Lecture flow

Market dynamics, expectations – stability discussion

Macroeconomic stability – Multiplier/Accelerator model

The dynamics of distribution of income to the level of income

- Profit share, output dynamics or wage share and employment dynamics (example of Goodwin's Predator – Prey model. (See Goodwin's "Growth cycle" paper)

- From the *stylized facts* on income distribution discussed in my first lecture, think of the dynamics of *inter* and *intra* distribution of income in the categories of profits and wage

Modelling instability, abrupt transition from bull to bear market, described by Christopher Zeeman's paper.

Using a stock flow consistent accounting framework, we define a macroeconomy, which includes financial sector including both commercial banks and financial periphery (shadow banking sector).

Assuming that the stylized macroeconomic model captures not only the institutional details of the financial sector and also incorporate the *stylized facts* of income distribution between the profit and wage categories.

Using this macroeconomic model, we can now study the dynamics of the system. I.e. can we understand why economies after a financial crash linger in the recessionary state for a long time, or economies experience what is called as *secular stagnation*

Economic dynamics

L1: Market dynamics

Introduction to Economic Dynamics

- We study models because they suggest how real-world processes behave.
- Every model of a physical process is at best an *idealization*. The goal of a model is to capture some feature of the real-world process.
- In this course the real-world process that we study is an *economic system*

- Since there are multitudes of actors (consumers, investors, governments, banks and other institutions) in an economy and they all interact through markets our study will focus on the functioning of the markets.
- In order to capture some features of the markets, as we can't capture the complexities of the markets in its entirity, we necessarily need to **model** markets and study these features of the markets.
- As we have seen in the recent **Great Crash 09** markets are suceptible to crashes and booms. One of the fundamental aims of the subject economics is to understand the *stability* of the markets.
- In order to understand the stability of markets, we model some key features of the markets and study their evolution over time. Thus the study of markets naturally lends itself to the study of dynamical systems, as the latter subject provides with methods, tools and techniques to analyze *any* physical system that evolves over time.

- However, there is one significant difference between an economic system and other physical systems. As you know that the economic systems are made up of multitude of actors, who are people with emotions, **the behaviour** of these actors play an important role in determining the stability of such a system.
- For instance, consumers would behave differently than investors and other actors during a recession, and similarly each actor would **behave** differently vis-a-vis the other, which has great implications for the future evolution of the system.
- This is a crucial difference between an economic system and any other physical system, and makes it more difficult to model the former and study its stability under various assumptions of human behaviour.

An Illustration

- To illustrate the use of dynamical systems in the study of stability of markets, let us consider this simple model of Demand and Supply.

$$\begin{aligned} D_t &= a + b p_t, \qquad b < 0, \qquad (1) \\ S_t &= a_1 + b_1 p_{t-1}, \quad b_1 > 0. \qquad (2) \end{aligned}$$

- Where D_t equation represents the demand at time t as a function of price at time t and S_t equation represents the supply at time t as a function of price at time $t-1$.
- The economic intuition behind such a supply function is that suppliers (for instance, farmers) would supply more in the present period t, if they had received a favourable price in the last time period $t-1$.

Equilibrium

- The equilibrium condition, where neither the demander nor the supplier are unsatisfied in the exchange, is

$$D_t = S_t,$$

i.e. it is the point where both consumer and the supplier agree to exchange their goods.

- Substituting equations (1) and (2) in the equilibrium condition would yield

$$bp_t - b_1 p_{t-1} = a_1 - a, \tag{3}$$

which is a first-order difference equation.

Analysis

- The general solution of the homogenous part of the equation is

$$A\left(\frac{b_1}{b}\right)^t.$$

- The homogenous part of the equation is

$$bp_t - b_1 p_{t-1} = 0 \quad \Rightarrow \quad bp_t = b_1 p_{t-1} \quad \Rightarrow \quad p_t = \frac{b_1}{b} p_{t-1}.$$

- To see the solution of the homogenous part, let us iterate the above solution

$$p_1 = \frac{b_1}{b} p_0, \quad p_2 = \left(\frac{b_1}{b}\right)^2 p_0, \quad \dots \quad p_t = \left(\frac{b_1}{b}\right)^t p_0.$$

By letting $p_0 = A$, we have

$$p_t = \left(\frac{b_1}{b}\right)^t A,$$

which is the general solution of the homogenous part of equation (3).

