Lectures on Discrete Groups and Riemann Surfaces

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Abstract

Lecture I: Riemann surfaces, Holomorphic maps, The Riemann-Hurwitz relation, Singularities.

Lecture II: Galois coverings, Automorphisms with fixed points, Monodromy, Permutation groups, Galois coverings of the line, The Galois extension problem.

Lecture III: Uniformization, The Dirichlet region, Surface groups, Triangle groups, Uniformization of automorphisms.

Lecture IV: Teichmüller spaces of Fuchsian groups, The extension problem, Generalized Lefschetz curves, Accola-Maclachlan and Kulkarni curves, Appendix: *Dessins d'enfant*.

1 Lecture I

1. Riemann surfaces
2. Holomorphic maps
3. The Riemann-Hurwitz relation
4. Singularities

1.1 RIEMANN SURFACES

A Riemann surface "looks locally" like an open subset of the complex plane.

Definition. A Riemann surface X is a second-countable, connected, Hausdorff space with an atlas of charts $\{\phi_\alpha : U_\alpha \to V_\alpha \mid \alpha \in \mathcal{A}\}$.

U_α is an open subset of X, V_α is an open subset of $\mathbb{C}$, and ϕ_α is a homeomorphism. For any two charts ϕ_α, ϕ_β with overlapping domains, the **transition map**,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is **bianalytic**, that is, analytic with analytic inverse. (It is a bijection by construction.)

1.2 EXAMPLES OF RIEMANN SURFACES

1. The Riemann sphere. A two-chart atlas on $S^2 = \{(x, y, w) \in \mathbb{R}^3 \mid x^2 + y^2 + w^2 = 1\}$ is given by **stereographic projection** from the north and south poles: Define charts

$$\phi_1 : S^2 \setminus (0,0,1) \to \mathbb{C} \text{ by } (x,y,w) \mapsto \frac{x}{1-w} + i\frac{y}{1-w}$$
$$\phi_2 : S^2 \setminus (0,0,-1) \to \mathbb{C} \text{ by } (x,y,w) \mapsto \frac{x}{1+w} - i\frac{y}{1+w}$$

The inverse chart maps are

$$\phi_1^{-1}(z) = \left(\frac{2\text{Re}(z)}{|z|^2+1}, \frac{2\text{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$
$$\phi_2^{-1}(z) = \left(\frac{2\text{Re}(z)}{|z|^2+1}, \frac{-2\text{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right).$$

The transition map $\phi_2 \circ \phi_1^{-1}$ is simply $z \mapsto 1/z$.

2. The graph of an analytic function. The graph $\{(z, g(z)) \mid z \in U\} \subseteq \mathbb{C}^2$, where g is an analytic function whose domain contains the open set $U \subseteq \mathbb{C}$, is a Riemann surface with the single chart $\pi_z : (z, g(z)) \mapsto z$ (projection onto the z coordinate).

3. Smooth affine plane curves. **Definition.** An affine plane curve X is the zero locus of a polynomial $f(z, w) \in \mathbb{C}[z, w]$. It is **non-singular** or "smooth" if, for all $p = (a, b) \in X$, $\left.\frac{\partial f}{\partial z}\right|_p$ and $\left.\frac{\partial f}{\partial w}\right|_p$ are not simultaneously zero.

By the implicit function theorem, in a neighborhood of every p on a smooth affine plane curve, at least one of the coordinates z, w is an analytic function of the other, depending on which partial derivative is $\neq 0$.

If $\left.\frac{\partial f}{\partial w}\right|_p \neq 0$, there is an open set U containing p such that, for all $q = (z, w) \in U$, $w = g(z)$, an analytic function of z. Thus $\pi_z : U \to \mathbb{C}$ is a local chart. If, also, $\left.\frac{\partial f}{\partial z}\right|_p \neq 0$, there is an open set V containing p such that, for all $q = (z, w) \in V$, $z = h(w)$, an analytic function of w. Then $\pi_w : V \to \mathbb{C}$ is also a local chart. The transition functions,

$$\pi_w \circ \pi_z^{-1} : z \mapsto g(z)$$
$$\pi_z \circ \pi_w^{-1} : w \mapsto h(w),$$

defined on $\pi_z(U \cap V)$ and $\pi_w(U \cap V)$, resp., are, by construction, analytic. Thus a **connected** smooth affine plane curve is a Riemann surface. [Connectivity can be guaranteed by taking the polynomial $f(z, w)$ to be irreducible.]

