Credit Derivatives

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December 28, 2015

Definition

- Credit Risk: The risk arising due to the possibility of default in any financial agreement
- Credit Derivatives: Instruments that allow for hedging against credit risk
- ISDA: International Swaps and Derivatives Association
- Total Notional: 62 trillion as of January 2008; Down to 20 trillion by end of November 2008.

Examples

- Credit Default Swap
- Forward Credit Swap
- Basket CDS, Index Swaps
- Credit Swaption
- Collateralized Debt Obligations (CDOs)
- Tranche Swaps, Options on Tranches

Credit Default Swaps

- CDS is a contract to provide insurance against default
- Valid upto maturity T or default whichever is earlier
- Notional Principal: Total par value
- Protection Buyer: Pays premium at regular times
  1. Percentage of Principal notional
  2. Spread over LIBOR
- Protection Seller: Pays on Default
  1. $R\%$ or $(100 - Z)\%$ of notional; $Z-$ mid market recovery rate
  2. Physical Delivery
- Problem: What is the fair value of the premium?

CDS: Example 1

- 5 yr. CDS with notional principal 100 million entered on March 1, 2002. Buyer pays 90 basis points annually.
- **No Default:** Buyer pays $900,000$ on March 1 of each year from 2002 to 2006.
- **Default event:** Credit event on September 1, 2005. Buyer hands over 100 million worth of bonds and receives 100 million and pays the accrued payment amount (approx. $450,000$).

CDS: Example 2

- CDS allow for active management of credit risk.
- Bank has several million dollars loan to Enron on Jan 1, 2001.
- Buy a 100 million 5 yr. CDS on Enron for 135 bps or 1.35 million per year.
- Exchange part of exposure to a company in a totally different sector: Sell a 5 yr. 100 million CDS on Nissan for 1.25 million.
- Net Cost: $100,000$ per year.

Model Requirements

- Default Probability
  1. A model for investor uncertainty.
  2. A model for evolution of information over time.
  3. A model definition of default event.
- Model for the risk-free interest rate.
- Model for recovery upon default.
- Model for the risk premium that investors require for taking on the risk of default.

Two Approaches to Pricing

- Specify model under the Real world probability measure ($\mathbb{P}$).
  1. Derive a martingale measure equivalent to physical measure (eg. using Girsanov's transformation).
  2. Model calibrated on historical data.
- Specify the Model in a Risk-Neutral World or under an Pricing Martingale Measure ($\mathbb{Q}$)
  1. All discounted traded securities are Martingales.
  2. The price of any derivative is the Expected Discounted Payoff under the risk-neutral measure.
  3. Model calibrated based on liquid instruments.

Three Categories of Models

- Structural Models (eg. Merton's approach).
- Reduced Form Models (eg. Dynamic Intensity Models).
- Incomplete Information Models.

Structural Models

- Fundamental approach that seeks to price all debt instruments and credit derivatives of a firm using information from the balance sheet and the share price. Dates back to Black and Scholes (1973), Merton (1974).
- Objectives:
  1. Understand the link between debt and equity and the implications thereof to default risk.
  2. Optimizing the capital structure of the firm: Endogenous default boundaries, strategic debt service etc.
  3. Pricing Convertible and Callable Bonds.
  4. Pricing portfolio products using copulas and simulations.

Reduced-Form Models

- Objective is to be able to consistently price complex credit derivatives.
- Model calibrated based on liquid instruments (Dai and Singleton (2003)).
- Default occurs without warning according to a stochastic intensity process.
- Default dynamics are specified exogenously directly under a pricing measure.

Incomplete Information Models

- Combines aspects of both structural and reduced form models.
- Delayed information: Information available at time t is the asset price at time $t - \epsilon$.
- Information on asset prices available only at fixed times, say quarterly (Duffie and Lando(2001)).
- Uncertain information about the default barrier (Kay Giesecke (2006, 2007)).

A Simple Reduced-form Model Based Valuation

1. Assume notional to be 1.
2. Default events, interest rates and recovery rates are independent.
3. T : Life of credit default swap in years.
4. q_t : Risk neutral default probability density at time t.
5. $R(t)$: Recovery rate when default happens at time t.
6. $u(t)$: present value of payments made at the rate of 1 per year on payment dates between 0 and t.
7. $v(t)$: present value of 1 received at time t.
8. w : Payments per year made by CDS buyer.
9. s : Value of w that causes the CDS to have a value of zero.
10. π : The risk neutral probability of no credit event during the life of the swap.
11. $A(t)$: Accrued interest on the reference obligation at time t.