- The particular solution of the equation is

$$p_e = \frac{a_1 - a}{b - b_1}.$$

- The particular solution is obtained by substituting

$$p_t = p_{t-1} = p_e$$

in equation (3), since the right hand side of the equation is constant.[1]

[1] Refer to Chapter 3 in Gandolfo for solving the first-order difference equations for the case where the RHS is a function of time or exponential function or some other generic function of time

- Now combining both the solutions, the general solution of equation (3) is

$$p(t) = \frac{a_1 - a}{b - b_1} + A\left(\frac{b_1}{b}\right)^t \quad \Rightarrow \quad p(t) = p_e + A\left(\frac{b_1}{b}\right)^t.$$

- Finally, we have to find the value of A. To do that, let $t = 0$ in the above general solution

$$\Rightarrow p(0) = p_e + A \quad \Rightarrow \quad A = p(0) - p_e.$$

- And hence, the general solution of equation (3) is

$$p(t) = (p(0) - p_e)\left(\frac{b_1}{b}\right)^t + p_e. \tag{4}$$

Stability Analysis

- The stability of the Demand-Supply model can be understood from equation (4). In the equation

$$\frac{b_1}{b}$$

represent the ratio of the slope of the supply curve and the demand curve and p_e is the equilibrium price.

- Note that the demand curve has a negative slope $b < 0$ and the supply curve has a positive slope $b_1 > 0$. Then $\frac{b_1}{b} < 0$, so the price will have an oscillatory movement around its equilibirum value p_e
- Three cases arise depending upon the value of the ratio of the slope.

- **The case when** $|\frac{b_1}{b}| < 1$

  This implies the slope of the supply curve is less than that of the demand curve, i.e. $|b_1| < |b|$. In this case, you can very well infer from the equation (4) that for any initial condition the distance $(p_0 - p_e)$ will vanish and the succession of prices over time will converge towards the equilibrium price p_e. This is the case of **Oscillatory Convergence**.

- **The case when** $|\frac{b_1}{b}| > 1$

  This case implies that $|b_1| > |b|$, i.e. the slope of the supply curve is greater than that of the demand curve. From the solution one can infer that for any intital condition p_0 the distance $(p_0 - p_e)$ will become greater and the succession of prices over time will diverge from the equilibrium price p_e. This is the case of **Oscillatory Divergence**.

- **The case when** $|\frac{b_1}{b}| = 1$
  In this case the succession of prices over time will have a constant amplitude oscillations (**Limit Cycle**)

A General Demand-Supply Model

- Let us consider the previous model in a more general setting. This would generalization would help you to see the role of **behaviour** of human agents in the simple model that we discussed in the last section.
- The simple model of demand-supply analyzed in the last section can be modified in the following way:

$$D_t = a + bp_t, \qquad b < 0$$
$$S_t = a_1 + b_1 p_t^e, \quad b_1 > 0$$

- Note that in this model supply is a function of expected price (p_t^e). With b_1, the slope of the function being positive this means that the supply of a good is positively related to the expected price, i.e. if the supplier expectes that he might get a higher price for his product in the next period ($t + 1$), he will increase his supply in this period (t).
- The equilibrium condition again is the same, i.e

$$D_t = S_t.$$

Specifying Expectation Rules: Naive Expectation

- Now in order to complete the model, we need to specify the rule for expectations i.e. we need to specify how do the supplier form his expectations about the future price, p_t^e.
- This is one of the most challenging problems in economics and in fact it is this behavioural aspect that distinguishes economics from other scientific disciplines. As you would agree that forming expectations of about the future can vary anything from tossing a coin to studying the past time series using the frontier techniques, and articulating a theory of expectations has been one of the most toughtest challenges for economics.
- Nonetheless, we can introduce simple rules of expectations and study the dynamics of the model.
- In the fist instance, let us specify that the supplier has naive expectations,

  $$p_t^e = p_{t-1},$$

  i.e. he expects the next period's price to be the same as the current period's price.

