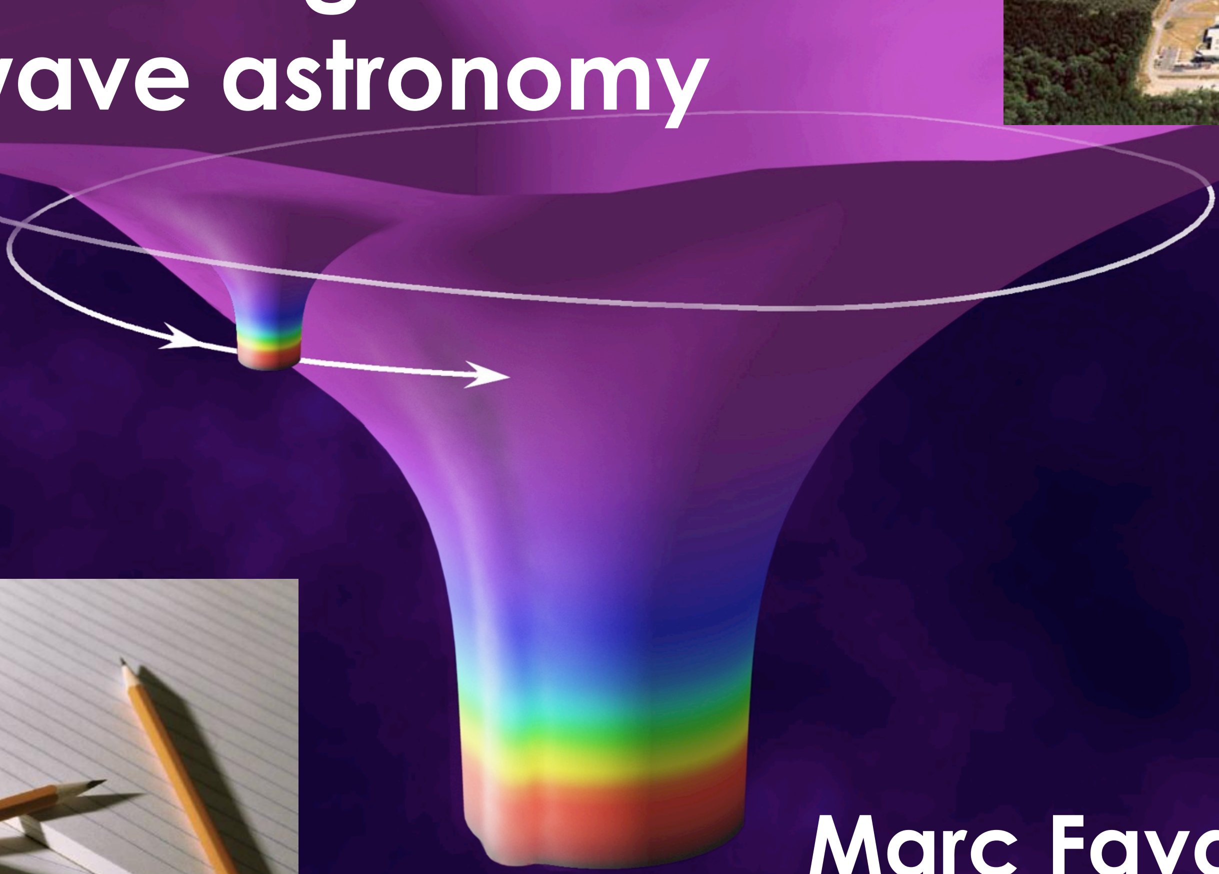
Analytic approximation methods for gravitational-wave astronomy



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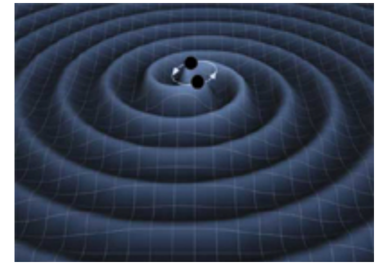
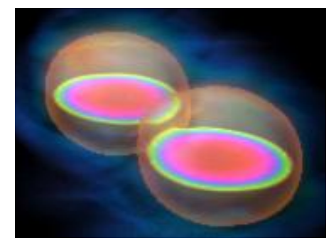
Outline for this talk:



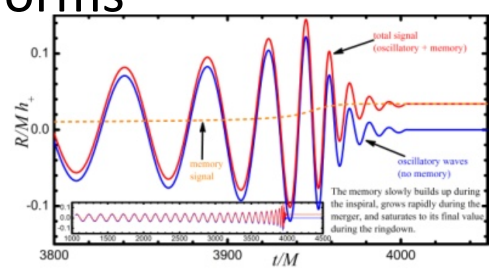
1. Brief overview of gravitational waves, their properties, and their sources.

2. Motivation for needing accurate models of GW sources.

3. Overview of various analytic techniques (focused on coalescing binaries only):

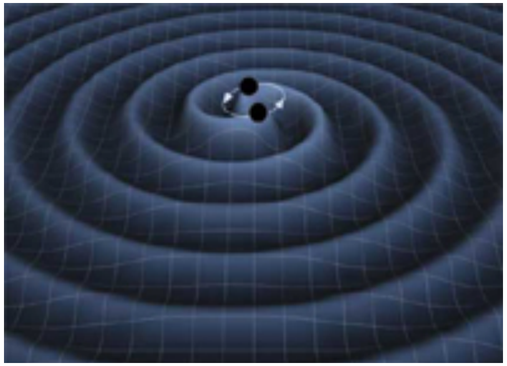


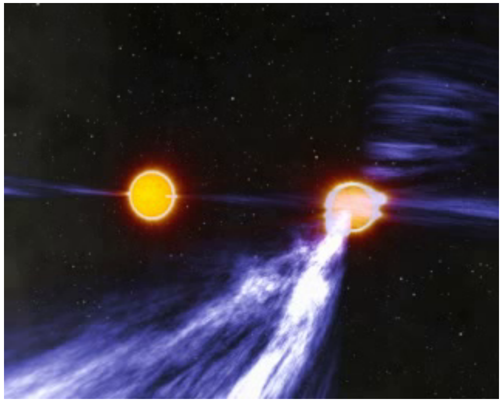
   - why we need analytic methods
   - post-Newtonian
   - BH perturbation theory + self-force approach
   - effective-one-body & phenomenological waveforms
   - areas requiring future work



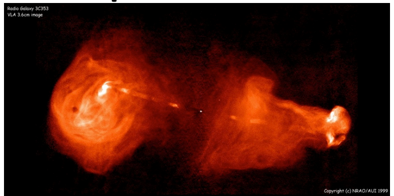
4. Summary

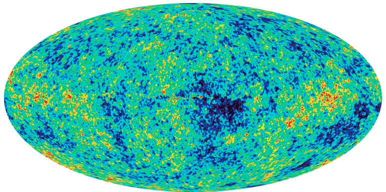
Gravitational-waves in brief:

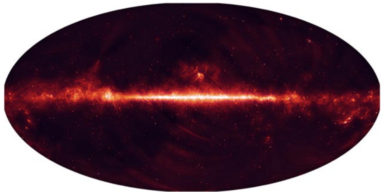


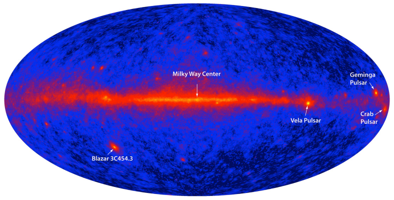


- Ripples in the curvature of spacetime that propagate at the speed of light.
- Produced by matter and energy moving at relativistic speeds.
- Existence confirmed by timing orbital decay of binary pulsars.
- Direct detection on Earth will allow us to observe gravitational-wave events throughout the universe.
- Every time a new waveband has been opened, new phenomena have been discovered.



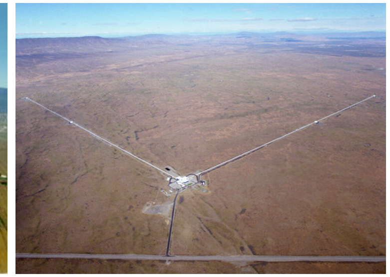






Gravitational-wave detector networks:

- AdvLIGO/Virgo+:
  ~2017

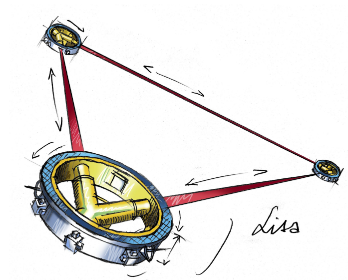

- Kagra:
  ~2018+

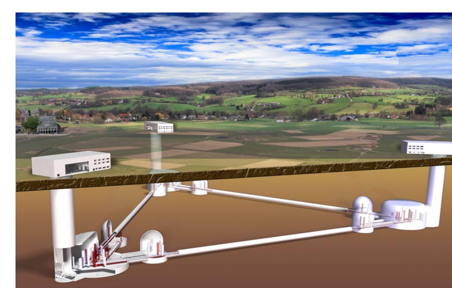

- LIGO-India:
  ~2020+

- Pulsar Timing Arrays:
  (NANOgrav, EPTA, PPTA)
  ~now



- Future detectors:
  ~2025+++??





Property	EM	GW
Substance	Oscillations of EM fields that propagate through spacetime	Oscillations of spacetime itself
Sources	Small-scale motion of microscopic charges	Large-scale motion of macroscopic masses
Speed	300,000 km/s	300,000 km/s
Wavelength λ compared w/ source size L	$\lambda << L$ (allows imaging of source)	$\lambda \geq L$ (no imaging of source components)
Propagation through matter	Significant absorption, scattering, dispersion	Negligible absorption, scattering, dispersion
What they tell us	Thermodynamic state of diffuse matter	Bulk motion of dense concentrations of matter & energy

Sources of gravitational waves: What will we learn?

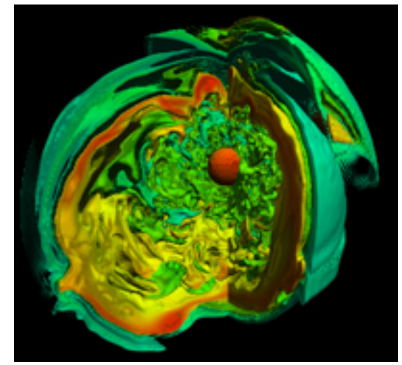
Merging stellar-mass compact-object binaries [pairs of neutron stars (NS) or black holes (BH)]

- Measure the masses and spins of NSs and BHs.
- Determine the rates at which they merge.
- Probe the central engine of short gamma-ray bursts (NS/NS or NS/BH).
- Learn about the properties of matter at ultra-high densities in the centers of neutron stars.
- Test if general relativity is correct in the strong-gravity, highly nonlinear regime of the theory; put constraints on competing theories of gravity.

Sources of gravitational waves: What will we learn?

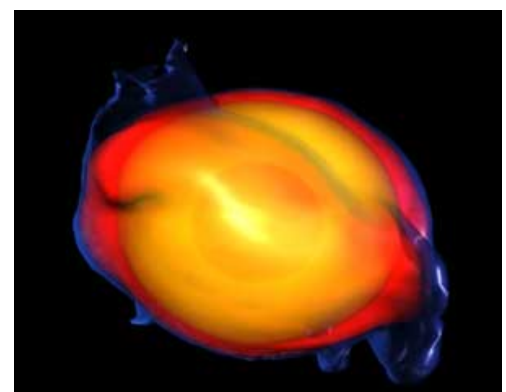
Individual compact objects

Supernova explosions



- Probe the inside of a supernova explosion.
- Study nuclear physics at high densities.

Distorted neutron stars

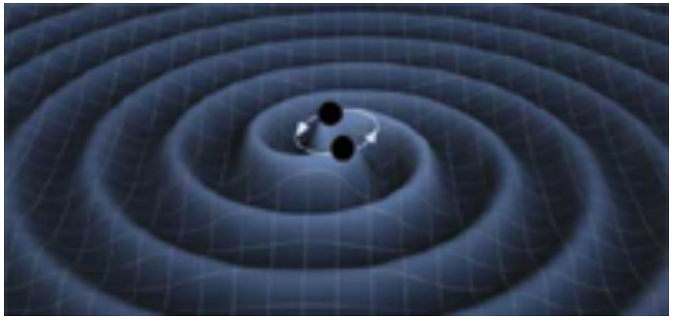


- Rotating neutron stars w/ mountains/bumps.
- Fluid instabilities in neutron stars.

Sources of gravitational waves: What will we learn?

