# Networks and Credit Contagion

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# Introduction

- Credit Contagion: Propagation of economic distress among firms.
- Systemic Risk or Measures of Contagion.
- Network Effect: Financial Institutions are interconnected as they hold debt claims against each other
- Liquidity Channel:
    1. Hold common pool of assets and thus affected by fluctuations in market price.
    2. Financial Institutions in distress selling assets in the market depress prices of assets thereby affecting solvency of other institutions.
- Goal: To understand how these two channels interact to propagate individual defaults to a system-wide catastrophe.
- Understanding how idiosyncratic risks evolve and propagate in a tightly coupled financial system is crucial to designing regulatory tools to measure, monitor, mitigate and manage systemic risks.

# Approaches to Modeling Credit Contagion

- One of the first attempts at modeling network dependence: Systemic risk in financial systems, Eisenberg and Noe, Management systems (2001). Does not take into account the liquidity channel. Many works improving and motivated by this paper. Will present the model proposed by Chen, Liu and Yao (2014) which incorporates this aspect in the Eisenberg-Noe framework.
- Random graph models: *Robust-yet-fragile* feature. Probability of contagion is low in diversified networks, but when contagion occurs effects can be quite widespread.
- Three aspects:
  1. Network structure
  2. Network based measures
  3. Leveraging the above to reduce systemic impact of defaults
- Research focussing on contagious effect of asset price due to fire sales leading to adverse welfare consequences such as high price volatility, more defaults and market inefficiency.

**Notations:**

- $uv$: inner product of two vectors.
- $\mathbf{1}, \mathbf{0}$: vector of all $1'$s and $0'$s.
- For two index sets $I, J$ and matrix $M$, $M_{I,J}$ denotes the sub-matrix consisting of the rows and columns indexed by $I$ and $J$ respectively. $M_{I,I} = M_I$.
- $u > v$: $u_i > v_i$ for all $i$. $u \wedge v = (u_1 \wedge v_1, u_2 \wedge v_2, ...)$.

# The Eisenberg-Noe Model

- Financial system consisting of $n$ banks.
- Interconnectedness of balance sheets represented via a liability matrix $L = (L_{ij})$.
- $L_{ij}$ = liability of bank $i$ to bank $j$. $L_{ij} \geq 0$ if $i \neq j$ and $L_{ii} = 0$.
- $b_i$ = external liabilities of bank $i$. $b_i \geq 0$.
- $\ell = (\ell_i)$ : Liability vector and $P = (p_{ij})$ : Matrix of relative liabilities

$$\ell_i = b_i + \sum_{j \neq i} L_{ij} \qquad \text{and} \qquad p_{ij} = \frac{L_{ij}}{\ell_i}.$$

- Assets: $\alpha = (\alpha_i)$ is the value of exogenous assets invested by bank $i$.

# Problem Statement

- All creditors of a bank are assumed to have the same seniority.
- Given a realization $\alpha$ find a repayment vector $x = (x_i)$ such that it complies with the limited liability principle.
- Bank $i$ must pay all its liabilities $\ell_i$, if it can, and if unable to pay, declare default and pay all that it receives from external and internal sources:
- Receipts of Bank i: $\alpha_i + \sum_{j \neq i} x_j p_{ji}$.
- Limited Liability: $x_i = \ell_i \wedge (\alpha_i + \sum_{j \neq i} x_j p_{ji})$.

# Alternate Formulation as an Optimization Problem

- In matrix form, the problem is to find $x$ so that

$$x = \ell \wedge (\alpha + xP) : \qquad \text{fixed point formulation.} \tag{1}$$

- If $b_i > 0$ then the mapping is a contraction and hence has a unique solution.
- Linear programming formulation: Let $|x| = x\mathbf{1} = \sum_{i=1}^{n} x_i$.

$$\max_x |x|, \qquad \text{s.t.} \qquad x(I-P) \leq \alpha, \qquad 0 \leq x \leq \ell. \tag{2}$$

- For any optimal solution, either $x(I-P) \leq \alpha$ or $x \leq \ell$ is binding.
- Any solution to (2) is also a solution to (1).
- Solution to (2) is one in which a minimum number of banks will default. Banks that default in the optimal solution will default in any other equilibrium solution - consequence of the partition algorithm.

# Reformulating the Problem

- $\mathcal{D} = \{i : x_i < \ell_i\}$ : Set of defaulted banks.
- $\mathcal{N} = \{i : x_i = \ell_i\}$ : Set of non-defaulted banks.

$$P = \begin{pmatrix} P_{\mathcal{D}} & P_{\mathcal{D},\mathcal{N}} \\ P_{\mathcal{N},\mathcal{D}} & P_{\mathcal{N}} \end{pmatrix}.$$

