

# Fluids in Relativity

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# Introduction

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# The Ideal Fluid

The most popular EOS in relativistic fluid dynamics has been the ideal fluid EOS, but that is changing as more realistic models are desired. However, this is still the place to begin.

The ideal fluid EOS is for a degenerate or near-degenerate classical gas. The pressure is simply

$$P = (\Gamma - 1)\rho_0\epsilon_{\text{int}} \quad (1)$$

where  $\Gamma$  is a constant.

ideal gas = systems of noninteracting particles. In practice this means viscosity is negligible. energy of particle interactions is much smaller than the kinetic energy

Perfect gasses are ideal gasses obeying Maxwell-Boltzmann statistics.

Perfect gas = nondegenerate ideal gas

Completely degenerate gases are ideal gases but not perfect gases.

# First Law of Thermodynamics

The particles in each fluid volume are described using thermodynamics. Their random motions and energies are averaged, and described by a few state variables.

In a comoving reference frame  $\mathcal{O}$ , the First law of thermodynamics is

$$dU = \delta Q - \delta W \quad (2)$$

$\delta W$  represents a change in mechanical or chemical work, and  $\delta Q$  represents difference in heat energy.  $\delta Q$  and  $\delta W$  are not perfect differentials because they depend on how work is done or heat flows. The general work term could include

$$\delta W = PdV - JdL - \sigma dA - \mathbf{E} \cdot d\mathbf{P} - \mathbf{H} \cdot d\mathbf{M} - \phi de - \sum_j \mu_j dN_j \quad (3)$$

We simplify and write the First Law as

$$dU = \delta Q + PdV + \sum_j \mu_j dN_j \quad (4)$$

## The Second Law of Thermodynamics

The entropy  $S$  plays a central role in thermodynamics. It can be thought of as a measure of disorder in a system. For an isolated system, the entropy is the logarithm of the number of available states. Entropy is related to heat energy

$$dS \geq \frac{1}{T} dQ, \quad (5)$$

however, for a *reversible* process

$$dS \equiv \frac{1}{T} dQ \quad (\text{reversible process only}) \quad (6)$$

The combined First and Second Laws of thermodynamics give the

### Thermodynamic Identity

$$dU = T dS - P dV + \sum_j \mu_j dN_j \quad (7)$$

# Intensive and Extensive Variables

When the size of the system changes, some state variables are modified proportionally, while others are constant. The former are *extensive* variables, and the latter are *intensive* variables. Consider two systems, only larger (or smaller) than the other by a factor  $\lambda$

- Extensive variables
  - Internal Energy,  $\tilde{U} = \lambda U$
  - Volume,  $\tilde{V} = \lambda V$
  - Entropy,  $\tilde{S} = \lambda S$
- Intensive variables
  - Pressure,  $\tilde{P} = P$
  - Temperature,  $\tilde{T} = T$
  - Chemical Potential,  $\tilde{\mu} = \mu$

# The Fundamental Equation

We write

$$S(\lambda U, \lambda V, \lambda N_j) = \lambda S(U, V, N_j) \quad (8)$$

Differentiate  $(\lambda S)$  w.r.t.  $\lambda$

$$\begin{aligned} \frac{d}{d\lambda}(\lambda S) &= \left( \frac{\partial S}{\partial \lambda U} \right)_{V, N_j} \frac{d}{d\lambda}(\lambda U) + \left( \frac{\partial S}{\partial \lambda V} \right)_{U, N_j} \frac{d}{d\lambda}(\lambda V) \\ &\quad + \sum_j \left( \frac{\partial S}{\partial \lambda N_j} \right)_{U, V, N_{k \neq j}} \frac{d}{d\lambda}(\lambda N_j) \end{aligned} \quad (9)$$