The points of the **complex projective plane** $\mathbb{P}^2$ are the one-dimensional subspaces of $\mathbb{C}^3$. The span of $(x, y, z) \in \mathbb{C}^3$, $(x, y, z) \neq (0, 0, 0)$, is denoted $[x : y : z]$. For $\lambda \in \mathbb{C}$, $\lambda \neq 0$,

$$[x : y : z] = [\lambda x : \lambda y : \lambda z] \quad \text{("homogeneous coordinates").}$$

$\mathbb{P}^2$ is a complex manifold of dimension 2. It is covered by three sets, defined by $x \neq 0$, $y \neq 0$, $z \neq 0$, respectively. In homogenous coordinates, we may assume that $|x|^2 + |y|^2 + |z|^2 = 1$; in particular, that $|x|, |y|, |z| \leq 1$. Thus $\mathbb{P}^2$ is **compact**.

Definition. A polynomial $F(x, y, z) \in \mathbb{C}^3$ is **homogeneous** if, for every $\lambda \in \mathbb{C}^*$, $F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x, y, z)$, where d is the degree of the polynomial.

The zero locus of a homogeneous polynomial $F(x, y, z)$ is well-defined on $\mathbb{P}^2$.

4. Smooth projective plane curves. **Definition.** A **projective plane curve** X is the zero locus in $\mathbb{P}^2$ of a homogeneous polynomial $F(x, y, z) \in \mathbb{C}[x, y, z]$. It is **non-singular** (smooth) if, at all $p = [x : y : z] \in X$, the three partial derivatives $\left.\frac{\partial F}{\partial x}\right|_p$, $\left.\frac{\partial F}{\partial y}\right|_p$ and $\left.\frac{\partial F}{\partial z}\right|_p$ are not simultaneously zero.

An affine plane curve $f(x, y) = 0$ is "projectivized" (hence, compactified) by multiplying each term of f by a suitable power of z so that all terms have the same (minimal) degree. The affine portion of the projectivized curve corresponds to $z = 1$, and the "points at infinity" correspond to $z = 0$.

Theorem. A nonsingular projective plane curve is a compact Riemann surface.

Proof. Let $U_i = \{[x_0 : x_1 : x_2] \subseteq \mathbb{P}^2 \mid x_i \neq 0\}$, $i = 0, 1, 2$. (Up to homogeneous coordinates, $x_i \neq 0$ is equivalent to $x_i = 1$.) Let X be a smooth projective plane curve defined as the zero locus of the homogenous polynomial $F(x_0, x_1, x_2)$, and let $X_i = X \cap U_i$. Each X_i is an affine plane curve, e.g.,

$$X_0 = \{(a, b) \in \mathbb{C}^2 \mid F(1, a, b) = 0\}.$$

With Euler's formula for homogeneous polynomials of degree d,

$$F(x_0, x_1, \ldots x_k) = \frac{1}{d} \sum_{i=0}^{k} x_i \frac{\partial F}{\partial x_i},$$

it can be shown that X is nonsingular if and only if **each X_i is a smooth affine plane curve.**