CDS Rate

- Premium Leg:

$$w \int_0^T q(t)u(t)dt + w\pi u(T).$$

- Default Leg:

$$\int_0^T [1 - R(t) + A(t)]q(t)v(t)dt.$$

- Credit Default Swap Spread Rate

$$s = \frac{\int_0^T [1 - R(t) + A(t)]q(t)v(t)dt}{\int_0^T q(t)u(t)dt + \pi u(T)}.$$

Implying Default Probabilities from CDS Swaps

- Suppose CDS spreads for maturities $t_1, t_2, \ldots, t_n$ are $s_1, s_2, \ldots, s_n$
- Assume default probability density $q(t)$ satisfies

$$q(t) = \sum_{i=1}^{N} q_i 1_{\{t_{i-1} < t \leq t_i\}}$$

$$s_i = \frac{\sum_{k=1}^{i} q_k \int_{t_{k-1}}^{t_k} [1 - R(t) + A_i(t)] v(t) dt}{\sum_{k=1}^{i} q_k \int_{t_{k-1}}^{t_k} u(t) dt + u(t_i) [1 - \sum_{k=1}^{i} q_k (t_k - t_{k-1})]}$$

$$\delta_k = t_k - t_{k-1}, \qquad \alpha_k = \int_{t_{k-1}}^{t_k} (1 - R(t)) v(t) dt$$

$$\beta_{k,i} = \int_{t_{k-1}}^{t_k} A_i(t) v(t) dt, \qquad \gamma_k = \int_{t_{k-1}}^{t_k} u(t) dt$$

- Note that s_i depends only on $q_1, \ldots, q_i$.
- So we can solve recursively for the q_i:

$$q_i = \frac{s_i u(t_i) + \sum_{k=1}^{i-1} q_k [s_i(\gamma_k - u(t_i)\delta_k) - \alpha_k + \beta_{k,i}]}{\alpha_i - \beta_{i,i} - s_i(\gamma_i - u(t_i)\delta_i)}$$

- The assumed structure of the default probability density depends on the available data and is too restrictive.
- Can fit a smooth curve that approximates the default probability density estimated above.
- Desirable to have a more flexible model.
- Ref: Options, Futures and Other Derivatives, by John C. Hull.

Structural Approach

- Model for uncertainty: $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which all process will be defined. $W = \{W_t : t \geq 0\}$ be a standard Brownian motion.
- V: Market value of the assets of a firm.

$$\frac{dV_t}{V} = \mu dt + \sigma dW_t$$

$$V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

- Money-market account with constant riskless rate r.
- Firm has a simple capital structure: Debt and Equity. Debt is a zero coupon bond with face value D and maturity T. Firm is run by equity owners.

Corporate Liabilities as Contingent Claims

- Merton's Model (Merton 1974): Payoffs to Debt and Equity:

$$
\begin{aligned}
D_T &= min(D, V_T) = D - (D - V_T)^+ \\
S_T &= (V_T - D)^+
\end{aligned}
$$

- Think of the firm as being run by equity owners.
- At maturity of the bond, equity owners pay the face value of the bond precisely when the asset value is higher than the face value of the bond.

Merton's first passage model (Black and Cox 1976)

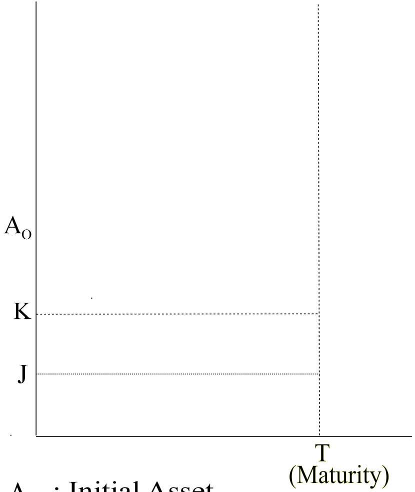
- Bond has safety covenants. Bond holders take over and liquidate the firm if asset value falls below some barrier $C(t)$.
- Default time τ is given by

$$\tau = \begin{cases} \inf\{t \in [0,T] : V_t < C(t)\} & \text{if the set is non-empty} \\ \infty & \text{otherwise.} \end{cases}$$

- Payoffs at maturity

$$\begin{aligned} D_T &= D - (D - V_T)^+ + (V_T - D)^+ \mathbf{1}_{\{\tau < T\}} \\ S_T &= (V_T - D)^+ \mathbf{1}_{\{\tau \geq T\}} \end{aligned}$$

Default Behavior



A_O : Initial Asset

K : Face Value ; J : Barrier Level

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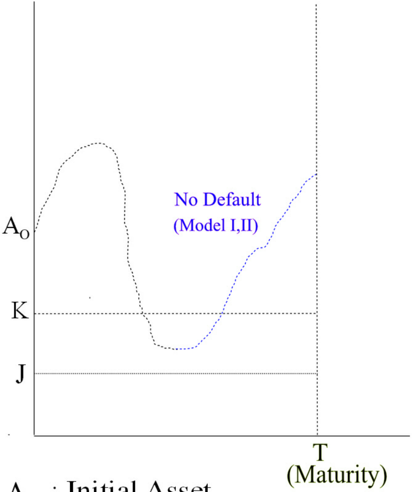