- Can reformulate the problem as: Find $\mathcal{D}^*, \mathcal{N}^*$ optimal partition that solves

$$\max_{x_{\mathcal{D}},\mathcal{D},\mathcal{N}} |x| \qquad \text{s.t.}$$

$$x_{\mathcal{D}} = \alpha_{\mathcal{D}} + x_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}} + x_{\mathcal{D}} P_{\mathcal{D}}, \tag{3}$$

$$x_{\mathcal{N}} \leq \alpha_{\mathcal{N}} + x_{\mathcal{N}} P_{\mathcal{N}} + x_{\mathcal{D}} P_{\mathcal{D},\mathcal{N}}, \tag{4}$$

$$x_{\mathcal{N}} = \ell_{\mathcal{N}}, \qquad x_{\mathcal{D}} < \ell_{\mathcal{D}}. \tag{5}$$

# The Partition Algorithm

- Start with setting $\mathcal{D} = \phi$ and $\mathcal{N} = \{1, 2, \ldots, n\}$. Equivalent to setting $x = \ell$.
- This satisfies (3), (5). If (4) is satisfied, then we have a feasible solution. Since $x = \ell$ is larger than any other solution, it is optimal.
- If (4) is violated for a subset of banks, then include them in $\mathcal{D}$ and update $\mathcal{N} = \{1, 2, \ldots, n\} \setminus \mathcal{D}$.

$$x_{\mathcal{D}} = (\alpha_{\mathcal{D}} + x_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}})(I_{\mathcal{D}} - P_{\mathcal{D}})^{-1}, \qquad x_{\mathcal{N}} = \ell_{\mathcal{N}}. \tag{6}$$

- Since (4) is violated for the banks in $\mathcal{D}$ we have

$$\ell_{\mathcal{D}}(I_{\mathcal{D}} - P_{\mathcal{D}}) > \alpha_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}}$$

  and hence it follows from (6) that $\ell_{\mathcal{D}} > x_{\mathcal{D}}$.
- If (4) is satisfied, then we have a feasible solution. Else repeat the above procedure.

# Optimality of the Solution

- Algorithm keeps reducing $|x|$ of the infeasible solutions by identifying more and more defaulting banks. When it terminates we have a feasible solution. Algorithm takes at most $n$ iterations.
- Suppose $y$ is any other solution. Let $x = (x_{\mathcal{D}}, x_{\mathcal{N}})$ be an intermediate solution produced by the algorithm with partition $(\mathcal{D}, \mathcal{N})$.

$$y_{\mathcal{N}} \leq \ell_{\mathcal{N}} = x_{\mathcal{N}}.$$

$$y_{\mathcal{D}} \leq \alpha_{\mathcal{D}} + y_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}} + y_{\mathcal{D}} P_{\mathcal{D}} \leq \alpha_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}} + y_{\mathcal{D}} P_{\mathcal{D}}.$$

$$y_{\mathcal{D}} \leq (\alpha_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}})(I_{\mathcal{D}} - P_{\mathcal{D}})^{-1} = x_{\mathcal{D}}.$$

- This proves that the final solution obtained by the partition algorithm is indeed optimal. Further any bank defaulting in the optimal solution will default in any other equilibrium.

# The Network Multiplier

- The matrix $P$ captures all the information about the network.
- By shadow-price interpretation of dual variables, $e_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}$ and $e_i P_{\mathcal{N}^*,\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}$ represent the sensitivities of the repayment vector $x^*$ with respect to $\alpha_i$ and $\ell_j$ for $i \in \mathcal{D}^*$ and $j \in \mathcal{N}^*$.
- Network Multiplier:

$$(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} = I_{\mathcal{D}^*} + P_{\mathcal{D}^*} + P_{\mathcal{D}^*}^2 + \dots . \tag{7}$$

- Captures the network effect on the payment vector $x_{\mathcal{D}^*}^*$.
- Suppose the exogenous asset of bank $i \in \mathcal{D}^*$ is less by 1. Then
  1. The payment to bank $i$ reduces by 1.
  2. Payment to immediate creditors of $i$ receive $e_i P_{\mathcal{D}^*}$ units less.
  3. Creditors to the immediate creditors of $i$ receive $e_i P_{\mathcal{D}^*}^2$ units less and so on.
  4. Aggregate effect of this is given by left hand side of (7).

# An aside on computation

$$x_{\mathcal{D}} = (\alpha_{\mathcal{D}} + x_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}})(I_{\mathcal{D}} - P_{\mathcal{D}})^{-1}, \qquad x_{\mathcal{N}} = \ell_{\mathcal{N}}.$$

- Typical attributes of banking systems data are $\mathcal{D}$ large and $P_{\mathcal{D}}$ sparse $(I_{\mathcal{D}} - P_{\mathcal{D}})^{-1}$ can be computed faster using the following iterative technique.
- Let $F(y) = \alpha_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}} + y P_{\mathcal{D}}$. Solution to above equation is the fixed point for $F(y)$.
- Let $y^{(0)} = \ell_{\mathcal{D}}$. For $n \geq 1$ let $y^{(n)} = F(y^{(n-1)})$. The sequence $\{y^{(n)}\}$ is decreasing and hence converges.