## Fundamental Equation II

From the Thermodynamic Identity

$$\left(\frac{\partial S}{\partial U}\right)_{V, N_j} = \frac{1}{T}, \quad (10)$$

$$\left(\frac{\partial S}{\partial V}\right)_{U, N_j} = -\frac{P}{T}, \quad (11)$$

$$\left(\frac{\partial S}{\partial N_j}\right)_{U, N_{k \neq j}} = -\frac{\mu_j}{T}. \quad (12)$$

These are the thermal, mechanical, and chemical *Equations of State*. Combined with the previous slide, we obtain

### Fundamental Equation of Thermodynamics

$$U = TS + PV + \sum_j \mu_j N_j. \quad (13)$$

## Relativistic Considerations

Note that in relativity,  $e$  as the total energy also contains the mass energy. We write

$$\epsilon = \sum_j m_j n_j c^2 + \epsilon_{\text{int}} \quad (14)$$

where  $m_j$  is the rest mass of single particle of species  $j$  and  $\epsilon$  is the internal energy.  $\epsilon$  contains the thermal energy, kinetic energy of random, thermal motion, energies of any internal states.

$\mu_j$  is the relativistic chemical potential for particles of species  $j$ , and it also contains the rest mass energy. We can express this in terms of the classical chemical potential,  $\mu'_j$  as

$$\mu_j = \mu'_j + m_j. \quad (15)$$

For a system with a single particle species, we can simplify the equations by using densities. Let  $\lambda = 1/V$ . Then

$$\tilde{S} = s, \quad \tilde{V} = 1, \quad \tilde{N} = n, \quad \tilde{U} = \epsilon \quad (16)$$

where

- $s$  is the entropy per unit volume
- $n$  is the number density
- $\epsilon$  is the total energy per unit volume.

$$\epsilon = \tilde{U}(\tilde{S}, \tilde{V}, \tilde{N}) = \tilde{U}(s, 1, n) = \tilde{U}(s, n). \quad (17)$$

The Thermodynamic Identity becomes

$$d\tilde{U} = \tilde{T} d\tilde{S} - \tilde{P} d\tilde{V} + \sum_j \tilde{\mu}_j d\tilde{N} = T ds + \sum_j \mu_j dn_j \quad (18)$$

which becomes

$$d\epsilon = T ds + \sum_j \mu_j dn_j. \quad (19)$$

We identify

$$T = \left( \frac{\partial \epsilon}{\partial s} \right)_{n_j}, \quad \mu_j = \left( \frac{\partial \epsilon}{\partial n_j} \right)_{s, n_k \neq j} \quad (20)$$

Combining with the Fundamental Equation

$$P = -\epsilon + s \left( \frac{\partial \epsilon}{\partial s} \right)_{n_j} + \sum_j n_j \left( \frac{\partial \epsilon}{\partial n_j} \right)_{s, n_k \neq j} \quad (21)$$

## The Ideal Fluid

The perfect gas is valid for both relativistic and non-relativistic, non-degenerate, non-interacting particles. Consider a system of  $N$  particles with mass  $m$ . Each particle has  $f$  degrees of freedom, which includes translational, and any rotational or other internal degrees of freedom. The ideal gas law is

$$PV = Nk_B T \quad (22)$$

and the internal energy is

$$U = \frac{f}{2} Nk_B T. \quad (23)$$

Here we only need to know that  $U \propto T$ , so that  $dU/dT = U/T$ . The fluid temperature  $T$  does not appear explicitly in our fluid equations, so we cannot use the EOS in this form. We therefore rewrite this EOS in terms of other variables.

## Ideal Fluid: Heat Capacity at Constant Volume

The heat capacities at constant volume and pressure are

$$C_V = \left( \frac{\delta Q}{dT} \right)_V, \quad C_P = \left( \frac{\delta Q}{dT} \right)_P \quad (24)$$

From the first law of thermodynamics

$$\delta Q = dU + P dV \quad (25)$$

we have that

$$C_V = \left( \frac{\delta Q}{dT} \right)_V = \left( \frac{dU}{dT} \right)_V \quad (26)$$