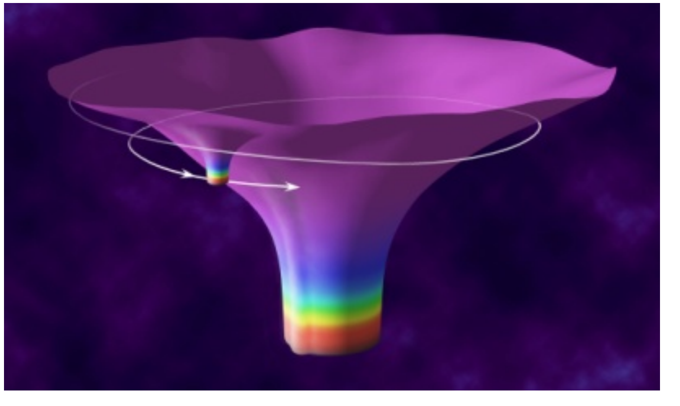
Mergers involving supermassive black holes [space-based detectors only]

Supermassive BH binaries



- Learn how massive BHs grow.
- Test relativity with very high precision.
- Constrain nature of dark energy.

Extreme-mass-ratio inspirals



- Determine the number density of compact stars in the centers of galaxies.
- Make a precision “map” of the spacetime around a BH (analog of geodesy).

Sources of gravitational waves: What will we learn?

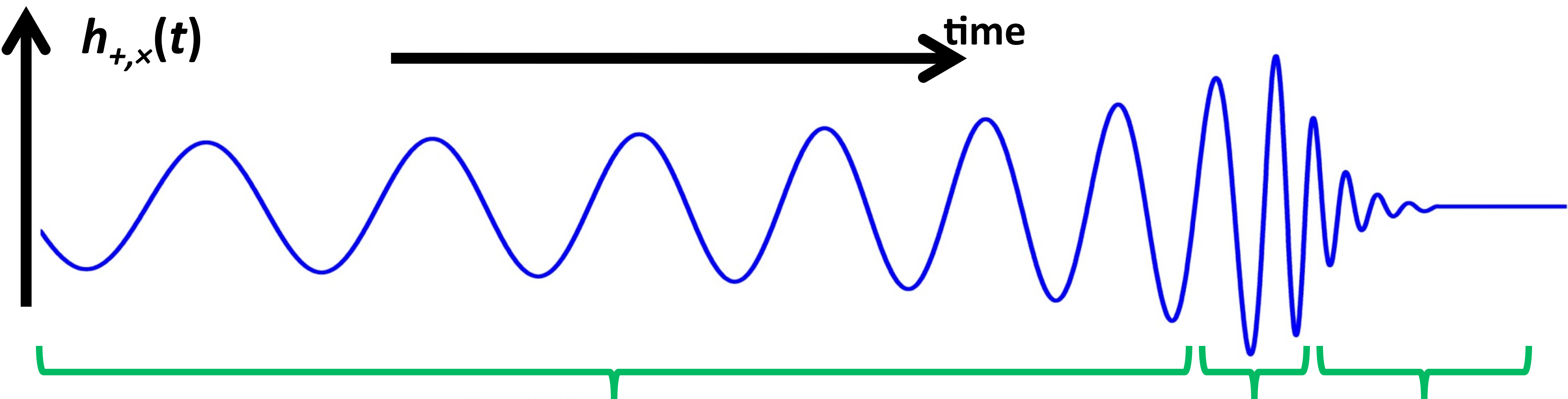
Exotic sources

- Cosmic strings
- GWs from the Big Bang.
- Unexpected sources ??

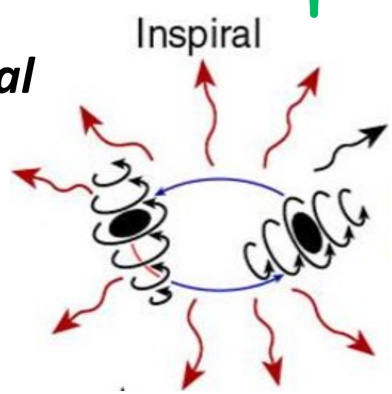
Mathematical-physics questions

- What types of interesting nonlinear effects manifest themselves? Are they observable?
- In what regimes are different analytic approximations valid?

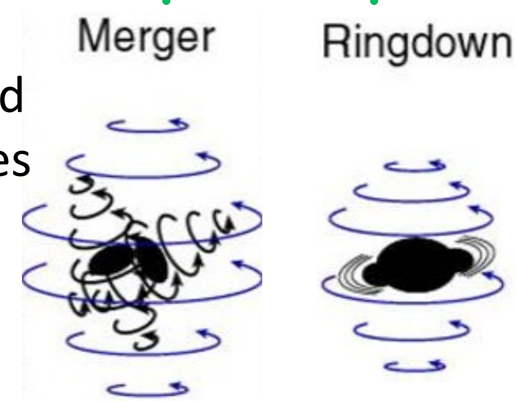
Gravitational-wave signal: coalescing binaries



During the ***inspiral*** the binary orbit slowly decays…

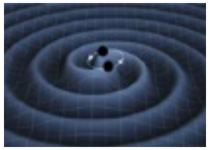


…during the ***merger*** and ***ringdown***, the two holes merge, & the remnant experiences damped oscillations.

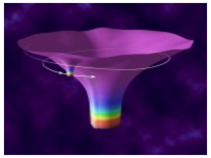


Sounds of spacetime:

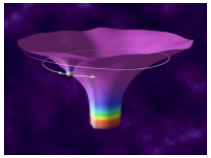
BH/BH, 3:1, no spin



BH/BH, 3:1, high spin



EMRI, 1:10^4, low spin



EMRI, 1:10^4, high spin,
high eccentricity

[Sounds from Scott Hughes, gmunu.mit.edu]

Why is gravitational-wave astronomy hard?

Experimental side: GWs are really weak!

- A typical source changes the LIGO arms by ~10^{-9} nm (1/1000 proton diameter)

- Many sources of noise in a detector:
  - seismic noise, gravity gradient noise
  - laser noise, shot noise
  - radiation pressure from laser on the test-masses
  - light scattering in the beam tube
  - vibrations in the test-masses and their suspensions
  - cosmic rays hitting the test masses
  - wind, tumble weed, airplanes, tree loggers, trains…

Why is gravitational-wave astronomy hard?

Theoretical side: Einstein's equations are hard to solve!

- Accurate waveform models needed to pick out signals from noise.
- Computing waveforms requires solving for the motion of the binary.
- Relativity *much* more complicated than Newtonian gravity...

Why is gravitational-wave astronomy hard?

Newton **vs.** **Einstein**

Equations are much more complex:

1 eq., 1 variable ($\boldsymbol{\Phi}$), simple differential operator

6 indep. eqs., 6 indep. variables ($g_{\mu\nu}$), complicated differential operator; many, many terms…

$$\nabla^2\Phi(\boldsymbol{x},t) = 4\pi\rho(\boldsymbol{x},t)$$

$$G_{\mu\nu}[g_{\alpha\beta}(x^\gamma)] = 8\pi T_{\mu\nu}[\rho(x^\gamma),\cdots]$$

Why is gravitational-wave astronomy hard?

Newton **vs.** **Einstein**

There are more sources of gravity:

$$\nabla^2\Phi(\boldsymbol{x},t) = 4\pi\rho(\boldsymbol{x},t)$$

Only mass density

$$G_{\mu\nu}[g_{\alpha\beta}(x^\gamma)] = 8\pi T_{\mu\nu}[\rho(x^\gamma),\cdots]$$

$$T_{\mu\nu}[\rho(x^\gamma), v_j, \rho v^2/2, P, t_{ij}, F_{\alpha\beta}, g_{\alpha\beta}, \cdots]$$

density, velocity, kinetic energy, pressure, internal stress, EM fields, …

Why is gravitational-wave astronomy hard?

Newton **vs.** **Einstein**

Gravity is a source for gravity (non-linearity)

Highly non-linear differential operator

$$G_{\mu\nu}[g_{\alpha\beta}] \sim g\partial^2 g + (\partial g)^2 + (g\partial g)^2 + g^3\partial^2 g + g^4(\partial g)^2 + \cdots$$

$$\nabla^2\Phi(\boldsymbol{x},t) = 4\pi\rho(\boldsymbol{x},t)$$

$$G_{\mu\nu}[g_{\alpha\beta}(x^\gamma)] = 8\pi T_{\mu\nu}[\rho(x^\gamma),\cdots]$$

Linear differential operator

Why is gravitational-wave astronomy hard?

Newton vs. **Einstein**

Equation of motion is also more complex:

$$\nabla^2\Phi(\boldsymbol{x},t) = 4\pi\rho(\boldsymbol{x},t) \qquad G_{\mu\nu}[g_{\alpha\beta}(x^\gamma)] = 8\pi T_{\mu\nu}[\rho(x^\gamma),\cdots]$$

$$\ddot{x}_i = -\nabla_i\Phi(t,x_j)$$

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}[g_{\mu\nu}]\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma})$$

How to solve Einstein's equations:

Exact solutions:

- Only known for highly symmetric spacetimes.
- Only 2 astrophysically useful solutions: Kerr and FRW.

Perturbation theory:

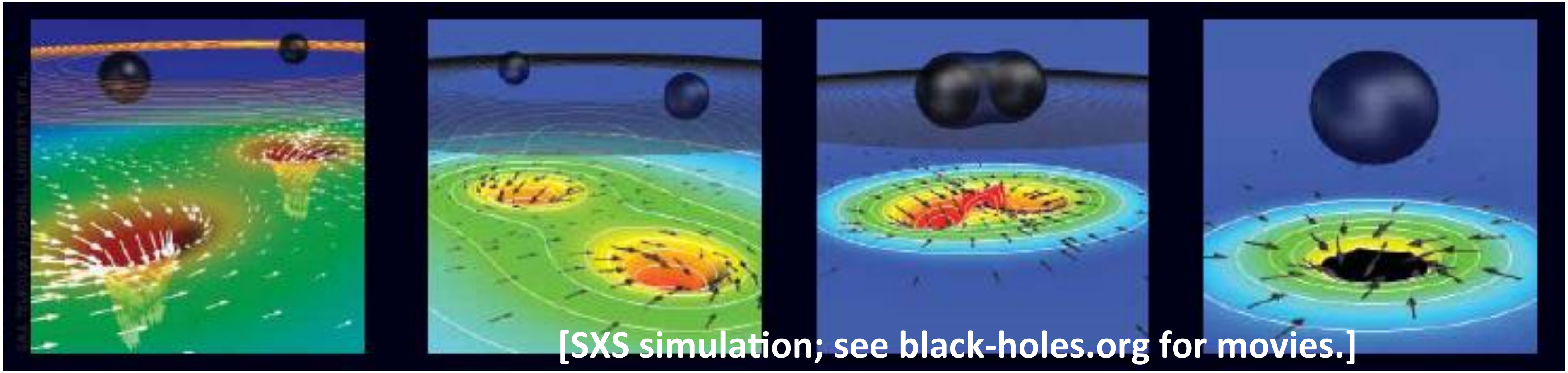
- Expand about an exact solution in terms of a small parameter:
  1. post-Newtonian (PN) theory
  2. BH perturbation theory

Numerical relativity:

- Solve equations numerically on a computer.
- No symmetries or approximation.
- Limited by finite-resolution, inexact initial conditions, and time/computing power.

Numerical relativity:

- Many groups have been trying to numerically solve the binary BH problem for over 30 years.
- Success was finally achieved in 2005 & 2006. [Pretorius, Goddard, RIT/ Brownsville]



[SXS simulation; see black-holes.org for movies.]

- BH/BH, NS/NS, BH/NS simulations now achieved by multiple groups.
- Recent/future work:
  - BH/BH: detailed exploration of parameter space (NRAR collab.)
  - NS/BH, NS/NS: realistic EOS, magnetic fields, EM signal, ...

Numerical relativity: limitations

- Time: computational costs become excessive beyond ~10 of orbits

- # of orbits in LIGO band:

NS/NS	NS/BH	BH/BH
8000	1800	300

- Orbital and radiation-reaction timescales:

$$T_{\mathrm{orb}} \sim \frac{r^{3/2}}{M} \qquad T_{\mathrm{rr}} \sim \frac{M}{\eta}\left(\frac{r}{M}\right)^4$$

- Larger orbits take longer to orbit and much longer to inspiral.

- Small mass ratio orbits ($\eta < 1/10$) are very costly. EMRI orbits ($\eta \sim 10^{-5}$) or IMRIs (intermediate mass ratio inspirals, $\eta \sim 10^{-2}$) impossible.