# On Uniqueness

- If $P$ is strictly sub-stochastic, then $I_{\mathcal{D}} - P_{\mathcal{D}}$ and all principal sub-matrices are invertible. Consequently there is a unique solution. May not hold if some banks do not have external liabilities or in presence of senior debts.
- Example: Suppose we have three banks $\{1, 2, 3\}$. Suppose that none of them have exogenous assets or liabilities. Assume that their nominal liabilities are all $1 (= \ell_i)$ and relative liability matrix is given by

$$(\alpha_1, \alpha_2, \alpha_3) = (b_1, b_2, b_3) = (0, 0, 0). \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- $x = (1, 1, 1)$ is a solution and so is any $\theta x$ for any $\theta \in (0, 1)$. In the first solution no bank will default while in the others all the banks default.
- Example of network interdependence causing a self-fulfilling phenomenon. Expectation of a bad equilibrium may cause every bank to pay partially and consequently everyone may default in the equilibrium.
- The partition algorithm gives the best possible solution, that is, one in which the least number of banks will default.

# Senior Debts

- Outside debts held by depositors or other senior-debt holders have priority over inter-bank claims when a bank files for bankruptcy.
- Assume all inter-bank claims have same seniority.
- Modified clearing payment equation:

$$x_i = \ell_i \wedge (\alpha_i + \sum_{j \neq i} (x_j - b_j)^+ p_{ji}),$$

where $x^+ = \max(0, x)$ and $p_{ji} = \frac{L_{ji}}{\sum L_{ji}}$.

- Change variable to $\tilde{x}_i = x_i - b_i$, $\tilde{\ell}_i = \ell_i - b_i$ and $\tilde{\alpha}_i = \alpha_i - b_i$ to get

$$\tilde{x}_i = \left[ (\ell_i - b_i) \wedge (\alpha_i - b_i + \sum_{j \neq i} (x_j - b_j)^+ p_{ji}) \right]^+$$

$$= \tilde{\ell} \wedge (\tilde{\alpha}_i + \sum_{j \neq i} \tilde{x}_j p_{ji})^+.$$

# The Full Model: Incorporating Market Liquidity

| Assets | Liabilities and owner's equity |
|---|---|
| External Investments: $\beta_i$ | External debt claims: $b_i$ |
| Inter-bank Loans $L_{ki}$ for $k \neq i$ | Interbank liabilities: $L_{ij}$ for $j \neq i$ |
| Liquid Securities: $\bar{y}_i$ | Equity: $e_i$ |
| Illiquid Securities: $\bar{s}_i$ | |

- Bank first tries to meet its liabilities from selling external investments, liquid securities and payments received from other banks.
- Further shortfall is met by liquidating part or whole of illiquid assets. Suppose bank $i$ decides to sell face value $s_i \in [0, \bar{s}_i]$ of its illiquid holdings, then it receives $s_i q$ where

$$q = Q\left(\sum_{j=1}^{n} s_j\right),$$

where $Q$ is the inverse demand function and $\sum_j s_j$ is the aggregate liquidation amount from the banking system.

# The inverse demand function

- Assumption: $Q(0) = 1$, $Q(s) \geq 0$, and $Q$ is decreasing.

$$\gamma := -\frac{Q^{'}(s)}{Q(s)}.$$

- $\gamma$ is the relative price change at the supply level $s$ in response to the increment in the liquidation amount. Large $\gamma$ implies a high degree of illiquidity.

$$\alpha_i = \beta_i + \bar{y}_i.$$

- Assumption: No short sale is allowed. Bank allowed to sell illiquid assets until all holdings are exhausted after it has sold all its liquid holdings.

# Market Equilibrium Condition

$$x_i = \ell_i \wedge (\alpha_i + \sum_{j \neq i} x_j p_{ji} + s_i q). \tag{8}$$

$$y_i = \alpha_i \wedge (\ell_i - \sum_{j \neq i} x_j p_{ji})^+. \tag{9}$$

$$s_i = \bar{s}_i \wedge \left\{ \frac{[\ell_i - (y_i + \sum_{j \neq i} x_j p_{ji})]^+}{q} \right\} \tag{10}$$

$$q = Q(|s|). \tag{11}$$

**Remark:** Capital Adequacy requirements are not incorporated. These could put further pressure on the price resulting in additional spillover effects.

# Multiple Equilibria due to Illiquidity

**Example:** Consider two banks $1, 2$ with $\bar{s} = (1, 2)$. External debt $b_1 = b_2 = 1$. $Q(s) = e^{-s}$. $\alpha = (0.1, 0.9)$.

- No network effect. Interbank liabilities are zero.
- Both banks will have to liquidate part or all of their illiquid holdings to pay off debts.

$$s_1 = 1 \wedge \frac{0.9}{q}, \qquad s_2 = 2 \wedge \frac{0.1}{q}, \qquad q = e^{-(s_1+s_2)},$$

where 0.9 and 0.1 are the respective shortfalls for the two banks.