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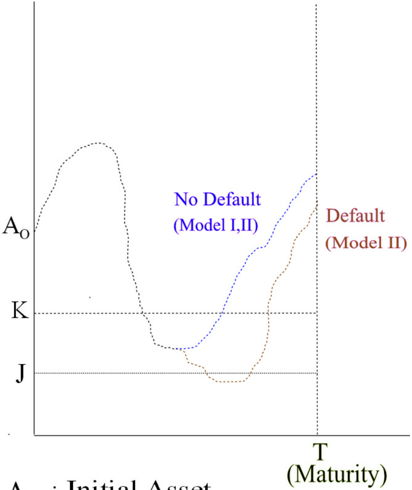

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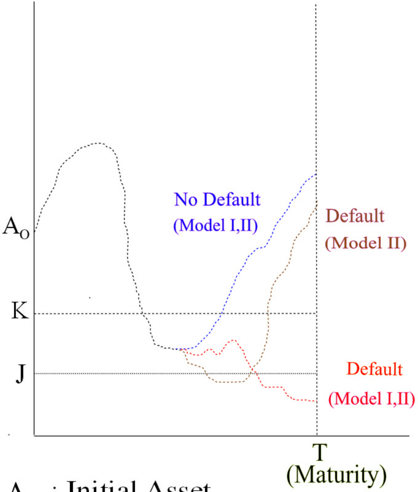

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Pricing Corporate Debt

- Merton's Model (Merton 1974): Payoffs to Debt and Equity:

$$\begin{aligned} D_T &= min(D, V_T) = D - (D - V_T)^+ \\ S_T &= (V_T - D)^+ \end{aligned}$$

- Equity is a call option on the firm's assets and thus is given by the Black-Scholes formula

$$C(V, D, T) = VN(d_1) - De^{-rT}N(d_2)$$

$$d_1 = \frac{\log(V/D) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}; \qquad d_2 = d_1 - \sigma\sqrt{T}$$

$$\begin{aligned} S_t &= C(V_t, D, T - t), \\ D_t &= De^{-r(T-t)} - P(V_t, D, T - t), \end{aligned}$$

Put-Call Parity: $C(V_t) - P(V_t) = V_t - De^{-r(T-t)}$.

Risk Neutral Probability of Default

- Under the real-world probability measure $\mathbb{P}$ the asset price follows

$$V_t = V_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

- Under the risk-neutral probability measure $\mathbb{Q}$ the asset price follows

$$V_t = V_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma \tilde{W}_t},$$

where $\tilde{W}$ is a $\mathbb{Q}$-Brownian motion.

$$\begin{aligned} PD^{rn} &= \mathbb{Q}\left(V_T < D\right) \\ &= \mathbb{Q}\left(\tilde{W}_T < \frac{\log(D/V_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma}\right) \\ &= N\left(\frac{\log(D/V_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

Advantages

- Treats the entire firm and its liabilities in one single consistent model.
- Provides a hedge based link between debt and equity.

Drawbacks

- Asset price is not traded, not observed and parameters are unknown.
- Specification of default barrier.

$$\text{Spread:} \qquad s(t,T) = -\frac{1}{T-t}\log\frac{B(t,T)}{D(t,T)}; \qquad B(t,T) = De^{-r(T-t)}$$

- Short Spreads: $s(t,T) \to 0$ as $t \to T$. Motivation for including jump risk.
- Company's balance sheet is typically more complicated; bonds may have covenants and may for example be callable etc.
- How to define the default point. Often it is due to drying up of refinance or liquidity issues.
- Measuring liabilities, including off-balancing sheet commitments accurately is difficult.

Extensions

- Discrete coupons
- Stationary Leverage Ratios
- Optimal capital structure
- Liquidity-driven default vs Barrier-driven default
- Game-theoretic approaches
- Models with strategic debt service

From Default Probabilities to Portfolios

- Modeling dependences between default events and between credit quality changes is one of the biggest challenges in credit risk models.
- All mechanisms for modeling dependences are mixtures of the following three themes:
  1. Factor Models: Default probabilities are influenced by a set of common background variables which are observable.
  2. Default probabilities depend on unobservable latent variables. In the event of a default, latent variables are updated which results in reassessment of default probabilities of other instruments in the portfolio.
  3. Direct contagion in which the actual default event causes a direct default of another firm or a deterioration of credit quality as a consequence of the default event.
- Most models use simulations to obtain the loss distribution (eg. Copulas, CPV, dynamic intensity models). Always desirable to have analytic solutions (CreditRisk$^+$, Top-Down Approach) since simulation is expensive, especially for large portfolios.