The Normal Price expectation case

- The second way to model expectations is to assume that the suppliers have some notion of a **normal price** or a **long-run price** to which the current market price will tend to over time.
- The simplest way to formalize this expectations rule is

$$p_t^e = p_{t-1} + c(p_N - p_{t-1}), \qquad 0 < c < 1. \tag{5}$$

- The intuition behind the rule is if the current price is lower than the normal price (p_N), the suppliers would expect an increase in the next period's price.
- And the fact that the positive constant c, which is called as the **speed of adjustment**, is assumed to be less than unity is equivalent to the assumption that the suppliers think that the convergence of the last time period's price to the normal price is not immediate and the process will take some time (measured by the reciprocal of c).

- However, we need to specify the **normal price** without which this specification would be a tautology. For this case, we assume that the normal price p_N is nothing but the equilibrium price p_e that we dervied in the earlier model, i.e

$$p_N = p_e = \frac{a_1 - a}{b - b_1}.$$

- Such an assumption could be justified by the reasoning that the suppliers have perfect information, but know that due to market frictions and other institutional rigidities the current price cannot immediately get back to the normal price, which is p_e.

- Now let us substitute these assumptions in the general model and study the stability properties

$$b_1(1-c)p_{t-1} - bp_t = a - a_1 - b_1 c p_e,$$

whose solution is

$$p(t) = (p_0 - p_e)\left(\frac{b_1(1-c)}{b}\right)^t + p_e. \tag{6}$$

The stability condition is

$$\left|\frac{b_1(1-c)}{b}\right| < 1 \qquad \text{or} \qquad |b_1(1-c)| < |b|.$$

- Analyze the solution
- When $\left|\frac{b_1(1-c)}{b}\right| < 1$, starting from any initial condition the succession of prices would converge to the equilibrium value.
- When $\left|\frac{b_1(1-c)}{b}\right| > 1$, starting from any initial condition the succession of prices would diverge away from the equilibirum value.
- When $\left|\frac{b_1(1-c)}{b}\right| = 1$, we should have constant-amplitude oscillations.

- How does this model compare with the earlier model?
- In the case $\left|\frac{b_1(1-c)}{b}\right| < 1$, the convergence is faster because

$$\left|\frac{b_1(1-c)}{b}\right| < \left|\frac{b_1}{b}\right| < 1$$

  and hence the absolute value of $\left(\frac{b_1(1-c)}{b}\right)^t$ tends to zero more rapidly than the absolute value of $(\frac{b_1}{b})^t$.
- The repeated oscillation can also becomes damped in this model. This is so because, if $|b_1| = |b|$, then $|b_1(1-c)| < |b|$.

- And the divergent movement is slower because the absolute value of $\left(\frac{b_1(1-c)}{b}\right)^t$ increases at a slower rate than the absolute value of $\left(\frac{b_1}{b}\right)^t$.

- However, for values of the parameter c is sufficiently colse to unity the divergent movement may become convergent because when c is greater $(1-c)$ is smaller so that it is more likely that $|b_1(1-c)| \leq |b|$ even if $|b_1| > |b|$. This makes economic sense becasue the larger the value of c means that the suppliers expect a faster approach to the current price towards its equilibrium price.

- To conclude, the introduction of expectations based on the normal price makes the model more stable.

Ways of Doing

L3: The Macroeconomy

Macrodynamics: The Multipliers

- So far we have seen examples in a single market or in a microeconomic setting, now we study an example in a macroeconomic context.
- The subject of Macroeconomics is the study of the economy as a whole encompassing all the markets, institutions etc in an economy. The subject is concerned with the stability and growth of the economy as a whole.

- In the literature, broadly, there are two methodological approaches for analyzing the macroeconomy -

  1. one is to study the overall structure using macroeconomic **aggregates** like consumption, investment, government expenditure and net exports of the economy as a whole.
  2. The second way, is to **build up** the economy by modelling individual behaviour and analyzing how the economy *evolves* when there is interaction between many individuals with either similar behaviour or different behaviour. For example, you can consider all individuals are **rational** and study their interaction to understand how such an economy evolves or you can consider individuals with different levels of rationality and study the implications for the evolution of the macroeconomy.