# No Interior Solution

- Total shortfall is $0.9 + 0.1 = 1$.
- Suppose both $s_1 < 1$ and $s_2 < 2$. Then

$$s_1 q = 0.9, \qquad s_2 q = 0.1,$$

$$\Rightarrow (s_1 + s_2)q = (s_1 + s_2)e^{-(s_1+s_2)} = 1,$$

leading to a contradiction because

$$\max_{0 \le s_1 \le 1, 0 \le s_2 \le 2} (s_1 + s_2)e^{-(s_1+s_2)} < 1.$$

- Therefore either $s_1 = 1$ or $s_2 = 1$ must hold in any equilibrium. Total shortfall exceeds the maximum liquidity that market can provide.

# Two Solutions

- $s = (1, 0.4092)$ is a solution with $q = 0.2443$ and $x_1 = 0.3443 < 1$ and $x_2 = 1$. So bank 1 defaults whereas bank 2 does not.
- $s = (1, 2)$ is a solution with $q = 0.0498$ and $x_1 = 0.1498 < 1$ and $x_2 = 0.9996 < 1$. So both banks default.
- Order of liquidation is important.
- If we liquidate bank 1 first, i.e., set $s_1 = 1$. Then we need to find $s_2$ such that $s_2 \exp(-(1 + s_2) = 0.1$ and this yields $s_2 = 0.4092$.
- If we set $s_2 = 2$, then there is no $s_1 \in [0, 1]$ for which $s_1 \exp(-(s_1 + 2)) = 0.9$. So system will end in second equilibrium.

# Market Clearing Vector

- The partition algorithm along with the iterative procedure for finding a fixed point can be adapted to find an optimal market clearing vector.
- The algorithm represents an approximation to describe the process of fire sales.
- In each iteration, the set of defaulting banks is augmented. The asset price falls as a result of more banks selling their illiquid assets.
- For those banks that enter the default set at an earlier stage, the feasibility condition is checked at a higher price. But algorithm does not to take this fact into account of locking in a higher price. So in this sense perhaps yields a sub-optimal solution. But a solution that takes that into account will satisfy a more complex set of equations.
- A bank that fails in the solution of this algorithm will also fail in any other equilibrium.

# The Liquidity Amplifier

- Suppose $(\mathcal{D}^*, \mathcal{N}^*)$ is the optimal partition obtained from the algorithm and $(x^*, y^*, s^*, q^*)$ be the optimal solution.
- Let $\mathcal{L}^* := \{i \in \mathcal{N}^* : s_i^* > 0\}$. This is the set of non-defaulted banks that had to dip into their illiquid assets to meet their liabilities.

$$\frac{\partial q^*}{\partial \alpha_i} = \frac{\gamma e_i (I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*} \mathbf{1}}{1 - \gamma(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*} \mathbf{1})}, \qquad i \in \mathcal{D}^*,$$

$$LA := \frac{\partial q^*}{\partial \alpha_i} = \frac{\gamma}{1 - \gamma(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*} \mathbf{1})}, \qquad i \in \mathcal{L}^*,$$

where $\gamma = -\frac{Q'(s)}{Q(s)}$.

# The First Order Liquidity Effect

- Suppose for some $i \in \mathcal{L}^*$, $\alpha_i$ is reduced by 1. Thus bank $i$ will have to sell an additional $1/q$ amount at unit price $q$ to make up for this shortfall.
- This will reduce the price by

$$Q(\cdot) - Q(\cdot + \frac{1}{q}) \approx -\frac{Q'(\cdot)}{Q(\cdot)} = \gamma.$$

# Second Order Liquidity Effect

- Price Effect on banks in $\mathcal{L}^*$: Since banks in $\mathcal{L}^*$ sell $s^*_{\mathcal{L}^*}$ the impact of the above sale reduces the amount they receive by $\gamma s^*_{\mathcal{L}^*}$.
- Price effect on banks in $\mathcal{D}^*$: Banks in $\mathcal{D}^*$ sell off all their assets to meet their liabilities. When price reduces by $\gamma$, the income of these banks shrink by $\gamma \bar{s}^*_{\mathcal{D}^*}$. This effect cascades via the network and its final effect is captured by the network multiplier and results in a reduction of $\gamma \bar{s}^*_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})$. So the second order effect from the two channels to banks in group $\mathcal{L}^*$ aggregates to

$$(\gamma s^*_{\mathcal{L}^*} + \gamma \bar{s}^*_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*})\mathbf{1},$$

causing a further price reduction of $\gamma$ times the above quantity.

# Higher Order Liquidity Effects

- Continuing this way we get a geometric series

$$\gamma + \gamma[\gamma(|s^*_{\mathcal{L}^*}| + \bar{s}^*_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})] +$$

$$\gamma[\gamma^2(|s^*_{\mathcal{L}^*}| + \bar{s}^*_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})^2] + \dots$$

which is precisely the expression given by

$$LA := \frac{\partial q^*}{\partial \alpha_i} = \frac{\gamma}{1 - \gamma(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})}, \qquad i \in \mathcal{L}^*.$$

# Intervention Policies

- Two Policies will be considered
  1. Direct purchase of illiquid asset by an external player, eg. the government
  2. Capital injection
- Federal Reserve purchased nearly \$1.25 trillion of mortgage-backed securities from August 2007 to August 2009.
- US Treasury injected \$205 billion in the form of preferred stock to the financial industry through a capital purchase program.
- Suppose the government injects $\Delta$ units of cash to one of the banks to mitigate its systematic impact.
- In direct asset purchase, cash is paid in exchange for some amount of illiquid securities. So assets of the bank increases to $\alpha_i + \Delta$ and its illiquid assets decreases by $\frac{\Delta}{q^*}$.
- In case of capital injection, liquid holdings increase to $\alpha_i + \Delta$.