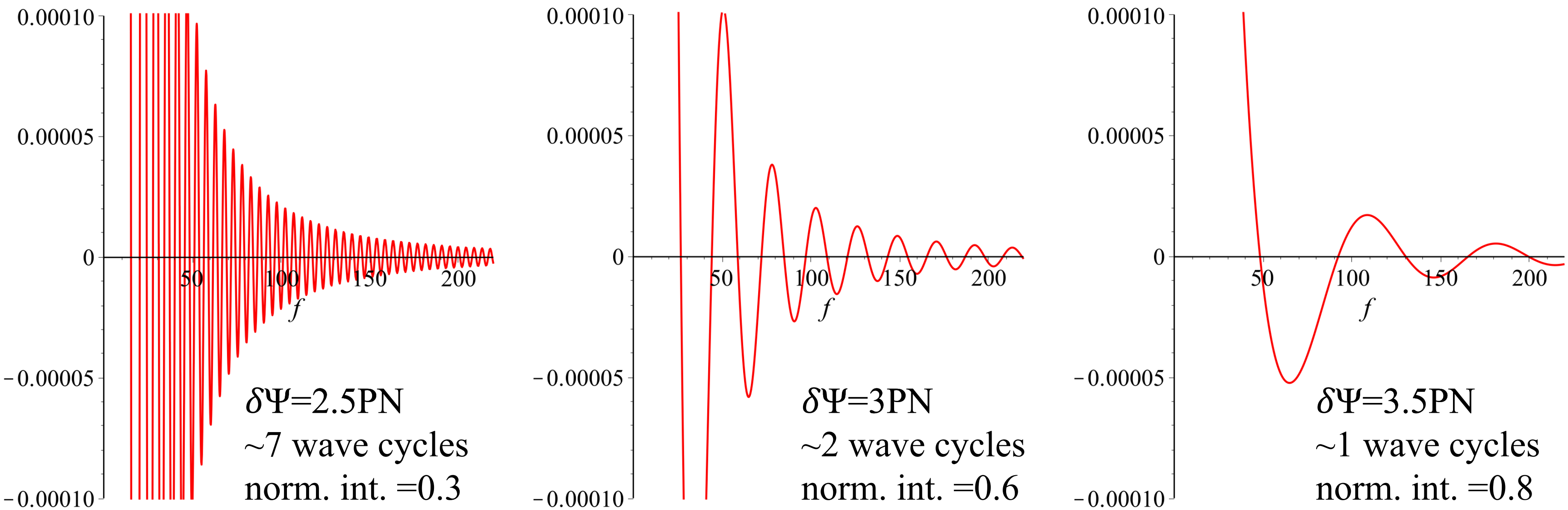
Need for phase accuracy:

- Matched filter involves comparing a "true signal" $h_{\rm T}$ with an "approximate template" $h_{\rm AP}$

$$
\begin{aligned}
(h_{\rm T}|h_{\rm AP}) &= 2\int_0^\infty \frac{df}{S_n(f)}(\tilde{h}_{\rm T}\tilde{h}^*_{\rm AP} + {\rm c.c}) \\
&\propto \int_0^\infty \frac{f^{-7/3}}{S_n(f)}\cos(\Psi_{\rm T} - \Psi_{\rm AP})
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_{\rm T}(f) &= \mathcal{A} f^{-7/6} e^{i\Psi_{\rm T}(f)} \\
\tilde{h}_{\rm AP}(f) &= \mathcal{A} f^{-7/6} e^{i\Psi_{\rm AP}(f)} \\
\Psi_{\rm T} &= \Psi_{\rm AP} + \delta\Psi
\end{aligned}
$$

Integrand [$S_n(f)$=1, 10+10 $M_\odot$, 10 to 220 Hz]



post-Newtonian theory in a nutshell

- Rewrite the full Einstein's equations in terms of a flat-space wave eqn:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$
$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$\Box h_{\mu\nu} = 16\pi T_{\mu\nu} + \mathcal{F}[h, h]$$

- Expand quantities in terms of small parameter: $\varepsilon \sim \frac{v_{\rm orb}}{c} \sim \sqrt{\frac{GM}{c^2 r}} \ll 1$

$$h_{\alpha\beta} = \varepsilon h^{(1)}_{\alpha\beta} + \varepsilon^2 h^{(2)}_{\alpha\beta} + \cdots$$
$$T_{\alpha\beta} = \varepsilon T^{(1)}_{\alpha\beta} + \varepsilon^2 T^{(2)}_{\alpha\beta} + \cdots$$

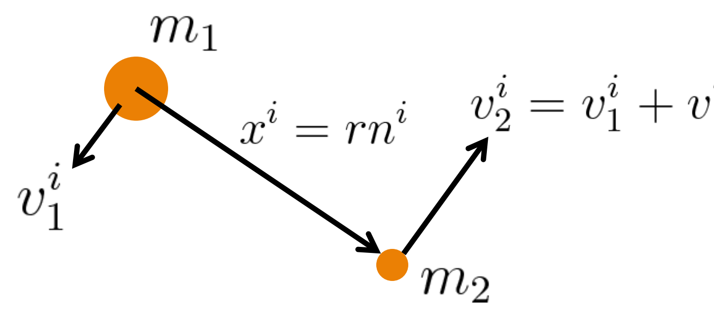
- Plug expansions back into equations, and solve at each order in ε:

$$\varepsilon^1 : \quad \Box h^{(1)}_{\mu\nu} = 16\pi T^{(1)}_{\mu\nu}$$
$$\varepsilon^2 : \quad \Box h^{(2)}_{\mu\nu} = 16\pi T^{(2)}_{\mu\nu} + \mathcal{F}[h^{(1)}, h^{(1)}]$$

- Metric near the source determines the equations of motion of the system; far from the source it determines the radiated GWs.

post-Newtonian theory (example):
equation of motion for two point masses

$$a^i = -\frac{GM}{r^2}\left[(1+\mathcal{A})\,n^i + \mathcal{B}\,v^i\right] + \mathcal{O}\left(\frac{1}{c^8}\right)$$



$$M = m_1 + m_2 \qquad \eta = \frac{m_1 m_2}{M^2}$$

[Blanchet '06, Liv. Rev. Rel.]

$$\begin{aligned}
\mathcal{A} \;=\; & \frac{1}{c^2}\left\{-\frac{3\dot{r}^2\eta}{2} + v^2 + 3\eta v^2 - \frac{Gm}{r}(4+2\eta)\right\} \\
& + \frac{1}{c^4}\left\{\frac{15\dot{r}^4\eta}{8} - \frac{45\dot{r}^4\eta^2}{8} - \frac{9\dot{r}^2\eta v^2}{2} + 6\dot{r}^2\eta^2 v^2 + 3\eta v^4 - 4\eta^2 v^4 + \frac{Gm}{r}\left(-2\dot{r}^2 - 25\dot{r}^2\eta - 2\dot{r}^2\eta^2 - \frac{13\eta v^2}{2} + 2\eta^2 v^2\right) + \frac{G^2m^2}{r^2}\left(9 + \frac{87\eta}{4}\right)\right\} \\
& + \frac{1}{c^5}\left\{-\frac{24\dot{r}\eta v^2}{5}\frac{Gm}{r} - \frac{136\dot{r}\eta}{15}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\left\{-\frac{35\dot{r}^6\eta}{16} + \frac{175\dot{r}^6\eta^2}{16} - \frac{175\dot{r}^6\eta^3}{16} + \frac{15\dot{r}^4\eta v^2}{2} - \frac{135\dot{r}^4\eta^2 v^2}{4} + \frac{255\dot{r}^4\eta^3 v^2}{8} - \frac{15\dot{r}^2\eta v^4}{2} + \frac{237\dot{r}^2\eta^2 v^4}{8} - \frac{45\dot{r}^2\eta^3 v^4}{2} + \frac{11\eta v^6}{4} - \frac{49\eta^2 v^6}{4} + 13\eta^3 v^6\right. \\
& \qquad + \frac{Gm}{r}\left(79\dot{r}^4\eta - \frac{69\dot{r}^4\eta^2}{2} - 30\dot{r}^4\eta^3 - 121\dot{r}^2\eta v^2 + 16\dot{r}^2\eta^2 v^2 + 20\dot{r}^2\eta^3 v^2 + \frac{75\eta v^4}{4} + 8\eta^2 v^4 - 10\eta^3 v^4\right) \\
& \qquad + \frac{G^2m^2}{r^2}\left(\dot{r}^2 + \frac{32573\dot{r}^2\eta}{168} + \frac{11\dot{r}^2\eta^2}{8} - 7\dot{r}^2\eta^3 + \frac{615\dot{r}^2\eta\pi^2}{64} - \frac{26987\eta v^2}{840} + \eta^3 v^2 - \frac{123\eta\pi^2 v^2}{64} - 110\dot{r}^2\eta\ln\left(\frac{r}{r'_0}\right) + 22\eta v^2\ln\left(\frac{r}{r'_0}\right)\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-16 - \frac{437\eta}{4} - \frac{71\eta^2}{2} + \frac{41\eta\pi^2}{16}\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{Gm}{r}\left(\frac{366}{35}\eta v^4 + 12\eta^2 v^4 - 114 v^2\eta\dot{r}^2 - 12\eta^2 v^2\dot{r}^2 + 112\eta\dot{r}^4\right) + \frac{G^2m^2}{r^2}\left(\frac{692}{35}\eta v^2 - \frac{724}{15}v^2\eta^2 + \frac{294}{5}\eta\dot{r}^2 + \frac{376}{5}\eta^2\dot{r}^2\right) + \frac{G^3m^3}{r^3}\left(\frac{3956}{35}\eta + \frac{184}{5}\eta^2\right)\right\}
\end{aligned}$$

post-Newtonian theory (example):
gravitational waveform for circular orbits

$$h_+(t) = -2\eta \frac{M}{R} v^2 \left[(1 + \cos^2\Theta) \cos[2\phi(t) - 2\Phi] + O(v) \right]$$

orbital phase evolution:

$$\begin{aligned}
\varphi(v) = \varphi_0 - \frac{1}{32\eta v^5} \Bigg\{ & 1 + v^2 \left(\frac{3715}{1008} + \frac{55}{12}\eta \right) - 10\pi v^3 + v^4 \left(\frac{15\,293\,365}{1\,016\,064} + \frac{27\,145}{1008}\eta + \frac{3085}{144}\eta^2 \right) \\
& + 2\pi v^5 \ln\left(\frac{v}{v_0}\right) \left(\frac{38\,645}{1344} - \frac{65}{16}\eta \right) + v^6 \Bigg[\frac{12\,348\,611\,926\,451}{18\,776\,862\,720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_E - \frac{856}{21}\ln(16v^2) \\
& + \left(-\frac{15\,737\,765\,635}{12\,192\,768} + \frac{2255}{48}\pi^2 \right)\eta + \frac{76\,055}{6912}\eta^2 - \frac{127\,825}{5184}\eta^3 \Bigg] + \pi v^7 \left(\frac{77\,096\,675}{2\,032\,128} + \frac{378\,515}{12\,096}\eta - \frac{74\,045}{6048}\eta^2 \right) \Bigg\}
\end{aligned}$$

$$v \equiv (M\omega)^{1/3} = \left(\frac{M}{r}\right)^{1/2} [1 + O(1/c)]$$

Contribution of each PN term to the number of wave cycles in the LIGO band [10+10 $M_\odot$ BH/BH, f_{end} = 220Hz]:

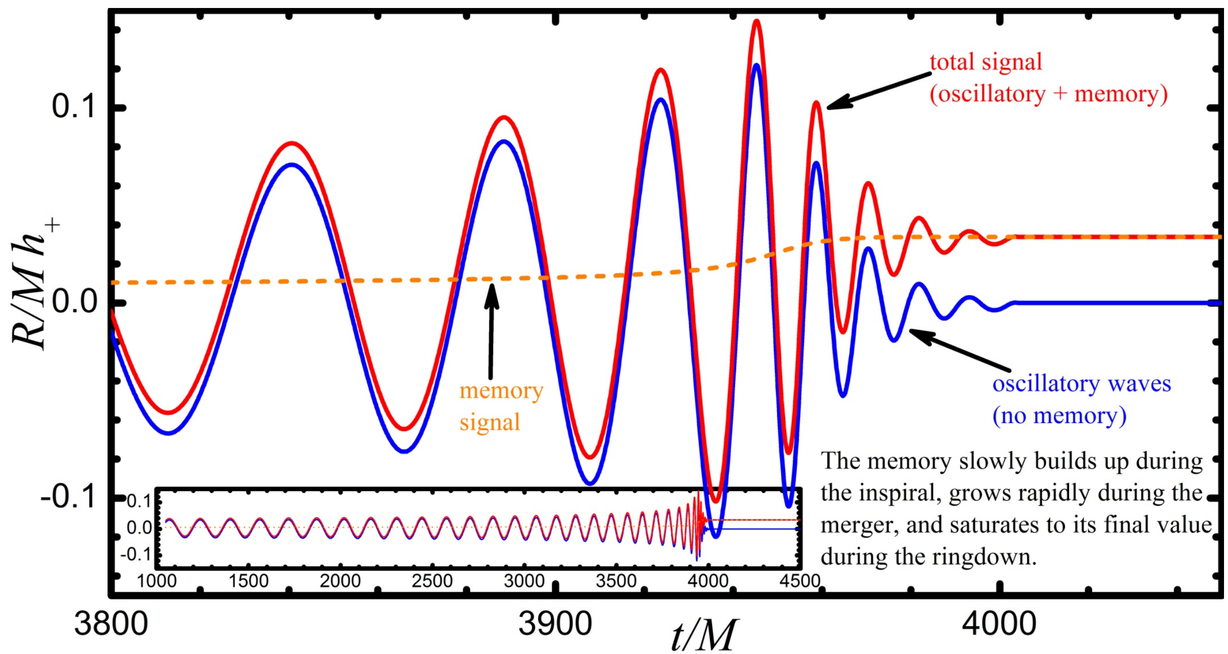
PN order	# of wave cycles
0PN [v^0]	602
1PN [v^2]	59.3
1.5PN [v^3]	-51.4
2PN [v^4]	4.06
2.5PN [v^5]	-7.14
3PN [v^6]	2.18
3.5PN [v^7]	-0.818

post-Newtonian waveforms: circular orbits, other harmonics

- Dominant waves produced at frequency $f_{gw} = 2\, f_{orb}$, but amplitude also has contributions at higher harmonics of the orbital frequency, $f_{gw} = n\, f_{orb}$, n=1, 3, 4, 5, ...