- We shall start with the first approach of studying the overall structure to analyze the stability and evolution of the macroeconomy.

The dynamics of Multipliers

- From an accounting perspective the macroeconomic income or the economy's income can be derived from either adding up the income of all the individuals in the economy or adding up their expenditures.
- For example, if we consider that there are only three groups in the economy, i.e. workers, firms and government, then the macroeconomic income can be derived by

$$Y = W + \Pi + NGR,$$

where W is the total wages in the economy, Π is the total profits in the economy and NGR is the net government revenue.

- We can also derive the national income from the expenditure side, i.e. if we add up all the expenditures of these three groups we should be able to get an *estimate* of the national income

$$Y = C + I + G,$$

  where C is the consumption expenditure, I is the investment expenditure and G is the government expenditure.
- If we consider that the economy in question trades with other countries, then we should add the net exports, i.e. exports minus imports to the above equation. That is, the national income in that case is

$$Y = C + I + G + (X - M).$$

- We shall work with the expenditure side of estimate of the national income rather than income estimate, because income reporting in any economy is not reliable.

Multiplier:An Introduction

- Let us first understand the notion of a Multiplier. Consider a closed economy, i.e. the country in question neither export nor import. In such a case, the national income from the expenditure side is

$$Y = C + I + G.$$

- Assume that the consumption expenditure is given by the following relation

$$C = a + bY, \qquad a \geq 0 \qquad 1 > b > 0,$$

  where a represents that part of cosumption that does not depend on income and b is called the marginal propensity to consume **MPC**. The MPC less than one means that individuals don't consume all their income, i.e. for instance if your income is 100 euros then you consume 60 euros when the MPC is equal to 0.6.

- Moreover, assume that the investment expenditure and government expenditure are constants, i.e. $\bar{I}$ and $\bar{G}$.
- Substituting these assumptions in the national income results in the following

$$Y = a + bY + \bar{I} + \bar{G}Y = \frac{a + \bar{I} + \bar{G}}{1 - b} \tag{1}$$

- Equation (1) implies that for a one unit increase in either autonomous consumption or investment expenditure or government expenditure, the national income would increase by $1 - b$ times, i.e let

$$Y = a + bY + \bar{I} + \bar{G}.$$

Then the change in the income for a change in investment is

$$\Delta Y = \frac{1}{1 - b}(a + \Delta I + \bar{G}).$$

Multiplier

- If we assume b to be equal to 0.6 then the above equation implies that one unit increase in the investment expenditure would lead to 2.5 times increase in the national income.
- This result, however, does not say anything about the **movement** from the old level of income to the new level of income, and in fact, we do not know if income will move towards (or away from) the new level of income.
- Moreover, we also need to know the nature of the movement from the old level to the new level, i.e. whether it would be smooth or monotonic, or fluctuating movement to the new level. This is important from the policy point of view, as the policy makers might want to know the cost of adjustment of the economy to the new level of income.

The dynamics of Multipliers

- In order to answer the questions about the **movement** from one level of income to another level of income, we need to resort to a dynamical model of national income.
- Let us restate the national income identity in time as

$$Y_t = C_t + I_t + G_t.$$

- Now let us assume that the consumption expenditure is a function of last period's income as given in the following

$$C_t = a + bY_{t-1}.$$

Let

$$I_t = \bar{I}$$

and

$$G_t = \bar{G}$$

be given constants as given above in the static formulation.

- Substituting these assumptions in the national income identity we have

$$\begin{aligned} Y_t &= a + bY_{t-1} + \bar{I} + \bar{G} \quad (2) \\ \Rightarrow \quad Y_t - bY_{t-1} &= a + \bar{I} + \bar{G} \quad (3) \end{aligned}$$

- With the simple assumptions about the consumption, investment and government expenditure, we have now set up a dynamic model of national income.
- Equation (3) is the first-order difference equation of the national income.