# Policy Effectiveness

$$PE^{I}_{\text{policy}} := \lim_{\Delta \to 0} \frac{q^*_{\text{policy}}(\Delta) - q^*}{\Delta} \qquad PE^{II}_{\text{policy}} := \lim_{\Delta \to 0} \frac{x^*_{\text{policy}}(\Delta) - x^*}{\Delta}$$

$$PE^{I}_{\text{DAP}} = LA, \qquad PE^{II}_{\text{DAP}} = LA \cdot \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}.$$

$$PE^{I}_{\text{Capital}} = LA \cdot e_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1},$$

$$PE^{II}_{\text{Capital}} = e_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} + PE^{I}_{\text{Capital}} \cdot \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}.$$

# Policy Effectiveness

- It can be shown that $e_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*} \mathbf{1} < 1$ and hence $PE^I_{\text{DAP}} > PE^I_{\text{Capital}}$. The effect of the direct asset purchase on the market price exceeds that due to capital injection.
- However $PE^{II}_{\text{Capital}}$ can exceed $PE^{II}_{\text{DAP}}$ if the network multiplier $e_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}$ is sufficiently large.
- The asset purchase program focuses mainly on the liquidity channel whereas the capital injection program mainly uses the network channel.
- The direct asset purchase program does not change the asset value of the bank and hence does not alter the probability that it will default. The capital injection program increases the asset value of the bank by $\Delta$ and in a highly leveraged banking system will impact the total payments in the equilibrium.
- The liquidity ratio under the direct asset purchase program is larger than that under capital injection.

$$\frac{\alpha_i + \Delta}{TAV + \Delta} < \frac{\alpha_i + \Delta}{TAV}.$$

# Numerical Results and Some Questions

- Three types of networks are considered, viz., complete, ring-type and core-periphery.
- Market or liquidity channel presents a far greater threat to contagion as compared to the network effect, especially since the
- Need to extend the model to capture incomplete information. Though interbank liabilities are relatively small compared to external liabilities, the interbank lending mechanism is not static. It can freeze in the face of uncertainties arising from incomplete information and heightened counter-party risk, which in turn can affect the liquidity in the market.
- Need to build a dynamic model to endogenize the decision process of network formation and illiquid asset holdings. Enable monitoring the accumulation of systemic risk within the system.

# Statistical Models

- Seek to understand behavior of financial networks having certain statistical properties.
- Results are statements about networks belonging to a particular class and not any particular network as network links may keep changing but overall statistical properties are retained.
- Interested in phase transition behavior, identifying critical nodes etc.
- One of the early theoretical works in this direction was the work of Gai and Kapadia titled “Contagion in financial networks”, Proc. of the Royal Soc., 2010 and a subsequent paper in 2011. The aggregate exposure is distributed equally among its counterparties.
- Recent work by Amini, Cont and Minca titled “Resilience to contagion in financial networks”, Mathematical Finance, 2013.
- An earlier work by Cont, Moussa, and Santos titled “Network Structure and Systemic Risk in Banking Systems”, 2010, derived various network based measures to study the problem of contagion and applied it to the Brazilian financial network.

# Contagion in Financial Networks

- Will consider a simple random graph model to demonstrate the presence of a phase transition behavior in financial networks.
- Random networks have been used extensively to model transmission of shocks, epidemics, rumors, information flow etc.
- Random network theory provide a parsimonious way of modeling a complex network.
- Simulations show that the insights gained from the theoretical analysis hold under liquidity shock due to fire sales etc.
- Gai and Kapadia (2010) show a surprising double phase transition that has not been reported in any other application of random network theory.

# Model Description

- Financial network is modeled as a weighted directed graph, the weights and direction representing interbank liabilities.
- Each node of the network represents a financial institution.
- Each node has an *in-degree* which is the number of links pointing into the node and an *out-degree*. Incoming links represent interbank assets/exposures. Outgoing links correspond to inter-bank liabilities.
- $\alpha_i$: Interbank assets of bank $i$. $\bar{s}_i$: Illiquid external assets.
- $\ell_i$: Interbank liabilities. $D_i$: Customer deposits.
- **Assumption:** Total inter-bank asset position is distributed uniformly among all the incoming links.

# Default Criterion

- Let $\phi$ be the fraction of banks with obligations to bank $i$ that have defaulted.
- Let $q$ be the resale price of illiquid assets.
- Condition for bank $i$ to be solvent is

$$(1-\phi)\alpha_i + q\bar{s}_i - \ell_i - D_i > 0,$$

- This can be rewritten as

$$\phi < \frac{K_i - (1-q)\bar{s}_i}{\alpha_i},$$

where $K_i = \alpha_i + \bar{s}_i - \ell_i - D_i$ is the bank's capital buffer (difference between book value of assets and liabilities.