- These higher harmonics are more pronounced for unequal-mass binaries, but are suppressed by factors of v/c. [e.g., Arun et. al, CQG, 2004]

- An exception is for n=0, the so-called ***memory*** modes [e.g., MF, PRD, 2009, Christodoulou '91]. These modes are zero-frequency but time-varying. They modify the waveform at leading-order. Interesting for third-generation detectors, LISA, and PTAs.



[MF, ApJ Lett., '09;
Phys. Rev. D, '09;
Pollney & Reisswig ApJL '11]

post-Newtonian waveforms: memory term

- Arises from the GW stress-energy tensor (GWs produced by GWs)
  [Blanchet & Damour '92, Christodoulou '91, Wiseman & Will '91]

$$\Box h_{\mu\nu} = 16\pi T_{\mu\nu} + \mathcal{F}[h,h]$$

$$\Box h_{\mu\nu} = 16\pi (T_{\mu\nu} + T^{\rm gw}_{\mu\nu}) + \mathcal{F}'[h,h] \qquad T^{\rm gw}_{\mu\nu} \propto \frac{1}{R^2}\frac{d^2 E_{\rm gw}}{dt\,d\Omega}[h,h]$$

- For inspiralling binaries the nonlinear memory modifies the waveform at ***leading*** (Newtonian) order:

$$h_+(t) = -2\eta \frac{M}{R} v^2 \left[(1+\cos^2\Theta)\cos[2\phi(t) - 2\Phi] + \frac{1}{96}\sin^2\Theta(17+\cos^2\Theta) + O(v) \right]$$

Why?

$$\Delta h^{\rm mem}_{jk} \sim \frac{\Delta E_{\rm gw}}{R} \qquad \Delta E_{\rm gw} \sim \Delta E_{\rm binding} \sim \frac{\mu M}{r} \sim \eta M v^2$$

$$h^{\rm oscil.}_{jk} \propto \frac{\ddot{\mathcal{I}}_{jk}}{R} \sim \frac{\eta M v^2}{R}$$

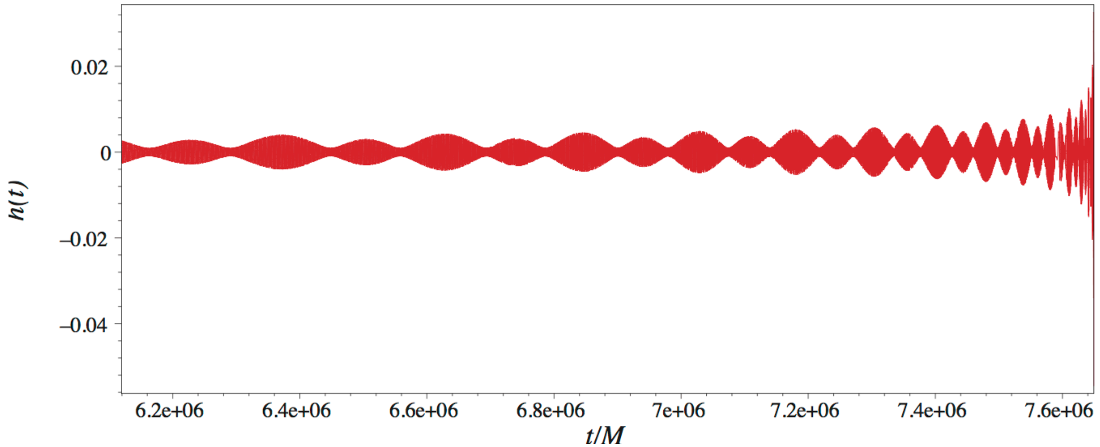
post-Newtonian waveforms: circular orbits, spin effects

- Equations of motion simplify for circular orbits, but spin introduces additional complications [Barker & O'Connel' '75; Kidder '95, Apostolatos+ '94].
- Spins aligned w/ orbital angular momentum: secular frequency and phase evolution modified (corrections at 1.5PN, 2PN, 2.5PN orders; partial results at 3PN, 3.5PN).
- Non-aligned spins: spin precession equations must now be solved. No analytic solutions in the general case.
- Precession causes amplitude (& phase) variations on precession timescale.

$$\frac{d\omega}{dt} = F[\omega, \mathbf{S}_1, \mathbf{S}_2]$$

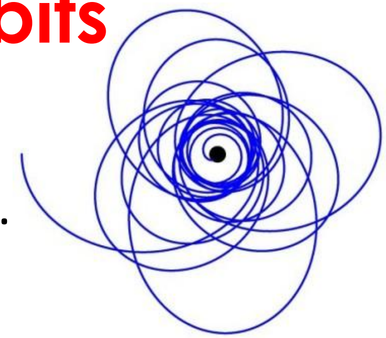
$$\frac{d\mathbf{S}_1}{dt} = \boldsymbol{\Omega}_1 \times \mathbf{S}_1$$

$$\frac{d\mathbf{S}_2}{dt} = \boldsymbol{\Omega}_2 \times \mathbf{S}_2$$



- Higher harmonics (including memory) are modified by spin effects [Arun et al '09, MF & Guo (in prep), Buonanno+ 2013]

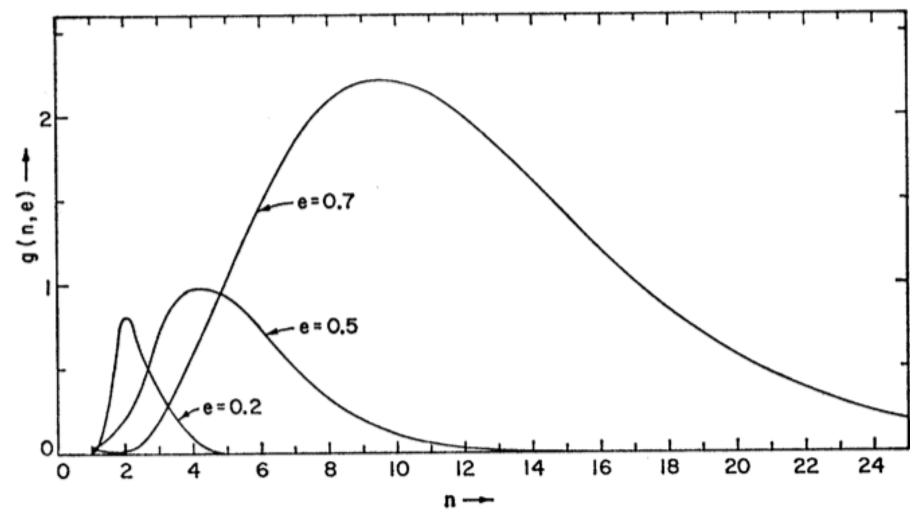
post-Newtonian waveforms: eccentric orbits



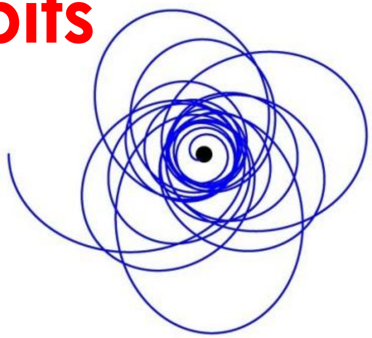
- Gravitational radiation typically damps eccentricity;
  e.g., most eccentric known binary pulsar will have $e \sim 10^{-5}$ at 10 Hz.
  But eccentric signals possible from binaries formed dynamically.

- New effects:
  --periastron precession (1PN+);
  --eccentricity-induced modulations to orbital phase & amplitude (0PN+) on the orbital timescale [Damour, Gopakumar, Iyer '04; Koningsdorffer & Gopakumar '06].

- Waveform amplitude corrections cause power at multiple harmonics of orbital frequency depending on eccentricity. [Peters &Mathews '63; Gopakumar & Iyer '02, MF '11]



post-Newtonian waveforms: eccentric orbits



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- Waveform amplitude corrections cause power at multiple harmonics of orbital frequency depending on eccentricity. [Peters &Mathews '63; Gopakumar & Iyer '02, MF '11]

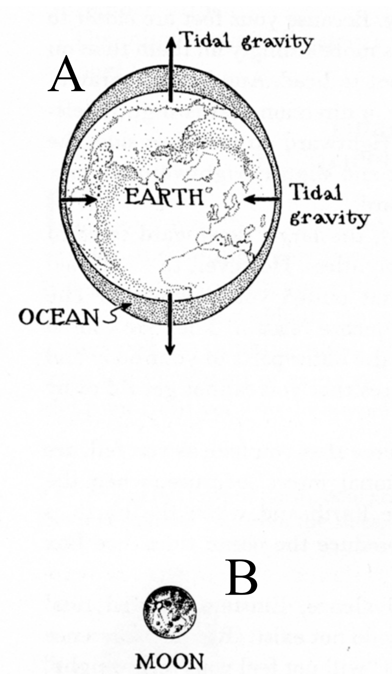
- Corrections to the energy, angular momentum flux (and GW phasing) are more complicated, especially tail effects. [Junker & Schaefer '92; Damour+ '04; Arun + '09]

- Equations must be solved numerically, except in small-eccentricity limit. [MF '06, Yunes+ '09]

post-Newtonian waveforms: tidal interactions

- Near the end of the inspiral, two neutron stars tidally distort each other. These distortions change their gravitational potentials, modifying their orbital motion (and gravitational-wave emission).

$$\Phi_A = -\frac{M_A}{r} - \frac{3}{2}\frac{\mathcal{I}^A_{jk} n^j n^k}{r^3} + O\left(\frac{1}{r^4}\right)$$



- This tidal distortion (and its effect on the waveform) is parameterized in terms of the tidal Love number:

$$\mathcal{I}^A_{jk} = -\lambda_A \mathcal{E}^A_{jk}$$

$$\mathcal{E}_{jk} \propto \frac{M_B}{d^3_{AB}}$$

$$\lambda_A = \frac{2}{3} k_2^A(\mathrm{EOS}) R_A^5$$

- Measuring the Love number will let us constrain the uncertain neutron star equation of state [e.g., Read+ '13; several other studies].

Measuring neutron star Love numbers: tidal corrections to PN waveform

- Leading order (quadrupolar, electric-type) tidal effect corrects the phasing at 5PN order. Corrections known (partially) to relative 1PN order (6PN). Octupole Love number enters at 7PN [Flanagan & Hinderer ’08; Hinderer+ ‘10; Vines+ ‘11].