The dynamics of Multipliers

- The solution of equation (3) is

$$Y(t) = Ab^t + \frac{a + \bar{I} + \bar{G}}{1 - b},$$

where the equilibrium value of income is

$$Y^e = \frac{a + \bar{I} + \bar{G}}{1 - b}$$

and

$$A = Y_0 - Y^e.$$

Therefore, the final solution of equation (3) is

$$Y(t) = (Y_0 - Y^e)b^t + Y^e.$$

Stability analysis

- The stability condition is simply $|b| \leq 1$, which will always be satisfied because of our assumption that the MPC will always be greater than zero and less than one.

- Hence, the solution means that starting from any arbitrary initial level of income (initial condition) Y_0 the system (economy) will monotonically move (since $b \geq 0$) to the equilibrium level of income Y^e.

- **That was no fun!**
- Suppose, in the above model if we assume

$$I_t = hY_{t-1} + I_0, \qquad h \geq 0,$$

where h is the marginal propensity to invest (MPI).

- Now, work out the solution and analyze the stability condition!

Homework

- Consider the following relations in an open economy

$$\begin{aligned} C_t &= a + bY_{t-1}, \\ I_t &= hY_{t-1} + I_0, \\ X_t &= X_0, \\ M_t &= mY_{t-1} + M_0. \end{aligned}$$

- Substitute these functions in the national income identity for an open economy given by

$$Y_t = C_t + I_t + X_t - M_t.$$

- Solve the system and analyze the stability condition.
- Compare the stability condition of this model with the previous model.
- This example will be dealt with in the next Practical.

Harrod's model of Growth

- The above multiplier models don't have growth, i.e. growth in output. One of the earliest models of growth in the literature is Roy Harrod's model.
- The assumptions of the model are

$$
\begin{aligned}
S_t &= sY_{t-1}, \\
I_t &= k(Y_t - Y_{t-1}),
\end{aligned}
$$

  where s is propensity to save and k is called the **acceleration** coefficient.
- The equilibrium condition for the macroeconomy is

$$S_t = I_t$$

Solution

- Substitution of the saving and investment functions in the equilibrium condition gives the following first-order homegenous difference eqution

$$Y_t - \frac{k+s}{k} Y_{t-1} = 0. \tag{4}$$

- and the solution is given by

$$Y_t = Y_0 \left(\frac{k+s}{k} \right)^t . \tag{5}$$

- The solution tells us that the equilibrium income (output) increases over time at the constant rate of growth $(\frac{s}{k})$.
- This rate is called by Harrod, the *warranted* rate of growth. It is a rate such that, when income grows accroding to it, ther is a continuous equality between saving and investment, i.e. a dynamic equilibrium obtains.

Extension of Harrod's model: Homework

- Modify Harrod's model by assuming that saving is in proportion to current income,

$$S_t = sY_t$$

while the investment function is the same.

1. Analyze the solution and discuss the economic intuition behind the stability condition.
2. Why is the warranted rate of growth greater than Harrod's $(\frac{s}{k})$?

Ways of Knowing

L5: Macroeconomic dynamics

Samuelson's Multiplier-Accelerator Economic Model (1939)

This is a model of a closed economy at time $t = 0, 1, 2, \ldots$ based on the following variables

- National income Y_t.
- Total investment I_t
- Government investment $I_t^{'}$
- Private investment $I_t^{''}$
- Consumption C_t

We study the interaction between multiplier and accelerator to the level of output (see below) based on the following model for positive constants b, G, k:

$$
\begin{aligned}
C_t &= bY_{t-1}, & (1)\\
I_t &= I_t^{'} + I_t^{''}, & (2)\\
I_t^{'} &= G & (3)\\
I_t^{''} &= k(C_t - C_{t-1}), & (4)\\
Y_t &= C_t + I_t. & (5)
\end{aligned}
$$

Explaining the Model

- Equation (1) is the consumption function seen before. Consumption is a function of last period's income and the coefficient b is the Marginal Propensity to Consume (MPC), with value range $0 < b < 1$.
- Equation (2) is the investment function. It has two parts, one is called the autonomous part, which is given in equation (3) i.e. we assume constant government investment $I_t^{'} = G$
- Equation (4) is called the acceleration relation. We assume that private investment takes place when consumer spending is increasing. $k > 0$ is called the accelerator coefficient.
- Equation (5) is the equilibrium condition matching the national income to consumption plus investment.