# Contagion Dynamics

- Suppose initially all banks are solvent. Then suppose a bank fails due to a purely idiosyncratic shock or as a result of an aggregate shock that has particularly adverse effect for this bank.
- A bank is said to be vulnerable if it will default even if one of its neighbors default.
- Our networks will locally tree-like, that is, there are no short cycles. Hence contagion spreads via vulnerable banks.
- Let $j_i$ be the in-degree of bank $i$. The condition for a bank to be vulnerable is

$$\frac{K_i - (1-q)\bar{s}_i}{\alpha_i} < \frac{1}{j_i}.$$

- So if we treat the capital buffer as a random variable, the probability that a bank with in-degree $j$ is vulnerable is

$$v_j = P\left(\frac{K_i - (1-q)\bar{s}_i}{\alpha_i} < \frac{1}{j}\right), \qquad \forall\ j \geq 1.$$

# Generating function for a Vulnerable Node

- Let $p_{jk}$ be the probability that a node in the network has in-degree $j$ and out-degree $k$.
- Since every in-coming link is an out-going link for some other node, the average degree $z$ is given by

$$z = \sum_{j,k} j\, p_{jk} = \sum_{j,k} k\, p_{jk}.$$

- $G_0(y)$ be the generating function for the number of links leaving a randomly chosen vulnerable node.

$$\begin{aligned} G_0(y) &= \sum_{k} P(\text{bank is vulnerable and has k outgoing links}) y^k \\ &= \sum_{jk} P(\text{bank vulnerable, has j incoming, k outgoing links}) y^k \\ &= \sum_{j,k} v_j\, p_{jk}\, y^k. \end{aligned}$$

- $G_0(1)$ is the fraction of banks that are vulnerable.

# Generating function for Second and Subsequent Generations

- Interest is in the propagation of shocks from one bank to another.
- We need the degree distribution of the number of links leaving a vulnerable bank reached by following a randomly chosen link going out of a vulnerable bank.
- Size Biasing: A bank with $j$ incoming links is $j$ times as likely to be chosen as a bank with 1 incoming link. So the generating function for the second and subsequent generations is given by

$$G_1(y) = \sum_{jk} v_j \, r_{jk} \, y^k = \frac{\sum_{j,k} v_j \, j \, p_{jk} \, y^k}{\sum_{j,k} j \, p_{jk}}$$

- $G_1(1)$ is the probability that the bank reached by following the random link is vulnerable.

# Vulnerable Cluster

- Start with a vulnerable bank. Follow a randomly chosen outgoing link to its end and from there to every other vulnerable bank that can be reached from that end.
- This set of banks will be called the vulnerable cluster at the end of the randomly chosen link of the vulnerable bank.
- We need the generating function for the size of the vulnerable cluster. This will help us characterize how default spreads.

# Generating Function of a Second Generation Vulnerable Cluster

- When we follow a random link out of a vulnerable bank the bank at the end of this link may
  1. be safe.
  2. have one, two, ... outgoing links to further clusters.
- Assume that the network is infinite and is tree-like and does not contain any closed loops or cycles.
- Let $Z$ be the size of the vulnerable cluster. Then the generating function of $Z$ is

$$H_1(y) = E[y^Z] = P(\text{bank is safe}) + \sum_{jk} v_j \, r_{jk} E[y^{1+Z_1+\cdots+Z_k}],$$

where $Z_1, Z_2, \ldots$ are i.i.d. copies of $Z$. Hence

$$\begin{aligned} H_1(y) &= 1 - G_1(1) + y \sum_{jk} v_j \, r_{jk} (H_1(y))^k. \\ &= 1 - G_1(1) + y G_1(H_1(y)). \end{aligned}$$

# Generating Function of a Vulnerable Cluster

- The Generating function $H_0$ of the size of a vulnerable cluster to which a randomly chosen bank belongs to, is given by

$$\begin{aligned} H_0(y) &= P(\text{bank is safe}) + y \sum_{jk} v_j \, p_{jk} (H_1(y))^k \\ &= 1 - G_0(1) + y G_0(H_1(y)) \end{aligned}$$

- In principle, we can compute the complete distribution of the size of a vulnerable cluster from the above equation for $H_0$ and $H_1$.

$$P(\text{size of cluster is m}) = \frac{H_0^{(m)}(0)}{m!}.$$

# Phase Transitions

- Average Size of the Vulnerable Cluster $\mathcal{S}$ is given by

$$\begin{aligned}\mathcal{S} &= H_0'(1) \\ &= G_0(H_1(1)) + G_0'(H_1(1))H_1'(1).\end{aligned}$$

$$H_1'(1) = G_1(H_1(1)) + G_1'(H_1(1))H_1'(1).$$

Since $H_1(1) = 1$ we get

$$H_1'(1) = \frac{G_1(1)}{1 - G_1'(1)}.$$

$$\mathcal{S} = G_0(1) + \frac{G_0'(1)G_1(1)}{1 - G_1'(1)}.$$

- Phase Transition at $G_1'(1) = 1$.