As  $U = U(T)$ , this becomes simply

$$C_V = \frac{dU}{dT} = \frac{U}{T}. \quad (27)$$

## Ideal Fluid: Heat Capacity at Constant Pressure

The heat capacity at constant pressure can be calculated by differentiating the equation of state

$$P dV + V dP = Nk_B dT \quad (28)$$

and combining with the first law

$$C_P = \left( \frac{dQ}{dT} \right)_P = \frac{dU}{dT} + Nk_B \quad (29)$$

$$= C_V + Nk_B \quad (30)$$

We often use the specific heat capacities,  $c_V$  and  $c_P$ , which are the heat capacities divided by the total mass,  $Nm$ . This gives

$$c_P - c_V = \frac{k_B}{m}. \quad (31)$$

The ratio of specific heat capacities is the *adiabatic exponent*

$$\Gamma = \frac{c_P}{c_V}. \quad (32)$$

## The Ideal Fluid EOS

The equation of state can be written in terms of  $\rho_0 = Nm/V$  as

$$P = \frac{k_B}{m} \rho_0 T. \quad (33)$$

Writing this in terms of adiabatic exponent gives

$$P = (c_P - c_V) \rho_0 T = (\Gamma - 1) c_V \rho_0 T. \quad (34)$$

We have

$$c_V = \frac{1}{Nm} \frac{dU}{dT} = \frac{1}{Nm} \frac{U}{T} = \frac{1}{Nm/V} \frac{U/V}{T} = \frac{\epsilon_{\text{int}}}{T}, \quad (35)$$

where we used  $\rho_0 = mN/V$ , the internal energy  $\epsilon = U/V$ , and the specific internal energy density  $\epsilon_{\text{int}} = \epsilon/\rho_0$ . Finally(!) we have

### Ideal Fluid EOS

$$P = (\Gamma - 1) \rho_0 \epsilon_{\text{int}}. \quad (36)$$



# The Ideal Fluid and Isentropic Processes

An adiabatic process satisfies

$$dU = T dS - P dV \quad (37)$$

If the process is also *reversible*,  $dS = 0$ , then  $dU = -P dV$ . Using  $U = \varepsilon V$  and the EOS  $P = (\Gamma - 1)\rho_0 \epsilon_{\text{int}}$ , we write

$$\begin{aligned} d((\rho_0 c^2 + \rho_0 \epsilon_{\text{int}})V) &= -P dV \\ c^2 d(\rho_0 V) + \frac{1}{\Gamma - 1} d(PV) &= -P dV \\ V dP + P dV &= (1 - \Gamma)P dV. \end{aligned} \quad (38)$$

In going from the second to third lines we used

$$V \propto \frac{1}{\rho_0} \quad \text{and} \quad dV \propto -\frac{1}{\rho_0^2} d\rho_0, \quad (39)$$

so  $d(\rho_0 V) = 0$ .

## The Ideal Fluid EOS in Polytropic Form

Now change variables from  $V$  to  $\rho_0$  to obtain

$$\frac{dP}{\rho_0} = \frac{\Gamma P}{\rho_0^2} d\rho_0, \quad (40)$$

which integrates to

$$\ln P = \Gamma \ln \rho_0 + \ln K(S) \quad (41)$$

with the integration constant  $\ln K(S)$ . I write explicitly  $K(S)$  to emphasize that we assumed  $S = \text{constant}$ .

The ideal fluid EOS in polytropic form

$$P = K(S)\rho_0^\Gamma. \quad (42)$$

This form of the EOS is valid for reversible processes only, and cannot be used for fluids with shocks, which are irreversible. We commonly use the polytropic form of the EOS for generating initial data.

## Simple Model for Nuclear Matter

The simplest model for cold nuclear matter is the degenerate Fermi gas. The Pauli Exclusion Principle gives rise to a pressure in the gas, as Fermions must occupy unique states. This nonthermal pressure is important in astrophysics because it is the primary source of pressure in cold stars. White Dwarfs are supported by electron pressure, and neutron stars by the Fermi pressure of the neutrons.