Copulas

- Technique for creating families of multivariate distributions with given marginals. Very popular for creating correlated defaults.
- Key idea: If Y is a continuous random variable with distribution $G(x) = P(Y \leq x)$, then $U = G(Y)$ has a uniform distribution.
- An n-dimensional copula function is a multivariate distribution function C defined on the unit cube $[0,1]^n$ with the additional requirement that all the marginals are uniform, that is,

$$C(1,\ldots,1,u_i,1,\ldots,1) = u_i, \qquad u_i \in [0,1].$$

- Suppose we are given n univariate distribution functions $F_1, F_2, \ldots, F_n$ and a copula function C. Then

$$F(x) = C(F_1(x_1), \ldots F_n(x_n)),$$

  is a n-dimensional distribution function with marginals F_i.
- Sklar's Theorem: Any joint distribution which has $F_1, F_2, \ldots, F_n$ as marginal distributions can be obtained by a suitable choice of copula function.

Simulations using Copulas

- Suppose $U_1, U_2, \ldots, U_n$ are uniform random variables with distribution function C.
- Then the random variables

$$X_1 = F_1^{-1}(U_1), X_2 = F_2^{-1}(U_2), \ldots, X_n = F_n^{-1}(U_n)$$

have distribution function F.

- Example: Gaussian Copula. Suppose the random variables $(Y_1, \ldots Y_n)$ have a multivariate normal distribution with mean vector $\mu = (\mu_1, \ldots, \mu_n)$ and variance-covariance matrix $\Sigma = (\sigma_{ij})_{1 \le i,j \le n}$.
- Then $U_1 = N\left(\frac{Y_i - \mu_i}{\sigma_i}\right)$ is uniformly distributed and the U_i's are dependent.
- Then $X_1 = F_1^{-1}(U_1), \ldots, X_n = F_n^{-1}(U_n)$ gives us a collection of random variables (Default Times) which are dependent and have the desired marginals F_i.

Disadvantages of Copulas

- A quick and dirty method of simulating joint default times.
- No clear connection between the asset correlations and the correlations obtained between the default times.
- Very little mathematical underpinning of this approach.

Credit Portfolio View (CPV)

- Each obligor in the portfolio belongs to one of S categories. Defaults are explained by a macroeconomic regression model. Macroeconomic model is calibrated by means of times series of empirical data.

$$Y_{s,t} = w_{s,0} + \sum_{j=1}^{M} w_{s,j} X_{j,t} + \epsilon_{s,t}, \qquad \epsilon_{s,t} \sim N(0, \sigma_s^2).$$

$$X_{j,t} = \theta_{j,0} + \sum_{k=1}^{t_0} \theta_{j,k} X_{j,t-k} + \eta_{j,t}, \qquad \eta_{j,t} \sim N(0, \tilde{\sigma}_j^2).$$

$$p_{s,t} = \frac{1}{1 + \exp(-Y_{s,t})}.$$

- Generate a realization of $p_{s,T}$ for a future time T based on a realization of $Y_{s,T}$. For each obligor in sector s generate a Bernoulli random variable (representing default) with parameter $p_{s,t}$.
- This gives a realization of the portfolio loss. Repeat large number of times to obtain loss distribution.

Reduced-Form Models

- Objective is to be able to consistently price complex credit derivatives.
- Model calibrated based on liquid instruments.
- Default occurs without warning according to a stochastic intensity process.
- Default dynamics are specified exogenously directly under a pricing measure, which we will denote by P.

- A Poisson process $N(t)$ with intensity $\lambda > 0$ is a counting process satisfying the following properties:
  1. $N(0) = 0$
  2. Process has Stationary and Independent Increments
  3. $P[N(h) = 1] = \lambda h + o(h)$.
  4. $P[N(h) = 2] = o(h)$.

Poisson Process Model for Default Timing

- The number of arrivals in any time interval of length T is Poisson distributed with mean λT:

$$P[N(T+S)-N(S)=k]=e^{-\lambda T}\frac{(\lambda T)^k}{k!}, \qquad k=0,1,2,\ldots.$$

- If default is modelled as the first arrival epoch of the process $N(t)$, then probability that no default happens by time t equals

$$p(t)=e^{-\lambda t}.$$

Non-homogenous Poisson Process

- If λ is not constant, but a function of t, then the above probability becomes

$$p(t) = e^{-\int_0^t \lambda(u)\, du}$$

- The Conditional Probability that the firm survives till time s given that it has survived till time $t < s$ is

$$p(s|t) = \frac{e^{-\int_0^s \lambda(u)\, du}}{e^{-\int_0^t \lambda(u)\, du}} = e^{-\int_t^s \lambda(u)\, du}$$

Doubly Stochastic Poisson Process

- Market conditions do not change in a deterministic manner.
- The intensity process λ is itself stochastic.
- Conditional on the path of the process λ, the process N is a non-homogenous Poisson process.
- In this case, the survival probability is given by

$$\begin{aligned} p(t) &= P[N(t) = 0] \\ &= E[P[N(t) = 0 | \lambda(u), 0 \leq u \leq t]] \\ &= E[e^{-\int_0^t \lambda(u)\, du}] \end{aligned}$$

- Question: What kind of intensity processes can we handle?