$$\sum_{jk} j\, k\, v_j\, p_{jk} = z.$$

# Implications of the Phase Transitions

- $G_1'(1)$ is the average out-degree of a vulnerable first neighbor.
- If $G_1'(1) < 1$ then all vulnerable clusters are small and contagion dies out quickly.
- If $G_1'(1) > 1$ then a giant vulnerable cluster whose size scales linearly with the size of the whole network exists.

$$\sum_{jk} j\, k\, v_j\, p_{jk} = z.$$

- As the average degree $z$ increases, more of the mass $p_{jk}$ shifts towards higher values of $j, k$. This increases the $jk$ term on the left, but $v_j$ is lower for higher $j$.
- So above equation will either have two solutions or none at all!

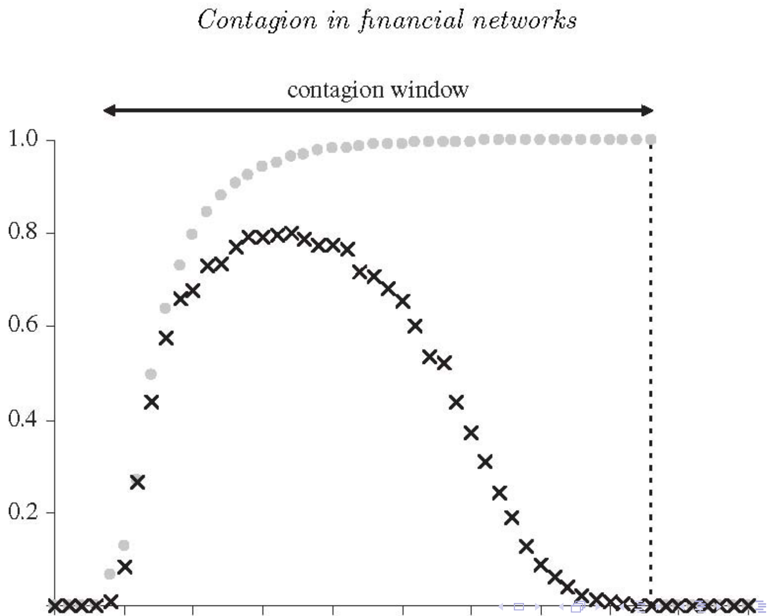
# Double Phase Transition

- In case there are two solutions there is a window of values of $z$ in which contagion occurs.
- For values of $z$ below the lower phase transition, the $jkp_{jk}$ terms are small. That is the network is insufficiently connected for contagion to spread.
- For values of $z$ above the upper end point of the window, $v_j$ is too small and so contagion cannot spread as there are too many safe banks.

# Numerical Simulation

- Edge between any two pair of nodes present in a particular direction present with probability $p$. Degree distribution then is approximately Poisson.
- Network of 1000 nodes with identical capital buffer and asset positons.
- Draw 1000 realizations of the network for each value of $z$. Assets of one bank wiped out for each draw. Failed bank and all subsequent failures default on all liabilities. Process continues until no new banks are pushed into default.
- Contagion: Over 5% of banks default.
- Extent of Contagion: The fraction of banks that default *conditional* on Contagion.

## Source: Gai and Kapadia (2010).



# Network Structure and Systemic Risk in Banking Systems

- R. Cont, A, Moussa, E. Santos, 2011.
- Quantitative methodology for analyzing contagion and systemic risk.
- Apply it to the Brazilian Financial system.
- Three aspects:
  1. Network Structure
  2. Network-based measures
  3. Leveraging the above to reduce systemic impact of defaults.

# Network Structure and Representation

- Complex heterogenous network structure: the distribution of in-degrees, out-degrees and mutual exposures are heavy tailed, exhibiting Pareto tails with exponents between 2 and 3.
- Qualitatively different from a small world network: many nodes with arbitrarily small clustering coefficient.
- The clustering coefficient of a node is the ratio of the number of links in the graph between the neighbors of that node to the total possible number of links.
- For a complete graph, the clustering coefficient is 1, whereas for the star graph, the clustering coefficient of the internal node is 0.

# Network Structure Representation

- Network: $I = (V, E, c)$, where
  1. $V$ is the set of financial institutions, $|V| = n$.
  2. $E$ is a matrix of bilateral exposures; $E_{ij}$ is the exposure of node $i$ to node $j$, i.e. the value of all liabilities of firm $j$ to firm $i$ or maximal short-term loss of $i$ if $j$ defaults.
  3. $c = (c_i, i \in V)$ where $c_i$ is the capital of firm $i$, or capacity to absorb losses.
- Interbank Assets: $A_i = \sum_j E_{ij}$.
- Interbank Liabilities: $L_i = \sum_j E_{ji}$.