There are only two parameters in the model, namely, the MPC parameter b and the Accelerator parameter k. The objective of the model is to study the interaction between the multiplier b and the accelerator k.

Difference equation for Y_t

From (2), (3) and (4) we have

$$\begin{aligned} I_t &= k(C_t - C_{t-1}) + G \\ &= kb(Y_{t-1} - Y_{t-2}) + G \end{aligned}$$

using (1). Hence (5) implies

$$\begin{aligned} Y_t &= C_t + I_t \\ &= bY_{t-1} + bkY_{t-1} - bkY_{t-2} + G \end{aligned}$$

Thus we have

$$Y_t - b(1+k)Y_{t-1} + bkY_{t-2} = G \tag{6}$$

(6) is the fundamental equation of the multiplier-accelerator model. It is a second-order non-homogeneous difference equation.

Analysis

Since the RHS of (6) is a constant, we can easily calculate the particular solution or the equilibrium value of output, which is given by

$$Y^* = \frac{G}{1-b}$$

The homogeneous equation

$$Y_t^H - b(1+k)Y_{t-1}^H + bkY_{t-2}^H = 0$$

has characteristic equation

$$\lambda^2 - b(1+k)\lambda + bk = 0$$

with roots $\lambda_1, \lambda_2 = \frac{-B \pm \sqrt{B^2 - C}}{2}$ for

$$B = -b(1+k), \quad C = bk.$$

Stability conditions

A necessary condition for convergence of the homogeneous solution $Y_t^H \to 0$ is $|C| = bk < 1$. Further analysis shows that this condition is in fact sufficient to guarantee convergence to quilibrium i.e.

$$bk < 1 \quad \Leftrightarrow \quad \text{Convergence: } Y_t \to Y^* = \frac{G}{1-b}$$

$$bk > 1 \quad \Leftrightarrow \quad \text{Divergence: } |Y_t| \to \infty$$

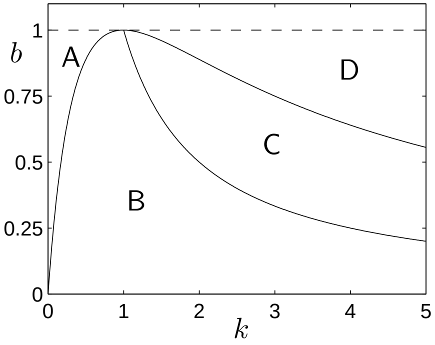
The sign of the discriminant $\Delta = B^2 - 4C$ determines whether the solution is oscillatory or monotonic. In particular we note that

$$\Delta = b^2(1+k)^2 - (4bk) < 0 \Leftrightarrow b < \frac{4k}{(1+k)^2} \Leftrightarrow \text{ Oscillatory}$$

$$\Delta = b^2(1+k)^2 - (4bk) > 0 \Leftrightarrow b > \frac{4k}{(1+k)^2} \Leftrightarrow \text{ Monotonic}$$

Dynamics of the model

We can graphically describe the stability of Y_t in terms of four regions in the parameter space (k, b) for $0 < b < 1$ and $0 < k$ determined by the above inequalities. The boundaries of these regions are given by the curves $b = \frac{4k}{(1+k)^2}$ and $b = \frac{1}{k}$. The boundary curve $b = 1/k$ is straightforward to plot. The curve $b = \frac{4k}{(1+k)^2}$ vanishes at $k = 0$ and has a maximum at $k = 1$ (check this). Thus we find



Qualitative Behaviour of the model

- In regions A and B the sequence converges to the equilibrium value Y^* the roots λ_1 and λ_2 are real, positive and less than 1 and thus there is a monotonic convergence .
- In region B the roots λ_1 and λ_2 are complex and less than 1 in magnitude and thus there is a oscillatory convergence to the equilibrium Y^*.
- In region C the roots λ_1 and λ_2 are complex and greater than 1 in magnitude and thus there is a oscillatory divergence from the equilibrium Y^*.
- In region D the roots $\lambda_1 > \lambda_2$ are real, positive and at least λ_1 is greater than 1 and thus there is a monotonic divergence from the equilibrium Y^*.