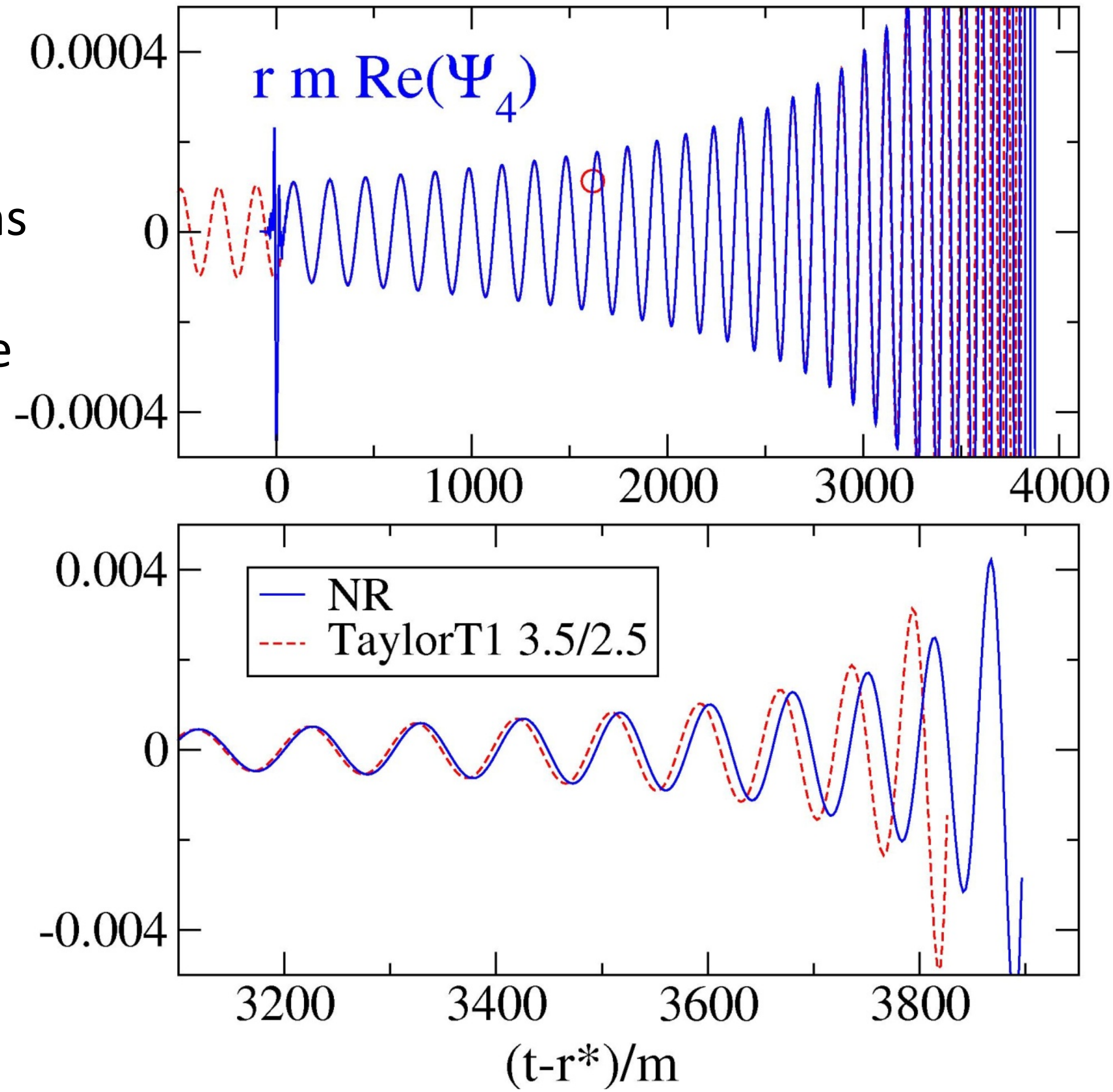
$$\phi^{\rm GW} = \frac{1}{32\eta v^5}\Big\{1 + c_{\rm 1PN}(\eta)v^2 + c_{\rm 1.5PN}(\eta)v^3 + c_{\rm 2PN}(\eta)v^4 + c_{\rm 2.5PN}(\eta)v^5 + c_{\rm 3.5PN}(\eta)v^7 \\ + c^?_{\rm 4PN}(\eta)v^8 + c^?_{\rm 4.5PN}(\eta)v^9 + [c^?_{\rm 5PN}(\eta) + F^{\rm 0PNtidal}(\lambda,\eta)]v^{10} \\ + c^?_{\rm 5.5PN}(\eta)v^{11} + [c^?_{\rm 6PN}(\eta) + F^{\rm 1PNtidal}(\lambda,\eta)]v^{12} + \cdots\Big\}$$

- However, many coefficients at the same (or lower) orders as the tidal terms are unknown.

- These terms can be comparable to the tidal terms, leading to very large systematic errors [MF (in prep)] if they are not accounted for.

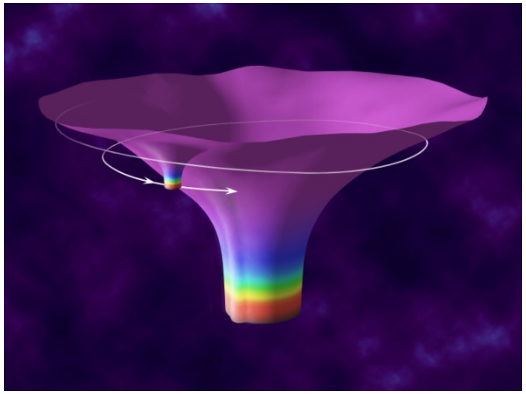
PN/NR comparison

- Nonspinning PN waveforms agree well with NR simulations for most of the inspiral.
- Agreement is worse for highly spinning systems [Lovelace et al CQG '12].



[Boyle, et al. '07]

Black hole perturbation theory
why needed?



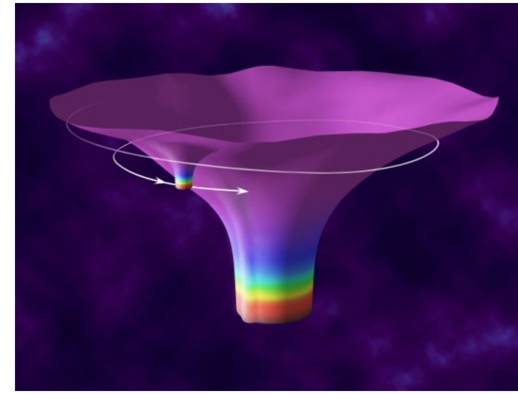
- Needed to treat inspirals of stellar-mass compact objects around SMBHs.

- Small mass ratio systems ($q = \mathrm{m}_1/\mathrm{m}_2 \lesssim 10^{-3}$) undergo many thousands of orbits in the strong field regime during an observation time.

- PN theory breaks down in the highly-relativistic region near a BH horizon (v/c ~ 1), and PN expansions converge more slowly for small mass ratio.

- NR modeling of >>100 orbits unfeasible in the near future.

Black hole perturbation theory

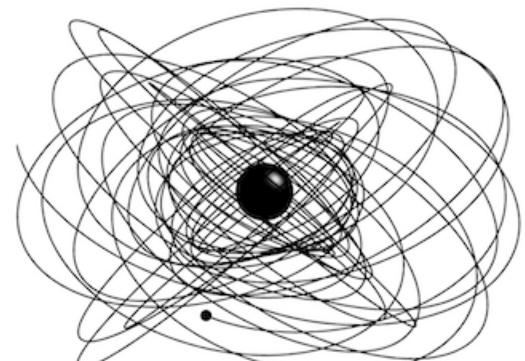


- Perturb about the metric of the exact Schwarzschild or Kerr solution

$$g_{\alpha\beta} = g^{\rm kerr}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + O(\varepsilon^2), \quad \varepsilon \sim q$$

- Results in a separable wave equation for the metric perturbation (in Schwarzschild) or the curvature perturbation (in Kerr) [Regge, Wheeler, Zerilli, Moncrief, Price, Bardeen, Press, Teukolsky, Chandrasekhar, ...]

$$\mathcal{D}^{\rm schw}[h^{(1)}] = T_{\alpha\beta}$$
$$\mathcal{D}^{\rm kerr}[\Psi_4] = F[T_{\alpha\beta}]$$



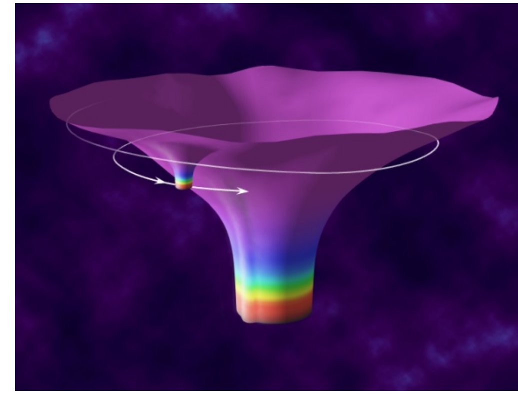
[Fig: Drasco]

- Waveform typically computed in **radiative approximation**:
  - Radiation field ($h^{(1)}$ or Ψ_4) computed along a geodesic (E, L_z, Q).
  - Change in constants of motion computed from radiation (dE/dt, dL_z/dt, dQ/dt).
  - Geodesic orbit updated [e.g., $E_i \to E_i + (dE/dt)\,\Delta t$].
  - Repeat to generate an inspiral sequence.

[Nakamura, Oohara, Detweiler, Shibata, Kojima, Sasaki, Tanaka, Mino, Poisson, Cutler, Kennefick, Hughes, Drasco, Sago, others...]

Black hole perturbation theory

analytic BH pert. theory/PN limit



- Solve BH perturbation theory equations in PN limit (radiative approx.)
  [Sasaki & Tagoshi Liv. Rev. Rel. ’03, Mino+ Prog. Theo. Phys. ‘97]

- Allows computation of energy flux to high PN order

[early results by Poisson, Cutler+, Shibata, Sasaki, Tagoshi, Tanaka, Mino,...]

--22PN for non-spinning binaries, Fujita (PN approximants, Varma+ ’13)

--4PN for spinning BHs Tagoshi+ ’96

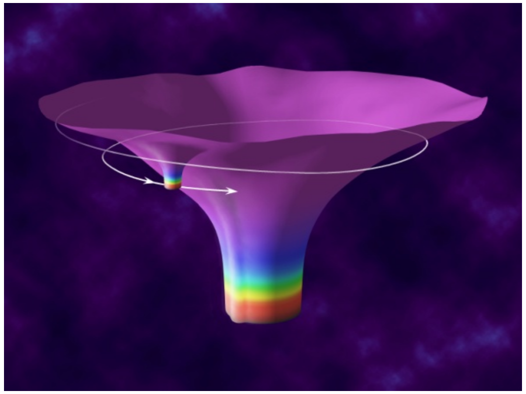
--4PN for small eccentricity, Schwarzschild [Mino+ ‘97]

--2.5PN for small eccentricity, Kerr [Tagoshi ’95]

Example:
6PN Talyor F2

$$
\begin{aligned}
c_{6\mathrm{PN}}^{(\mathrm{F2})} = & -\frac{24940759688510400996618326672 81}{35098206688634520081530880 0} + \left(\frac{219136}{441}\ln 2 - \frac{1525963498298779}{788628234240}\right)\pi^2 \\
& + \frac{1024}{21}\pi^4 - \frac{2323495644411647809}{370039154284800}\ln 2 + \frac{4043034784143}{108287635456}\ln 3 + \frac{188720703125}{913296384}\ln 5 + \frac{54784}{147}\zeta(3) + \frac{11723776}{15435}\ln^2 2 \\
& + \frac{2930944}{15435}\gamma_E^2 + \left(-\frac{10318037480141558 17}{370039154284800} + \frac{11723776}{15435}\ln 2 + \frac{109568}{441}\pi^2\right)\gamma_E \\
& + \left(-\frac{1031803748014155817}{370039154284800} + \frac{5861888}{15435}\gamma_E + \frac{109568}{441}\pi^2 + \frac{11723776}{15435}\ln 2\right)\ln v + \frac{2930944}{15435}\ln^2 v. \qquad \text{(B8e)}
\end{aligned}
$$

Black hole perturbation theory
self-force approach



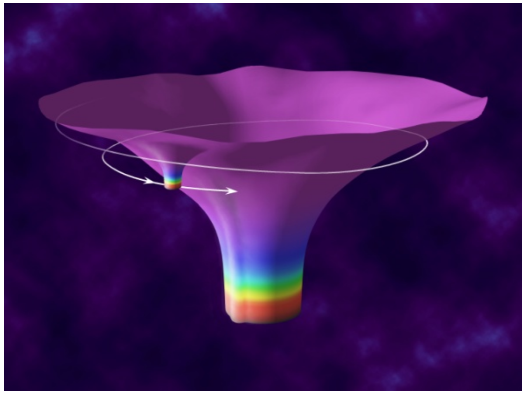
- Radiative approximation only accounts for dissipative changes to the orbit (ie., due to loss of energy or angular momentum to infinity or down the event horizon).

- But conservative forces proportional to small mass m also affects the orbit and GW phasing. This requires evaluation of the full self-force (which perturbs the test-mass from a geodesic).

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}[g^{\mathrm{kerr}}]\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \varepsilon a^\alpha_{\mathrm{sf}(1)} + O(\varepsilon^2)$$

- Example of conservative GSF (grav. self-force) effects:
  --correction to center-of-mass motion (Newtonian); i.e., $a=M/r^2 \;\rightarrow\; a=(M+m)/r^2$
  --correction to periastron advance rate and other orbital frequencies (small corrections to the effective potential).
  --correction to the location of the last stable orbit (LSO/ISCO).
  --interior light-cone effects (GWs emitted, scattered, & reabsorbed by test mass).

