Quantitative Geometry of Hyperbolic 3-Manifolds

For $n \geq 3$, the Mostow rigidity theorem asserts that the topological type of a closed hyperbolic n-manifold M determines M up to isometry. Thus any quantitative geometric invariant of a hyperbolic n-manifold, such as its volume, diameter, minimal covering radius, minimal or maximal injectivity radius, or Margulis number, is in principle a topological invariant. This raises the question of how these geometrically defined invariants compare with more classical topological invariants. My mini-course will focus on some aspects of this question for the case when $n = 3$ and M is orientable.

The $\log(2k-1)$ theorem, which follows from combining joint work by Anderson, Canary, Culler and myself with the Marden conjecture proved by Agol and Calegari-Gabai, has been a useful tool in addressing this question. The theorem asserts that if $x_1, \ldots, x_k$ are orientation-preserving isometries of $\mathbf{H}^3$ which generate a free, discrete group of rank k, then for any $z \in \mathbf{H}^3$ we have

$$\sum_{i=1}^{k} \frac{1}{1+e_i^d} \leq \frac{1}{2},$$

where $d_i = \mathrm{dist}(z, x_i \cdot z)$. This implies in particular that $\max_{1 \leq i \leq k} d_i \geq \log(2k-1)$.

My first lecture will be devoted to explaining the proof of the $\log(2k-1)$ theorem and giving a simple first application. As background for the ideas in the proof I will briefly explain the Patterson-Sullivan measure for a Kleinian group, the Banach-Tarski paradox, and the deformation space for a Kleinian group. The preliminary application asserts that if $\pi_1(M)$ has no finite-index subgroup of rank 2 then the minimal injectivity radius of M is at least $(\log 3)/2$. This topological hypothesis always holds if either $H_1(M, \mathbf{Q})$ has rank at least 3, or $H_1(M, \mathbf{Z}/p\mathbf{Z})$ has rank at least 4 for some prime p.

In my second lecture I will give applications involving deeper topological methods. An important role is played here by the notion of a k-free group: a group is said to be k-free if its subgroups of rank at most k are all free. The $\log(2k-1)$ theorem can be used, in combination with a topological argument involving coverings of $\mathbf{H}^3$ by cylinders, to show that if $\pi_1(M)$ is 3-free then $\mathrm{dist}\, M > 3.08$. A much deeper refinement of this argument, due to Culler and myself, shows that if $\pi_1(M)$ is 4-free then $\mathrm{dist}\, M > 3.44$. These results in turn can be combined with elaborate topological arguments about 3-manifolds, and with work by Agol, Storm and Thurston on volumes of Haken manifolds, to show that if $\mathrm{dist}\, M \leq 3.08$ then $H_1(M, \mathbf{Q})$ has rank at most 5, and that if $\mathrm{dist}\, M \leq 3.44$ then $H_1(M, \mathbf{Q})$ has rank at most 7.