Boundaries

- On the boundary between regions A and B we have monotonic convergence to Y^*.
- On the boundary between regions B and C we have periodic orbits about Y^*.
- On the boundary between regions C and D we have monotonic divergence from Y^*.

Ways of Doing

L6: Economic growth

Application: Solow's growth model

- As an application of the first-order differential equation, we shall study Solow's growth model.
- Solow's growth model was concerned about the growth in output in a macroeconomy. Solow's model modifed the Harrod's growth model by introducing unlimited substitutability between captial and labour.
- The assumptions of the model of the model are as follows.
- First, the production function is a homogenous of degree one with the usual properties of positive but decreasing marginal productivities and is given by

$$Y = F(K, L) \tag{5}$$

- Since we are interested in how ouput per labour (or in general how output per capita grows over time), we can write the production function in the 'intensive form' since the function is homogenous.

$$\frac{Y}{L} = f(r, 1), \quad r = \frac{K}{L} \tag{6}$$

Solow's growth model

- Therefore the production function is given by

$$Y = Lf(r, 1) \tag{7}$$

- The savings function is given by

$$S = sY \tag{8}$$

- The investment function is given by

$$I = K^{'} \tag{9}$$

That is, investment is nothing but a change in capital stock.

- The labour supply in the economy grows at a constant rate n, which is written as

$$L = L_0 e^{nt} \tag{10}$$

- Since we are considering a closed economy with no governement the macroeconomic equilibrium condition is

$$S = I$$

- Therefore from the above assumptions we see

$$K^{'} = sY \tag{11}$$

- Let us first calculate the left hand side of (14)

$$r = \frac{K}{L}$$
$$\Rightarrow K = rL_0e^{nt}.$$

Therefore,

$$K^{'} = r^{'}L_0e^{nt} + rnL_0e^{nt}$$

- The equilibrium condition (14) can be written as

$$r^{'}L_0e^{nt} + rnL_0e^{nt} = sL_0e^{nt}f(r,1)$$

Therefore,

$$r^{'} = sf(r,1) - nr \tag{12}$$

- Equation(15) is the fundamental dynamic equation of Solow's growth model. Let us now study the stability of the model for a special case where the function $f(r,1)$ is given by the so called Cobb-Douglas production function

$$Y = K^{\alpha}L^{1-\alpha}, \quad 1 > \alpha > 0$$

- Let us write the Cobb-Douglas production function in terms of output per labour to get the intensive form $f(r, 1)$, which is given by $f(r, 1) = r^{\alpha}$. So that equation(15) becomes

$$r^{'} = sr^{\alpha} - nr \tag{13}$$

- Hence in the case of Cobb-Douglas production function the fundamental dynamic equation of Solow's growth model is a first-order, but non-linear differential equation.
- However, equation (16) can be made linear by a simple transformation, which called the Bernoulli transformation, by defining a new variable

$$\kappa = r^{1-\alpha} \tag{14}$$

- Note that this transformation is not just a mathematical transformation, it has an economic interpretation. It is easy to check

$$\frac{K}{Y} = (\frac{K}{L})^{1-\alpha} = r^{1-\alpha} = \kappa$$

so that the transformation of variables simply amounts to pass from the capital/labour ratio to the capital/output ratio.

- Now we can restate the fundamental dynamic equation of Solow's model in liue with the transformation (κ) as follows: Let us first differentiate $\kappa = r^{1-\alpha}$, we get

$$\kappa^{'} = (1-\alpha)r^{-\alpha}r^{'} \tag{15}$$

$$i.e \frac{1}{1-\alpha}\kappa^{'} = r^{-\alpha}r^{'}. \tag{16}$$

Now multiply both sides of (16) by $r^{-\alpha}$ we get,

$$r^{-\alpha}r^{'} = s - nr^{1-\alpha}$$

Now, using equations (17) and (19) we have

$$\frac{1}{1-\alpha}\kappa^{'} = s - n\kappa \tag{17}$$

$$\text{or} \tag{18}$$

$$\kappa^{'} + n(1-\alpha)\kappa = s(1-\alpha) \tag{19}$$

which is linear in κ and with constant coefficients.