Affine Process

- A process X of state variables or risk factor process.
- Let λ be a non-negative function.
- Let N_t be the default counting process with intensity $\lambda(X_t)$.
- A function h is said to be affine if $h(x) = a + bx$.
- A Markov process X is said to be an affine process if for any affine function h we have

$$E_t\left[e^{-\int_t^T h(X_s)ds} g(X_T)\right] = e^{\alpha(t,T)+\beta(t,T)\cdot X_t}$$

Examples of Affine Processes

- Vasicek and CIR processes (Used extensively in modeling interest-rate derivatives)

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t$$

$$dX_t = \kappa(\mu - X_t)dt + \sigma\sqrt{X_t}\, dW_t$$

- Affine Jump Diffusion process

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t$$

- J_t is a pure jump process whose jump counting process M_t has a intensity γ and jump size distribution ν.
- For an affine jump diffusion, the functions $\alpha(t, T)$ and $\beta(t, T)$ are solutions of ordinary differential equations called the Ricatti equations.

Pricing a Defaultable Bond

- Suppose the short rate process is $r(X_t)$. Then the price of a default-free bond is

$$B(T) = E\left[e^{-\int_0^T r(X_s)ds}\Big|X_0\right]$$

- Let $\tau = \tau_1$ be the first arrival time of the default arrival process N_t with intensity process $\lambda(X_t)$.
- Then $1_{\{\tau>T\}}$ is the indicator function that no default occurs.
- The price of a zero-coupon zero-recovery risky bond is

$$\begin{aligned} D(T) &= E\left[e^{-\int_0^T r(X_s)ds}1_{\{\tau>T\}}\Big|X_0\right] \\ &= E\left[e^{-\int_0^T r(X_s)ds}E\left[1_{\{\tau>T\}}\Big|X_s, 0\le s\le T\right]\Big|X_0\right] \\ &= E\left[e^{-\int_0^T (r+\lambda)(X_s)ds}\Big|X_0\right] \end{aligned}$$

$$p(t) = E\left[e^{-\int_0^T \lambda(X_s)ds}\Big|X_0\right]$$

Pricing Formula

- Suppose $h(x) = (r + \lambda)(x) = a + bx$.
- X is the affine jump diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t$$

with jump transform

$$\theta(c) = \int e^{cZ}\nu(dz).$$

$$\mu(x) = \mu_0 + \mu_1 x; \qquad \sigma(x) = \sqrt{\sigma_0 + \sigma_1 x}.$$

$$D(T) = E\left[e^{-\int_0^T (r+\lambda)(X_s)ds} \Big| X_0\right] = e^{\alpha(0)+\beta(0)X_0}$$

$$\frac{d\beta(s)}{ds} = b - \mu_1\beta(s) - \frac{1}{2}\sigma_1\beta(s)^2; \qquad \beta(T) = 0.$$

$$\frac{d\alpha(s)}{ds} = a - \mu_0\beta(s) - \frac{1}{2}\sigma_0\beta_s^2 - \gamma(\theta(\beta(s)) - 1), \qquad \alpha(T) = 0.$$

Modeling Dependent Defaults in the Reduced Form Framework: Bottom-up Approach

- Suppose there are N firms in the portfolio, and S sectors in the economy.
- Let $s(i)$ be the sector to which firm i belongs.
- Let $X_i, X_{s(i)}, X_c$ be independent Basic Affine Processes representing the idiosyncratic risk, sectoral risk and common risk factors.
- Correlated Defaults:

$$\lambda_i(t) = (X_i + a_i X_{s(i)} + b_i X_c)(t)$$

$$P(\tau_i > T) = E[e^{-\int_0^T X_i(s)ds}]E[e^{-a_i \int_0^T X_s(i)(s)ds}]E[e^{-b_i \int_0^T X_c(s)ds}]$$

- Contagion Effect: If N_c is the arrival process with intensity X_c, then the arrival of a systematic shock can cause several firms to default simultaneously.

Computing the Loss Distribution

- Simulate a path of the basic affine processes $X_i, X_{s(i)}, X_c$.
- Compute λ_i for each firm i.
- Simulate $E_i \sim \exp(1)$ independent realizations for each i.
- If $E_i < \int_0^T \lambda_i(s)ds$, then firm i defaults.
- Based on this we can compute the total portfolio loss.
- Repeat large number of times to obtain the loss distribution.
- Works well for small N. For large N we take the top-down approach.