## Degenerate Fermi Gas

As an example of a degenerate Fermi gas, consider the matter in a cold white dwarf. The pressure comes from the electron gas, while most of the mass comes from the baryons. We assume:

- $T = 0$ , so that the electrons are in their lowest energy state. This is a degenerate Fermi gas.
- The proton and neutron masses are equal to the average baryon mass,  $m_p \simeq m_n = m_B$ .
- The matter is electrically neutral, so  $n_e = n_p$ .
- The average number of baryons per electron is  $A/Z$ , where  $Z$  is the atomic number and  $A$  is the mass number, the sum of neutrons and protons for an atom. The inverse ratio is the electron fraction  $Y_e = Z/A$ , or the number of electrons per baryon.

## Degenerate Fermi Gas: Electrons

We can analyze the degenerate Fermi gas using simple ideas from statistical mechanics. The number of free electrons states with a wave vector between  $k$  and  $k + dk$  per unit volume is

$$dn = \frac{4\pi k^2}{(2\pi\hbar)^3} dk \quad (43)$$

At  $T = 0$ , the number density of electrons is

$$n = 2 \frac{4\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^2 dk = \frac{1}{3\pi^2\hbar^3} k_F^3, \quad (44)$$

where  $k_F$  is the Fermi momentum. The initial factor of 2 accounts for the electron spin. While electrons supply the pressure, baryons contribute the mass. The rest mass density is

$$\rho_0 = \frac{m_B n}{Y_e}. \quad (45)$$

## Degenerate Fermi Gas: Electrons

The Fermi momentum can now be written

$$k_F = \hbar \left( \frac{3\pi^2 \rho_0}{Y_e m_B} \right)^{1/3}. \quad (46)$$

For a given momentum, the energy is  $E^2 = m^2 c^4 + k^2 c^2$ . The electron energy is

$$\epsilon_{\text{elec}}(k_F) = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} (k^2 c^2 + m_e^2 c^4)^{1/2} k^2 dk \quad (47)$$

This integrates to

$$\epsilon_{\text{elec}}(k_F) = \frac{m_e^4 c^5}{8\pi^2 \hbar^3} \left[ x(2x^2 + 1)(1 + x^2)^{1/2} - \sinh^{-1} x \right], \quad (48)$$

where  $x \equiv k_F / (m_e c)$ . The total energy density is

$$\varepsilon = \frac{m_B n c^2}{Y_e} + \epsilon_{\text{elec}}(k_F). \quad (49)$$

## Degenerate Fermi Gas: Electrons

From the first law of thermodynamics, it can be shown that at  $T = 0$

$$P = n \frac{d\epsilon}{dn} - \epsilon. \quad (50)$$

The pressure is then

$$P = \frac{m_e^4 c^5}{24\pi^2 \hbar^3} \left[ x(2x^2 - 3)(1 + x^2)^{1/2} + 3 \sinh^{-1} x \right]. \quad (51)$$

We now have parametric equations for  $\epsilon(\rho_0)$  and  $P(\rho_0)$ . Alternatively, the pressure can be calculated directly using statistical mechanics

$$P = \frac{1}{3} \int_0^{k_F} kv (2dn), \quad (52)$$

where the factor of  $1/3$  accounts for isotropy of space, and the factor 2 accounts for electron spin. To integrate this equation, write  $v = kc^2/E$ .

## Degenerate Fermi Gas: Electrons

The equations for  $\varepsilon$  and  $P$  are complicated, and it is helpful to look at the limiting cases of non-relativistic and relativistic electrons. In both cases we will be able to write the EOS in a particularly simple form

$$P = K\rho_0^\Gamma \quad (53)$$

called a *polytrope*, where  $K$  and  $\Gamma$  are constants.