- Hence we have transformed the fundamental dynamic equation of Solow's growth model to a first order linear differential equation.
- The general solution of the homogenous part is

$$Ae^{-n(1-\alpha)t} \tag{20}$$

and the particular solution, putting $\kappa^{'} = 0$, is

$$\kappa(\bar{t}) = \frac{s}{n}$$

- So the general solution is

$$\kappa(t) = Ae^{-n(1-\alpha)t} + \frac{s}{n}$$

- Assuming that $\kappa = \kappa_0$ at $t = 0$ we obtain $A = \kappa_0 - \frac{s}{n}$, and so the general solution is

$$\kappa(t) = (\kappa_0 - \frac{s}{n})e^{-n(1-\alpha)t} + \frac{s}{n} \tag{21}$$

- Since n and $(1-\alpha)$ are positive, the term $(\kappa_0 - \frac{s}{n})e^{-n(1-\alpha)t}$ tends to zero as t increases, so that $\kappa(t)$ tends to its equilibrium value, $\frac{s}{n}$.
- The difference between κ and its equilibirum value or steady-state value vanishes at a constant rate given by

$$\beta = n(1-\alpha)$$

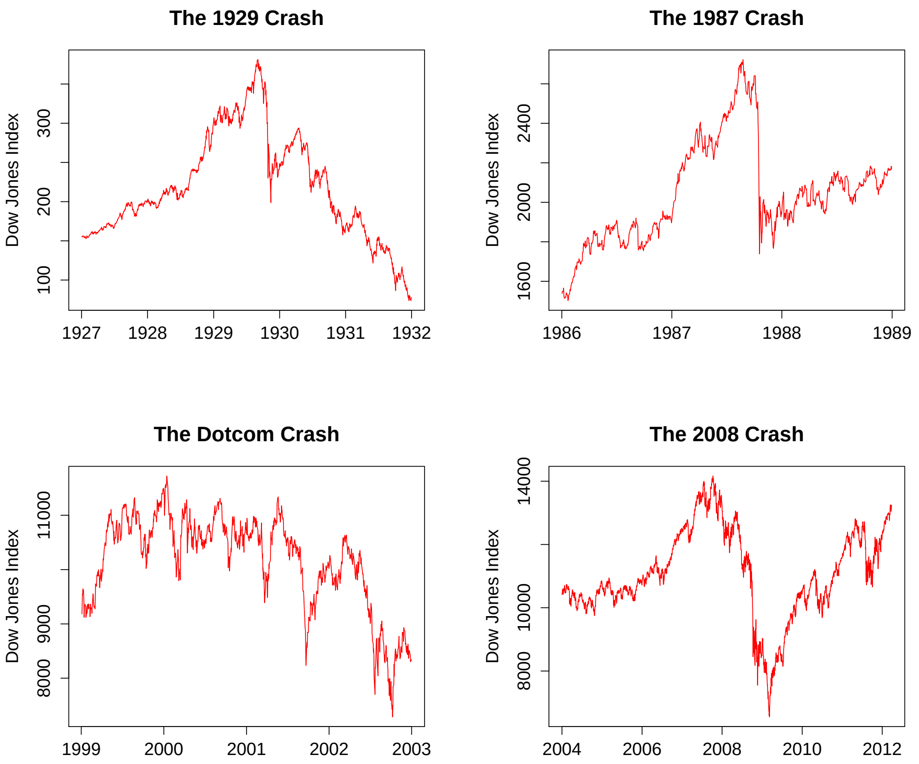
  where β is called the coefficient of convergence.
- What is the economic intuition behind this result?
- How can we understand the divergence of economic growht between developed countries and developing countries using this model?
- How does this model explain miracle growth in the post World war II era of Japan and Germany, whose capital stock was completely decimated in the war?

Homework

- Analyze the following situations:
  1. What happens when the saving rate s is increased?
  2. What happens when the population grow at a rate greater than n?
  3. What happens when you define investment as

$$I = K^{'} + \delta K$$

     to accord for depereciation (δ) of capital stock?



Visible fractures of the *invisible hand*