Collateralized Debt Obligations (CDOs)

- **CDO** is an asset backed security whose underlying collateral is typically a portfolio of corporate bonds or commercial loans
- It is a way of packaging credit risk
- Bank transfers the portfolio to an SPV and receives cash
- SPV issues fixed/floating rate notes to investors
- Notes are divided by order of seniority into tranches called equity tranche, mezzanine tranche(s) and senior tranche
- Investors receive the principal and interest from the portfolio in order of seniority

Types of CDOs

- **Cash Flow CDOs:** Collateral portfolio is not subject to active trading. All uncertainty regarding interest and principal payments is determined by timing of defaults
- **Market Value CDO:** Payments based on *mark-to-market* returns.

Economics of CDOs

CDOs become attractive due to market imperfections arising due to regulatory capital requirements, low valuations due to lack of liquidity, etc.

Two classes of CDOs

- Balance-sheet CDOs:
  - Collateralized Loan Obligations (CLO) – removes loans from the banks balance sheet providing capital relief and improve valuations.
  - Synthetic balance sheet CLO – No actual transfer of ownership due to client secrecy obligations or cost arising due to contractual restrictions. Use credit derivatives to transfer risk to the SPV.
- Arbitrage CDOs: Capture the difference in price due to lower cost of acquiring the collateral pool in the secondary market and the value received from management fees and sale of the CDO.

Example

Tranche	% Notional	Yield	SoL	Rating
Senior	15	6	100	Aa2
Mezzanine-I	40	7.5	250	Baa3
Mezzanine-II	40	15	550	Ba1
Equity	5	35	-	NR

Pricing: Bottom-up approach

- Model the correlated default process of individual securities comprising the portfolio:
  - Copulas – Static.
  - Dynamic intensity models – Simulation.
- Model calibrated based on market data.

Pricing: Top-Down approach

- Copula based approaches found highly inadequate.
- Need models to capture the dynamics in a more meaningful way that are also analytically tractable.
- Model the index on which options are to be priced.

Top-down Approach to Pricing CDOs

- Aim: To price a Tranche of the CDO.
- CDO/Portfolio has n firms.
- Let X be the risk factor process.
- Let λ be a non-negative function.
- Let N_t be the default counting process with intensity $\lambda(X_t)$.
- Loss Process:

$$L(t) = \sum_{n=1}^{N(t)} \ell^{(n)}$$

$\ell^{(n)}$ are independent of one another and drawn from a distribution η on $\mathbb{R}_+$, that has no mass at zero.

Top-down Approach to Pricing CDOs

- Risk Factor Process X_t is Self Affecting

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \delta dL_t.$$

- Default Probability Density:

$$q(t) = E\left[e^{-\int_0^t \lambda(X_s)ds}\lambda(X_t)\right].$$

Tranche Swap

- A Tranche is referenced on a portfolio and is specified by
  1. A lower attachment point $L^{(low)} \in [0, 1]$.
  2. An upper attachment point $L^{(up)} \in (L^{(low)}, 1]$.
- Default Leg: The protection seller covers tranche losses as they occur, i.e., the increments of the tranch loss process

$$U_t = (L_t - nL^{(low)})^+ - (L_t - nL^{(up)})^+.$$

- Tranche Notional: $K = L^{(up)} - L^{(low)}$. Upfront Rate: G.
- Premium Leg: The protection buyer pays GKn at inception and $SC_k(Kn - U_{t_k})$ at each date t_k.

Pricing Tranche Swap

- The value of the tranche swap default leg is given by

$$D = E\left[\int_0^T e^{-\int_0^s r(X_s)ds}\, dU_s\right].$$

- The value of the premium leg is given by

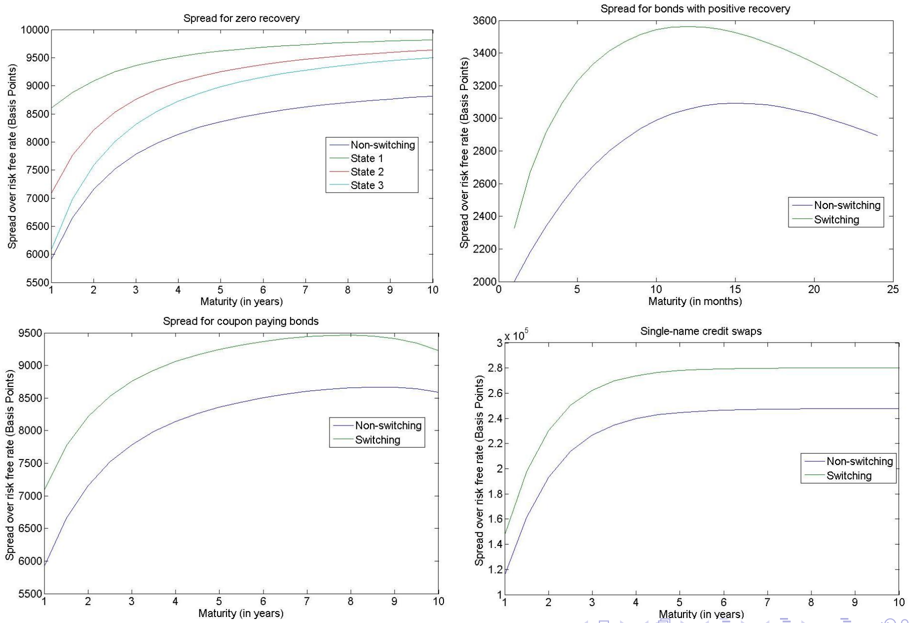
$$P = GKn + S\sum_{t_k \geq t} E\left[e^{-\int_0^{t_k} r(X_s)ds}\left(Kn - U_{t_k}\right)\right].$$