# Network Structure Statistics

| Statistics | In(Out)-Degree | Exposure (Relative) | Distance |
|---|---|---|---|
| Mean | 8 (8) | 0.08 (0.05) | 2.35 |
| Std. Dev. | 11 (9) | 0.54 (0.21) | 0.78 |
| 5% quantile | 0 (0) | 0 (0) | 1 |
| 95% quantile | 31 (27) | 0.35 (0.18) | 4 |
| Max. | 62 (44) | 16 (6) | 6 |

- Nodes with widely differing connectivity
- Most financial institutions hold more capital than exposures. Some have exposures more than 100 times their capital.
- More connected institutions have larger exposures.
- Clustering coefficient and Small Worlds: Financial institutions with few connections (small degree) have counterparties that have high clustering coefficient.

# Market Shock Model

- Initial Capital: $c(j), j \in V$.
- Bank $j$ hit by a market shock $\epsilon_j, j \in V$.
- $\{\epsilon_j, j \in V\}$ is a correlated collection of negative random variables.
- This leads to a Loss Cascade:

$$c_0(j) = (c(j) + \epsilon_j)_+.$$

$$c_{k+1}(j) = \max(c_0(j) - \sum_{i:c_k(i)=0} (1 - R_i)E_{ji}, 0)$$

# Model for Correlated Market Shocks

- System subject to correlated market shocks

$$\epsilon_i = \sigma_i F^{-1} \circ G(\rho S + (1-\rho) Z_i)$$

- $F$ cdf of a negative r.v., $G$ is cdf of a standard Cauchy and $\sigma_i$ is calibrated to the p.d. of firm $i$:

$$P\left[\epsilon_i < -c_i\right] = F\left(-\frac{c_i}{\sigma_i}\right).$$

$$\sigma_i = -\frac{c(i)}{F^{-1}(p_i)}$$

- Fundamental Defaults vs Defaults by Contagion

$$\mathbb{D}(c, E) = \{j : c_0(j) = 0\} \cup \{j : c_0(j) > 0, c_{n-1}(j) = 0\}$$

# Measures of Contagion

- Default Impact:

$$DI(i, c, E) = \sum_{j \in V} \left( c_0(j) - c_{n-1}(j) \right), \qquad \text{given } c_0(i) = 0.$$

- Contagion Index: How much would system suffer if $i$ fails

$$CI(i, c, E) = E\left[ DI(i, c_0, E) \middle| c_0(i) = 0 \right]$$

- Default Cascade: How many would fail if $i$ does? Expected number of defaults due to contagion conditional on failure of firm $i$: $\kappa(i, c, E)$.

$$\kappa(i, c, E) = E\left[ \sum_{j=1}^{n} 1_{\{c(j)+\epsilon_j > 0, c_{n-1}(j)=0\}} \middle| c(i) + \epsilon_i < 0 \right]$$

# Some Conclusions

- Most institutions do not generate other defaults. However some institutions can trigger up to 4 defaults which is about 3% of the financial system.
- A contagion index that significantly exceeds the size of an institution's inter-bank liability is a signature of contagion. Contagion index can be up to forty times the size of inter-bank liability.

# What makes an Institution Systemically Important: The Role of Balance Sheet

- Regression of the logarithm of the Contagion Index on the logarithm of the inter-bank liability size reveals the following.
- Interbank liabilities explain 96% of the variability of the Contagion Index. So size of balance sheet does matter.
- However size does not entirely explain the variations in the Contagion Index.
- Ranking of institutions according to their liabilities does not match their rankings according to their systemic impact.

# What makes an Institution Systemically Important: The Role of Network Structure

| Ranking | CI | Creditors | Liability |
|---|---|---|---|
| 1 | 20.77 | 8 | 23.27 |
| 2 | 4.95 | 32 | 1.57 |
| 3 | 4.58 | 13 | 2.96 |
| 4 | 3.85 | 14 | 1.95 |
| 5 | 3.40 | 21 | 0.97 |
| Median | 0.10 | 5 | 0.07 |
| 90% Quantile | 2.45 | 21 | 1.11 |

- The five most systemic nodes are not very connected, but their creditors are heavily connected and many of their cross exposures are are contagious.

# Determinants of High Contagion Index

- Counterparty Susceptibility

$$CS(i) = \max_{j: E_{ji} > 0} \frac{E_{ji}}{c(j)}$$

- Local Network Frailty

$$f(i) = \max_{j: E_{ji} > 0} \frac{E_{ji}}{c(j)} L(j)$$

- Institutions with a high Contagion Index tend to have large interbank liability, local network frailty and counterparty susceptibility.

# Does One Size Fit All?

- Consider two capital requirement policies:
  1. Minimum capital-to-exposure ratio:

$$\bar{c}(i) = \max(c(i), \theta A(i))$$

  2. Cap on susceptibility:

$$\bar{c}(i) = \max(c(i), \frac{\max_{j \neq i} E_{ij}}{\gamma})$$

- Apply the above policies to all institutions vs apply to the creditors of the 5% most systemic institutions.
- Targeted Capital Requirements achieve the same reduction in systemic risk, with same amount of capital, differently distributed across the network.