Black hole perturbation theory
self-force approach



- The SF can be split into dissipative and conservative parts at each order in the mass ratio.

$$a^\alpha_{\rm sf} = \varepsilon[a^\alpha_{1,\rm diss.} + a^\alpha_{1,\rm consv.}] + \varepsilon^2[a^\alpha_{2,\rm diss.} + a^\alpha_{2,\rm consv.}] + O(\varepsilon^3)$$

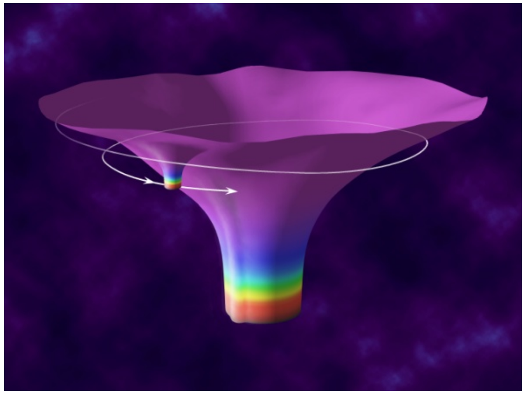
- The dissipative SF provides the dominant $O(1/\varepsilon)$ contribution to the phasing. The conservative piece provides a $O(1)$ contribution (which is non-negligible).

$$\varphi = \frac{1}{\varepsilon}\left[\varphi_1 + \varepsilon\varphi_2 + \cdots\right]$$

- Flanagan & Hinderer '12 also discovered a transient resonance phenomena sourced by the dissipative SF for generic orbits; this causes $O(1/\varepsilon^{1/2})$ phase shifts when the radial and polar orbital frequencies pass through a low-order rational ratio.

$$\varphi = \frac{1}{\varepsilon}\left[\varphi_1 + \varepsilon\varphi_2 + \varepsilon^{1/2}\varphi_{\rm reson.} + \cdots\right]$$

Black hole perturbation theory
self-force approach



- To implement self-force evolution, metric perturbation needs to be solved consistently with perturbed geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}[g^{\mathrm{kerr}}]\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \varepsilon a^\alpha_{\mathrm{sf}(1)} + O(\varepsilon^2)$$

- The self-force is (roughly) a gradient of the (regularized) retarded metric perturbation evaluated at the particle's position:

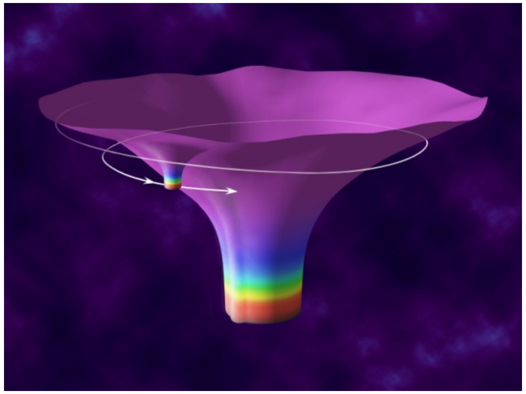
$$a^\alpha_{\mathrm{sf}(1)}(z) = \bar{\nabla}^{\alpha\beta\gamma}\bar{h}^{\mathrm{reg}}_{\beta\gamma}(z), \quad \bar{h}^{\mathrm{ret}}_{\alpha\beta} = \bar{h}^{\mathrm{sing}}_{\alpha\beta} + \bar{h}^{\mathrm{reg}}_{\alpha\beta}$$

- To determine the metric perturbation, one must solve the curved spacetime wave equation with a point-particle source:

$$\nabla^\gamma\nabla_\gamma\bar{h}_{\alpha\beta} + 2R^\mu{}_\alpha{}^\nu{}_\beta\bar{h}^{\mathrm{ret}}_{\mu\nu} = -16\pi T^{\mathrm{pp}}_{\alpha\beta}(z)$$

[Mino, Sasaki, Tanaka, Quinn, Wald; reviews by Detweiler '09, Barack '09; Poisson, Pound, Vega '04]

Black hole perturbation theory
self-force approach: results

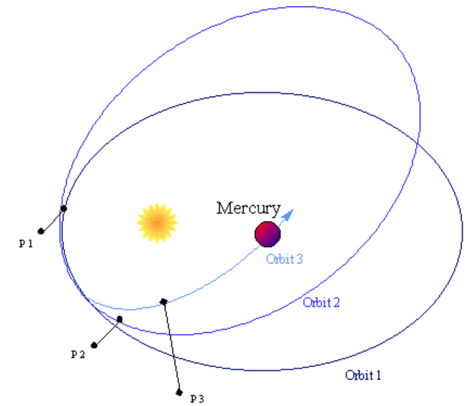


- Barack &Sago: implemented numerical evolution scheme for Schwarzschild—gives self-force along eccentric orbits.
- Other groups computed self-force along circular orbits [Detweiler &Whiting; Robinson; Friedman and collaborators.]. Checks between groups performed.
- Barack-Sago computed the shift in the inner-most stable circular orbit (ISCO).

$$M\Omega_{\rm isco}^{\rm BS} = 6^{-3/2}[1 + 1.251\eta + O(\eta^2)]$$

- Comparison studies between self-force ISCO shift, PN [MF '11a,b], and EOB [Damour '10]. Standard PN results perform surprisingly well.
- Self-force corrections to the periastron advance rate [Le Tiec+, PRL 2011] and Schwarzschild orbital energy and angular momentum computed and compared to NR, PN, and EOB calculations.
- Computation of conservative SF orbital de-phasing wrt radiative approx in Schwarzschild [Warburton et. al. 2012] (adiabatic orbital evolution).

Black hole perturbation theory
self-force approach: results



Example: SF corrections to the periastron advance rate [Le Tiec+, PRL 2011]. For Mercury:

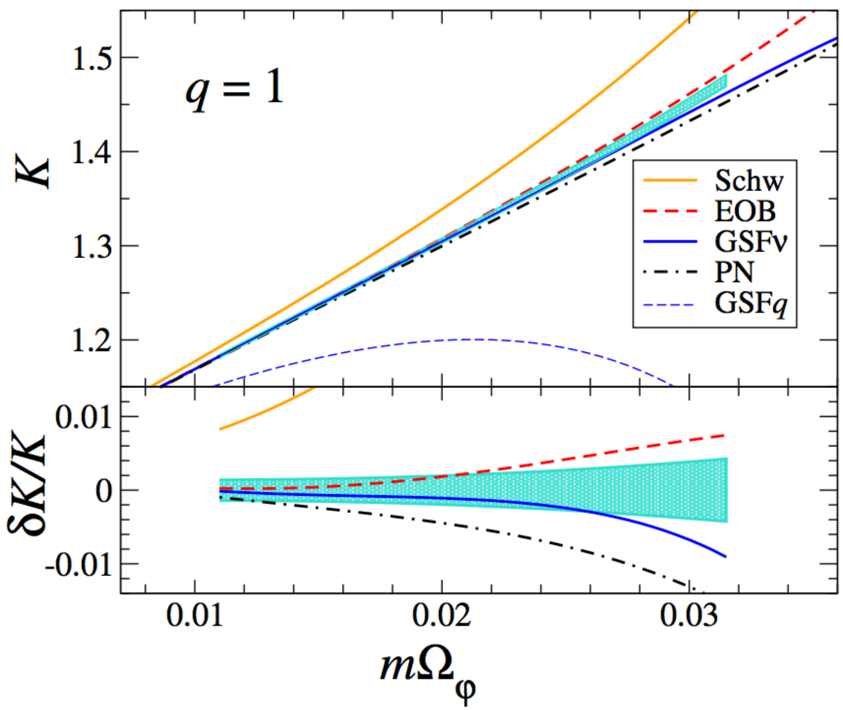
$$\Delta\Phi \approx \frac{6\pi M}{a(1-e^2)} \to 43''/100\text{yrs}$$

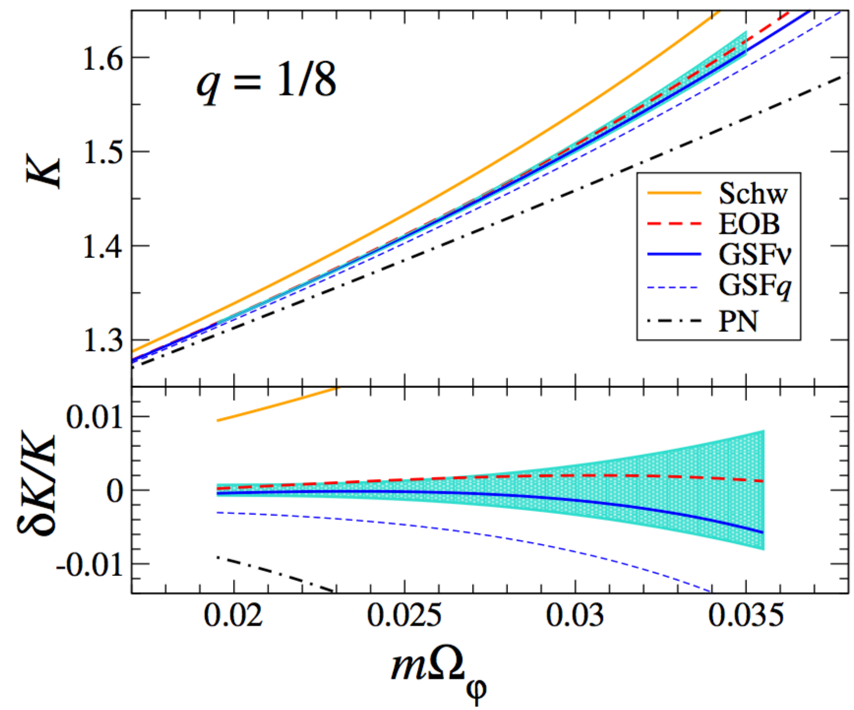
$$\frac{\Delta\Phi}{2\pi} = K - 1, \quad K \equiv \frac{\Omega_\varphi}{\Omega_r}$$

$$K_q^{\rm GSF} = \frac{1}{\sqrt{1-6(m\Omega_\varphi)^{2/3}}}\left[1 + qF_{\rm GSF}(m\Omega_\varphi) + O(q^2)\right]$$

$$q \to \eta = \frac{q}{(1+q)^2}$$

$$K_\eta^{\rm GSF} = \frac{1}{\sqrt{1-6(m\Omega_\varphi)^{2/3}}}\left[1 + \eta F_{\rm GSF}(m\Omega_\varphi) + O(\eta^2)\right]$$





Effective-one-body (EOB) approach

- A hybridization of PN, NR, and BH perturbation theory techniques (see Buonanno & Damour '99, '00; lecture notes by Damour '08).

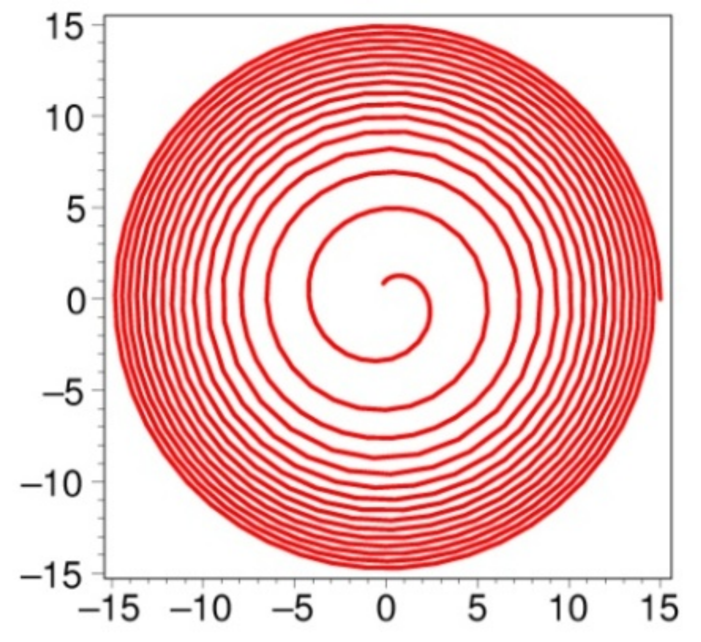
- **Motivation:** provide a quick, semi-analytic way to generate waveform templates that include the inspiral + merger + ringdown.

Contains input from three areas:

PN: An extension of the PN two-body dynamics to the non-adiabatic region (the transition from inspiral to plunge).

BH pert.: A matching of the "inspiral + plunge" waveform to a ringdown waveform.

NR: A variety of "flexibility" parameters that can be fit to NR simulations.