- For a fixed upfront payment rate G, the fair tranche spread S is the solution to the equation

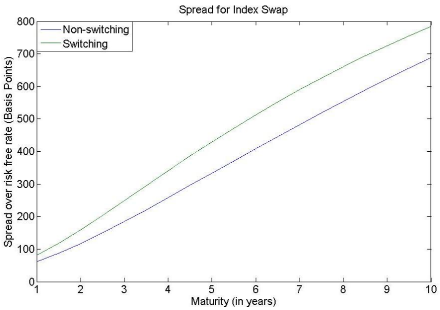
$$D = P.$$

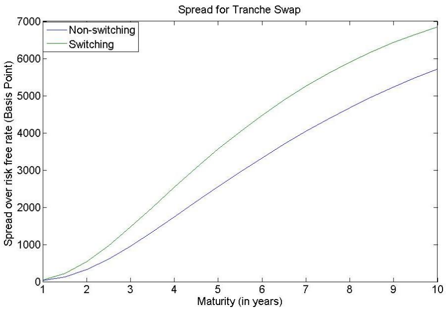
- For the affine self-affecting process D, P can be obtained as solutions of system of ODEs.

Spread Behavior for Single-Name Credit Derivatives



Spread Behavior for Multi-Name Credit Derivatives





We observed that Index and Tranche spreads are :

- decreasing in the parameter κ.
- increasing in the parameter δ.

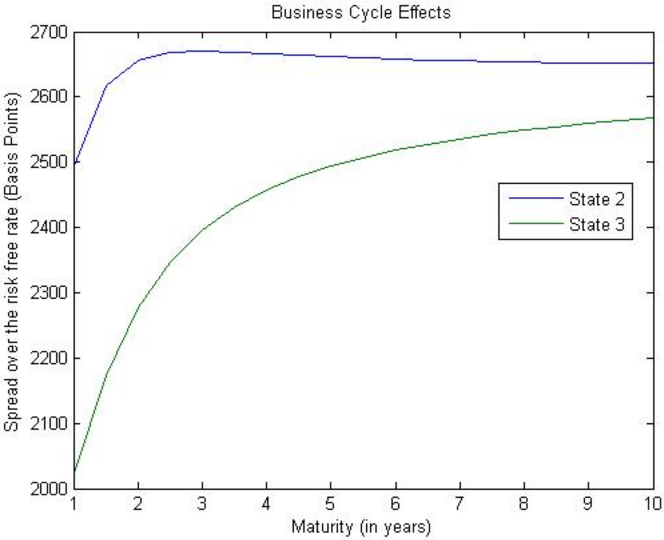
Example II (Business Cycle Effects)

- $$P_2 = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0.8 & 0.2 & 0 \end{bmatrix}.$$

- $\nu_1 = 2,\ \nu_2 = 6,\ \nu_3 = 6,\ \nu_4 = 2.$

- $$\left[r_0(k), \sigma_0(k), \Lambda_0(k), c(k)\right] = \begin{cases} (0.08,\ 0.6,\ 0.06,\ 0.8), & \text{if} \quad k = 1 \\ (0.07,\ 0.3,\ 0.03,\ 0.3), & \text{if} \quad k = 2 \\ (0.07,\ 0.4,\ 0.04,\ 0.6), & \text{if} \quad k = 3 \\ (0.06,\ 0.1,\ 0.01,\ 0.1), & \text{if} \quad k = 4. \end{cases}$$

Business Cycle Effects



Example III (Rare state effects)

- $$P_3 = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0.8 & 0 & 0.18 & 0 & 0.02 & 0 \\ 0 & 0.2 & 0 & 0.8 & 0 & 0 \\ 0 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$
- $\nu_k = 6,\ k = 1, 2, 3, 4,\ \nu_5 = \nu_6 = 2.$
- Rare states:- State 5 and 6.