- For relativistic electrons,  $k_F \gg m_e c$ , and we find

$$P_{\text{rel}} = K_{\text{rel}} \rho_0^{4/3}, \quad K_{\text{rel}} = \frac{\hbar c}{12\pi^2} \left( \frac{3\pi^2 Y_e}{m_B} \right)^{4/3} \quad (54)$$

- For nonrelativistic electrons,  $k_F \ll m_e c$ ,

$$P_{\text{non}} = K_{\text{non}} \rho_0^{5/3}, \quad K_{\text{non}} = \frac{\hbar^2}{15\pi^2 m_e} \left( \frac{3\pi^2 Y_e}{m_B} \right)^{5/3}. \quad (55)$$

(To calculate these limiting expressions, it may be easier to re-evaluate the integrals in the appropriate limits, rather than do a Taylor expansion of the final result.)



# Energy

The electron energy can be calculated in the same limits. We get

$$\epsilon_{\text{elec,non}} = \frac{Y_e m_e}{m_B} \rho_0 c^2 \quad (56)$$

It may seem strange that  $Y_e$  appears in this expression for the electron energy, but this is because it is written in terms of the baryon rest mass density,  $\rho_0$ . For relativistic electrons, we get the expected result

$$\epsilon_{\text{elec,rel}} = 3P_{\text{rel}}. \quad (57)$$

The total energy is, again,

$$\epsilon = \rho_0 c^2 + \epsilon_{\text{elec}}. \quad (58)$$

For non-relativistic electrons, and often even for relativistic electrons,  $\epsilon_{\text{elec}} \ll \rho_0 c^2$ , and we have

$$\epsilon \simeq \rho_0 c^2. \quad (59)$$

## Degenerate Fermi Gas: Neutrons

The simplest model for a neutron star consists of a degenerate pure neutron gas. The analysis of the neutron gas follows the electron case exactly, except that  $m_e \rightarrow m_B$ , we drop  $Y_e$ , and  $\varepsilon = \varepsilon_n$ .

The limiting cases are

- For relativistic neutrons

$$P_{\text{rel}} = K_{\text{rel}} \rho_0^{4/3}, \quad K_{\text{rel}} = \frac{\hbar c}{12\pi^2} \left( \frac{3\pi^2}{m_B} \right)^{4/3} \quad (60)$$
$$\epsilon_{\text{rel}} = 3P_{\text{rel}}$$

- For nonrelativistic neutrons

$$P_{\text{non}} = K_{\text{non}} \rho_0^{5/3}, \quad K_{\text{non}} = \frac{\hbar^2}{15\pi^2 m_B} \left( \frac{3\pi^2}{m_B} \right)^{5/3} \quad (61)$$
$$\epsilon_{\text{non}} = \rho_0 c^2$$

# Common Equations of State in Relativistic CFD

Different approximations for the equation of state are used depending on the system being modeled, the focus of the research, and the desired level of complexity. The most commonly used equations of state are:

- **Ideal Fluid** or **Gamma-Law EOS**. This is the classical ideal gas EOS. It is commonly used for (relatively) diffuse gases, such as accretion disks. This EOS is also widely used for nuclear matter because of its simple analytical description.
- **Cold Nuclear EOS**. Nuclear matter at  $T = 0$  is degenerate and has a relatively simple EOS. Many cold nuclear equations of state exist, and have been used to study questions of neutron star structure, the neutron star maximum mass, and constructing neutron star initial data.

## Common Equations of State in Relativistic CFD II

- **Hybrid EOS.** is a simple heuristic approach for simulating nuclear matter, where a cold nuclear EOS is combined with a simple thermal EOS

$$P = P_{\text{cold}} + P_{\text{thermal}} \quad (62)$$

The cold nuclear EOS more accurately represents stellar matter, while the thermal EOS allows for shock heating. The ideal fluid is often used for the thermal EOS.

- **Finite-temperature nuclear EOS.** These equations of state do it all—they accurately represent nuclear matter and allow for physical heating. They are derived from sophisticated models for nuclear matter, and are tabulated for use in evolution codes (see <http://stellarcollapse.org>). Examples include the Lattimer-Swesty EOS, the H. Shen EOS, etc.

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