Effective-one-body (EOB) approach

resumed PN piece (inspiral + plunge):

- Maps the 2-body PN Hamiltonian to an "effective" Hamiltonian equivalent to a point particle with mass equal to the reduced mass moving on a "deformed" Schwarzschild metric with BH mass equal to the total mass:

$$ds^2_{\rm eff} = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

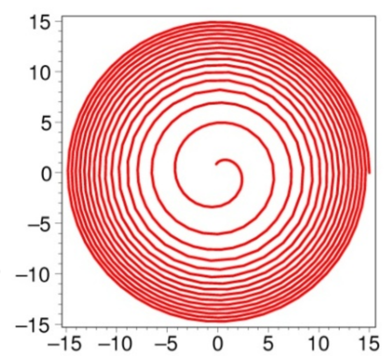
$$A(r) = 1 - \frac{2M}{r} + \eta\left[2\left(\frac{M}{r}\right)^3 + \tilde{a}_4\left(\frac{M}{r}\right)^4 + a_5(\eta,\lambda_1,\lambda_2)\left(\frac{M}{r}\right)^5\right]$$

- a_4 known from 3PN order dynamics; 4PN (and 5PN) "flexibility" parameters (a_5, a_6) introduced and adjusted to match phasing of NR waveforms.

- Effective Hamiltonian is constructed from this metric. Solving Hamilton's equations gives the conservative dynamics [$r(t)$, $\varphi(t)$, $p_r(t)$, $p_\varphi(t)$].

  – System must be supplemented by radiation-reaction force (based on 3.5PN *dE/dt* expressed in terms of waveform modes; also contains several adjustable parameters)

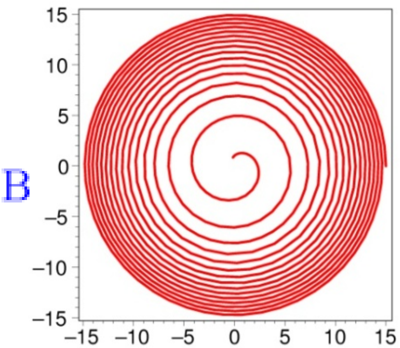
  (More complicated for spinning binaries: Barausse & Buonanno;Taracchini+ '12)



Effective-one-body (EOB) approach

EOB waveform:

Inspiral+plunge piece:



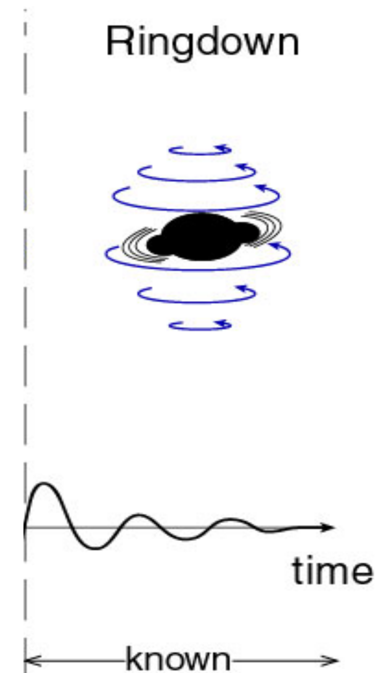
$$h_{lm}^{(\mathrm{EOB,insp+plunge})} = h_{lm}^{(\mathrm{Newt.})}[r(t), p_r(t), p_\varphi(t)]e^{-im\varphi(t)} F_{lm}^{\mathrm{EOB}}$$

Product of several terms:
Re-summed PN corrections, non-quasi-circular corrections, ... [Damour, Iyer, Nagar '08]

BH pert. piece: ringdown waveform

$$h_{lm}^{\mathrm{ringdown}} = \sum_n A_{lmn} e^{-i\omega_{lmn}^{\mathrm{QNM}}(t-t_{\mathrm{match}}^{lm})} e^{-(t-t_{\mathrm{match}}^{lm})/\tau_{lmn}^{\mathrm{QNM}}}$$

Depends on final BH mass & spin [Berti et. al '06; Tichy & Marronetti '08]



Match pieces at several times near merger to determine A_{lmn}

Effective-one-body (EOB) approach

Comparison of EOB w/ Caltech/Cornell NR simulation

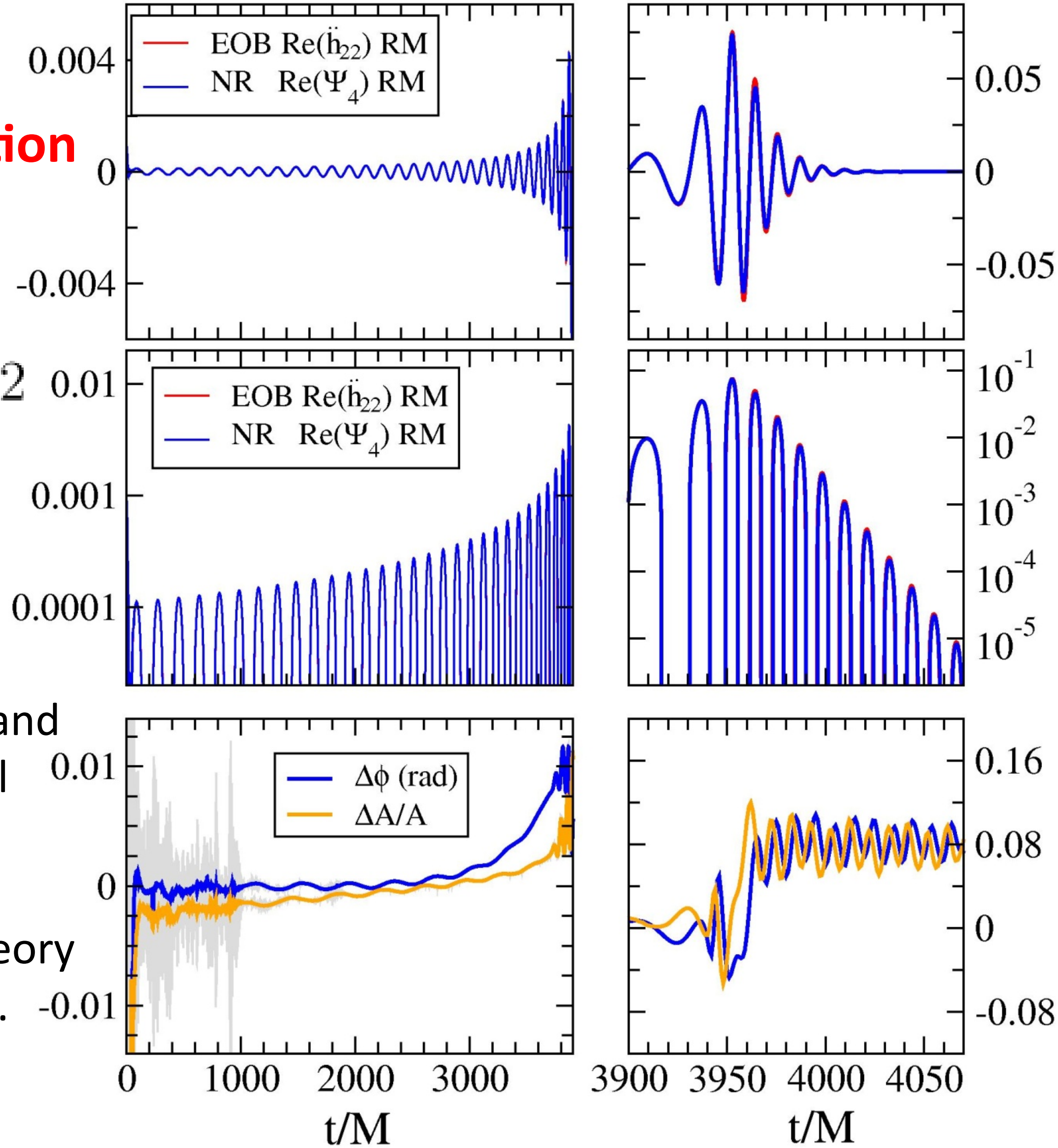
[Buonanno et. al; 0902.0790]

$$\Delta\phi\big|_{\rm insp.} \sim \frac{\Delta A}{A}\bigg|_{\rm insp.} \lesssim 0.002$$

$$\Delta\phi\big|_{\rm tot.} \sim \frac{\Delta A}{A}\bigg|_{\rm tot.} \lesssim 0.08$$

Similar results for higher modes and spinning systems [Taracchini et al ‘12, Pan et al ‘11]

Can also calibrate w/ BH pert. theory (radiative approx.) and SF results. [e.g., Barausse+ ’12; Barack+ ‘10]



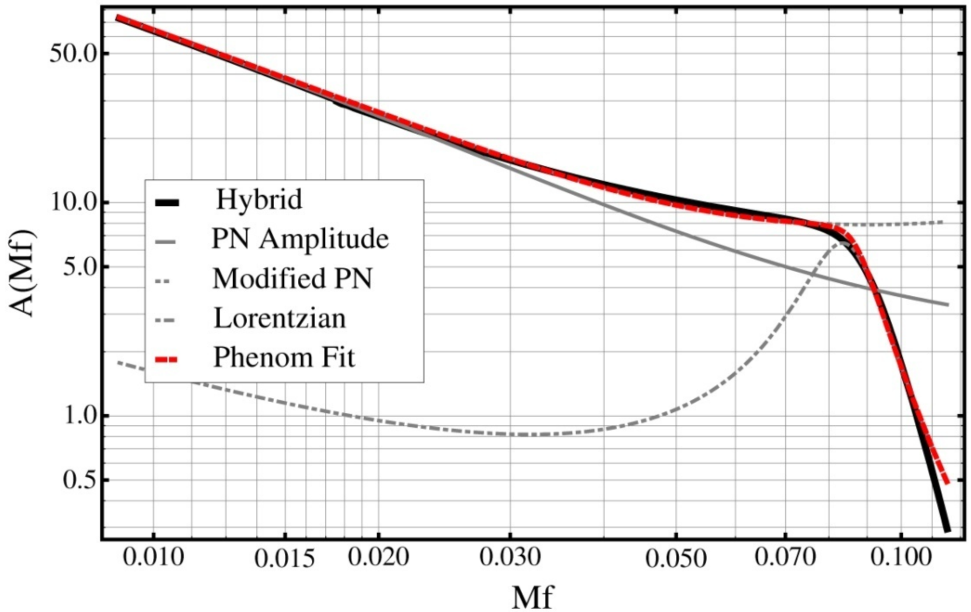
Phenomenological inspiral-merger-ringdown(IMR) templates:

- Match PN inspiral waveform to numerical relativity merger/ringdown waveform → "hybrid" waveform

- Fit this to an analytic function in the frequency domain:

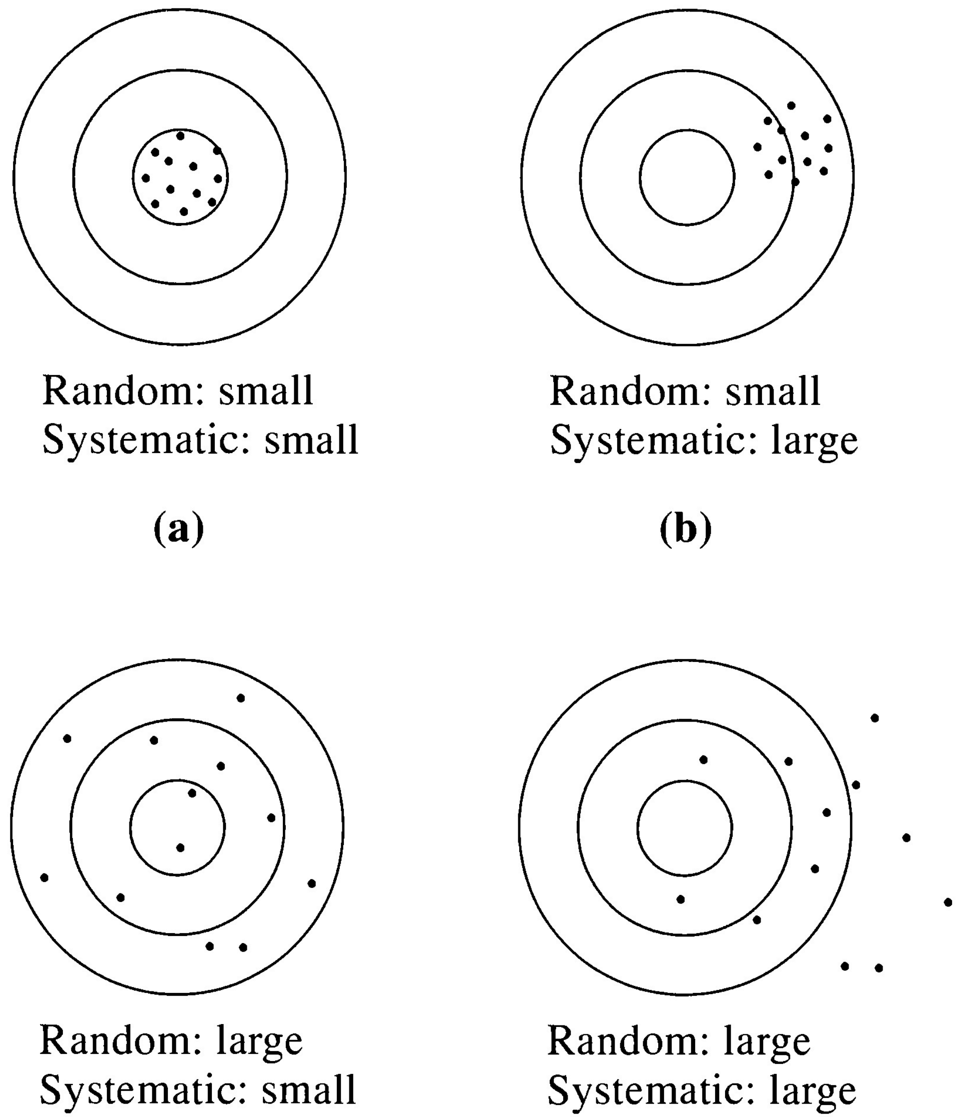
$$\tilde{h}_{\rm phen}(f) = A_{\rm phen}(f) e^{i\Psi_{\rm phen}(f)}$$

- Amplitude & phase are each split into inspiral + merger + ringdown pieces, with free parameters that are adjusted to:
    1. enforce continuity and smoothness
    2. match with the hybrid waveform.