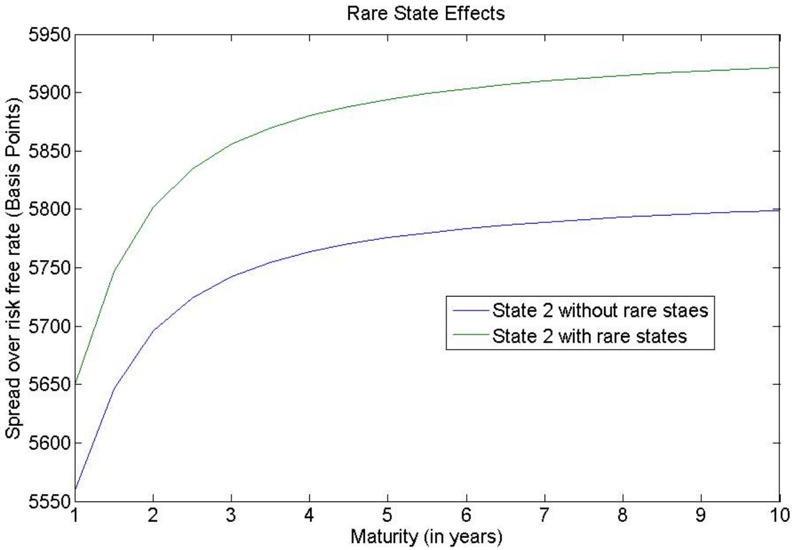
- Stationary distribution

$$\pi = \Big[0.219, 0.274, 0.268, 0.214, 0.021, 0.004\Big].$$

-

$$\Big[r_0(k), \sigma_0(k), \Lambda_0(k), c(k)\Big] = \begin{cases} (0.08,\ 0.6,\ 0.6,\ 0.8), & \text{if} \quad k = 1 \\ (0.06,\ 0.3,\ 0.3,\ 0.3), & \text{if} \quad k = 2 \\ (0.06,\ 0.4,\ 0.4,\ 0.6), & \text{if} \quad k = 3 \\ (0.04,\ 0.1,\ 0.1,\ 0.1), & \text{if} \quad k = 4 \\ (0.02,\ 0.7,\ 0.9,\ 0.7), & \text{if} \quad k = 5 \\ (0.01,\ 0.8,\ 1.0,\ 0.8), & \text{if} \quad k = 6. \end{cases}$$

Rare state effects



- With restructuring :

$$P_4 = \begin{bmatrix} 0 & 0.8 & 0.2 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.5 & 0.5 & 0 \end{bmatrix}.$$

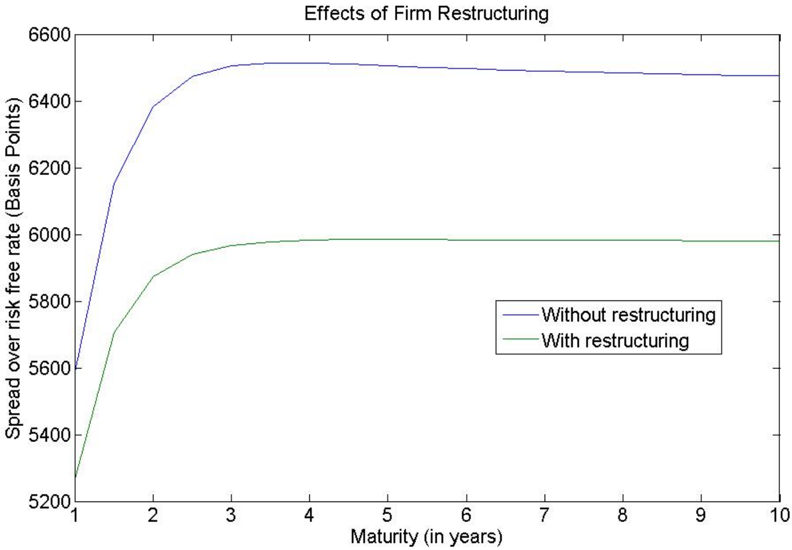
- $\nu_1 = 2,\ \nu_2 = 2,\ \nu_3 = 2,\ \nu_4 = 2.$
-

$$\left[r_0(k), \sigma_0(k), \Lambda_0(k), c(k)\right] = \begin{cases} (0.05,\ 0,\ 0,\ 0.1), & \text{if} \quad k = 1 \\ (0.05,\ 0,\ 0,\ 0.5), & \text{if} \quad k = 2 \\ (0.05,\ 0,\ 0,\ 2.0), & \text{if} \quad k = 3 \\ (0.05,\ 0,\ 0,\ 0.05), & \text{if} \quad k = 4. \end{cases}$$

- Without restructuring :

$$P_5 = \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 1 & 0 \end{bmatrix}.$$

Effects of Firm Restructuring



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