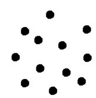
[Ajith et al, '07, '08, '11; Santamaria et al '10]

Motivation for future analytic modeling: systematic errors



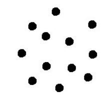
[from Taylor, Intro to Error Analysis]

Motivation for future analytic modeling: systematic errors



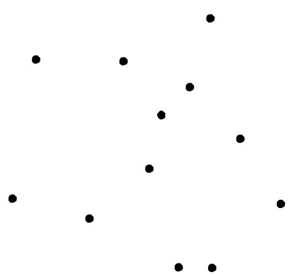
Random: small
Systematic: ?

(a)

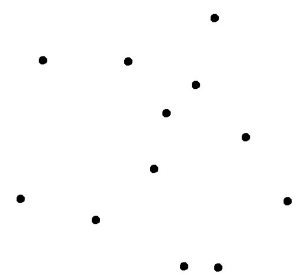


Random: small
Systematic: ?

(b)



Random: large
Systematic: ?



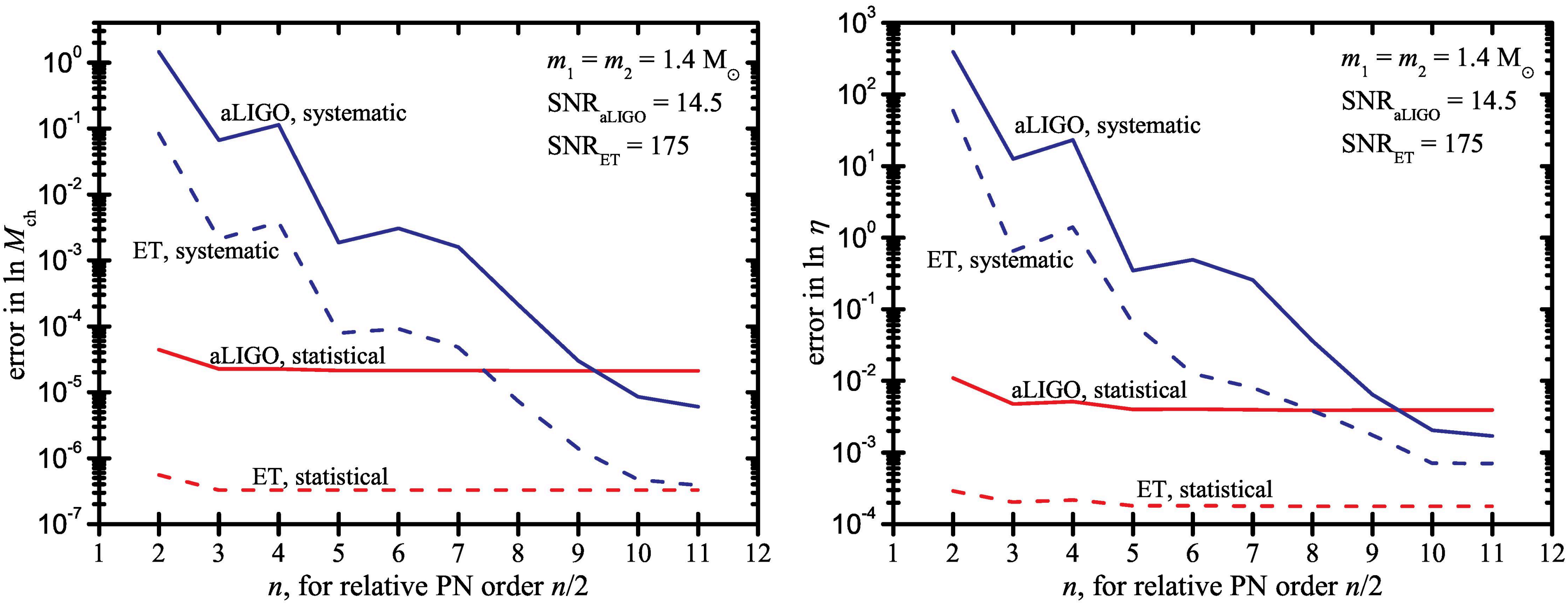
Random: large
Systematic: ?

[from Taylor, Intro to Error Analysis]

Motivation for future analytic modeling:
systematic errors: neglecting high PN terms

$$\Psi_{\mathrm{T}} \propto \Delta\Psi^{\text{point particle}}_{3.5\mathrm{PN}} + \Delta\Psi^{\text{test mass}}_{6\mathrm{PN}}$$

$$\Psi_{\mathrm{AP}} = \Psi_{\mathrm{T}} \ \text{ truncated at order } v^n$$



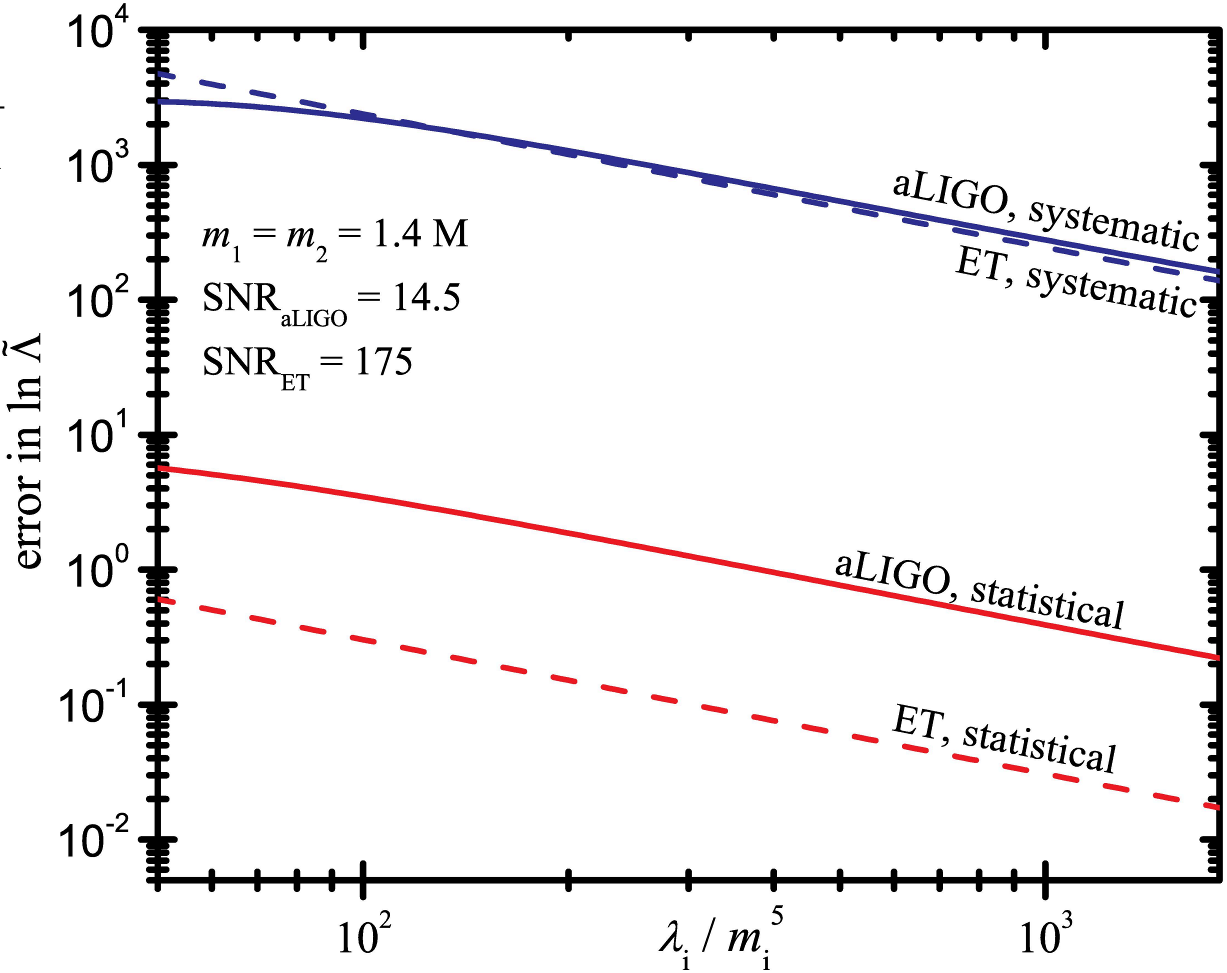
Motivation for future analytic modeling:
systematic errors: neglecting high PN terms

$$\Psi_{\rm T} \propto \Delta\Psi^{\rm point\ particle}_{\rm 3.5PN} + \Delta\Psi^{\rm tidal}_{\rm 6PN} + \Delta\Psi^{\rm test\ mass}_{\rm 6PN}$$

$$\Psi_{\rm AP} \propto \Delta\Psi^{\rm point\ particle}_{\rm 3.5PN} + \Delta\Psi^{\rm tidal}_{\rm 6PN}$$

$$\tilde{\Lambda} = F(\hat{\lambda}_A, \hat{\lambda}_B, \eta) \xrightarrow{(\eta=\frac{1}{4})} \frac{\lambda_A}{m_A^5}$$

- Tidal effects become unmeasureable with PN waveforms if high PN terms are not known.



Motivation for future analytic modeling: post-Newtonian waveforms

- Extending point-particle PN waveforms to 4PN order is desirable for accurate mass determination w/ AdvLIGO [partial results known Jaranowski & Scahefer ’12, ’13; Foffa & Sturani ‘13].

- 5PN point-particle waveforms desirable to improve determination of tidal effects.

- Spin effects to higher order in phasing (currently partially known to 3.5PN order, Bohe+ ‘13) are also likely needed for improved accuracy.

Motivation for future analytic modeling: extreme-mass ratio inspirals

- Computation of self-force in Schwarzschild case is understood, but many issues remain (needed primarily for LISA era).

- Kerr case still under development [Friedman+; Barack+].

- Fully self-consistent evolutions still not performed (beyond adiabatic approx).

- Resonance effect causes unexpected "glitch" in phasing when radial and polar frequencies form rational ratios [Hinderer, Flanagan, Hughes, Sago, others...]. Dominates conservative SF effect.

- Spin of compact object causes comparable dephasing to conservative SF (Binder & MF, Gair & Huerta).
$$m_1\left(\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha_{\beta\gamma}[g^{\rm kerr}]\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)=\frac{1}{2}\epsilon_{\lambda\mu\rho\sigma}R^{\alpha\nu\lambda\mu}u_\nu u^\sigma S_1^\rho+O(S_1^2)$$

- Possibly will need to go to 2nd order in pert. theory (still poorly developed).

- Analytic BH pert. calculations could be extended to higher PN orders for spinning or eccentric systems.

Motivation for future analytic modeling:
EOB and phenom waveforms

Effective-one-body:

- EOB well developed for BBHs w/ aligned spins [Buonanno, Pan, Taracchini +; Damour, Nagar+]
- Precessing case under development but likely completed soon.
- Some work on NS/NS [Damour, Nagar+], but not as developed; merger treatment more difficult. Likewise for NS/BH.

Phenomenological waveforms:

- Aligned spin case also generally understood, but room for model improvements and better fits.
- Precession effects not included.
- Some work on NS/BH case [Lackey+ '13].

Summary:

- 2nd generation network of gravitational wave interferometers will come online in 2017 – 2020. Hope to detect at least a few events per year.
- Combination of analytic and numerical models are needed to make accurate waveforms, which are important for determining parameters.
    - numerical relativity
    - post-Newtonian theory
    - BH perturbation theory + self-force approach
    - effective-one-body (EOB)
    - phenomenological waveforms
- Many cross-comparisons between these approaches have been performed. Excellent agreement in many cases.
- Higher-order PN waveforms needed to control systematic errors (also to improve Love number determination); spin effects needed to higher order.
- Much more work needed in small-mass-ratio case: self-force in Kerr; possibly will need to go to 2nd order in perturbation theory (for LISA).