Proof, cont. Coordinate charts on X_i are ratios of homogeneous coordinates on X. For example, charts on X_0 are x_1/x_0 or x_2/x_0, and charts on X_2 are x_0/x_2 or x_1/x_2. Transition functions are readily seen to be holomorphic, e.g., near $p \in X_0 \cap X_1$, where $x_0, x_1 \neq 0$, let $z = \phi_1 = x_1/x_0$ and $w = \phi_2 = x_2/x_1$. Then

$$\phi_2 \circ \phi_1^{-1} : z \mapsto [1 : z : h(z)] \mapsto \frac{h(z)}{z} = w,$$

where $h(z)$ is a holomorphic function, and $z \neq 0$, since $p \in X_1$. [Technical detail: connectivity is needed to make X_i (and hence X) a Riemann surface. This follows from a standard theorem in algebraic geometry: a nonsingular homogeneous polynomial is automatically irreducible.] **Q.e.d.**

Remark. Projective spaces $\mathbb{P}^n$ can be defined for all $n \geq 1$. The **complex projective line** $\mathbb{P}^1$ is the space of one-dimensional subspaces of $\mathbb{C}^2$,

$$\{[x : y] \mid x, y \in \mathbb{C}, (x, y) \neq (0, 0)\},$$

where $[x : y] = [\lambda x : \lambda y]$, $\lambda \in \mathbb{C}^*$. The two-chart atlas

$$\phi_0 : \mathbb{P}^1 \setminus \{[0:1]\} \to \mathbb{C}$$
$$\phi_1 : \mathbb{P}^1 \setminus \{[1:0]\} \to \mathbb{C},$$

defined by $[x : y] \mapsto y/x$, resp., $[x : y] \mapsto x/y$, has transition function

$$\phi_1 \circ \phi_0^{-1} : z \mapsto 1/z.$$

This makes $\mathbb{P}^1 \simeq \mathbb{C} \cup \{\infty\}$ a Riemann surface, where ∞ corresponds to $[1 : 0]$.

1.3 HOLOMORPHIC MAPS

Non-constant holomorphic maps between compact Riemann surfaces look locally (when "read through charts") like $z \mapsto z^m$, $m \geq 1$, and globally, like covering maps (except possibly at finitely many points).

Definition. A map $f : X \to Y$ between compact Riemann surfaces is **holomorphic** if, for every $p \in X$, there is a chart $\phi : U_p \to \mathbb{C}$ defined on a neighborhood of p, and a chart $\psi : V_{f(p)} \to \mathbb{C}$ defined on a neighborhood of $f(p) \in Y$, such that the map $\psi \circ f \circ \phi^{-1} : \phi(U_p) \to \psi(V_{f(p)})$ is analytic.

Two Riemann surfaces X and Y are **isomorphic** iff there exists a holomorphic bijection $f : X \to Y$ with a holomorphic inverse (a **biholomorphism**).

Exercise: Show that the complex projective line $\mathbb{P}^1$ is isomorphic to the Riemann sphere.

An isomorphism $f : X \to X$ of a Riemann surface *with itself* is called an **automorphism**. The automorphisms form a group, $\mathrm{Aut}(X)$, about which we will have much more to say.

Definition. A **meromorphic function** on a Riemann surface X is a holomorphic map $f : X \to \mathbb{P}^1$.

Examples. **1.** The meromorphic functions on $\mathbb{P}^1$ are the **rational functions** $r(z) = \dfrac{p(z)}{q(z)}$, where $p, q \in \mathbb{C}[z]$, $q \neq 0$.

2. The meromorphic functions on a smooth affine plane curve $f(x, y) = 0$ are the rational functions

$$r(x, y) = \frac{p(x, y)}{q(x, y)}, \quad p, q \in \mathbb{C}[x, y],$$

provided $q(x, y)$ is not a divisor of $f(x, y)$ (i.e, by Hilbert's *Nullstellensatz*, provided $q(x, y)$ does not vanish identically on the curve).

3. The meromorphic functions on a smooth projective plane curve defined by the nonsingular homogeneous polynomial $F(x, y, z)$ (as its zero locus) are rational functions

$$R(x, y, z) = \frac{P(x, y, z)}{Q(x, y, z)}, \quad P, Q \in \mathbb{C}[x, y, z],$$

where P and Q are homogeneous of the same degree, and Q is not a divisor of F.

Holomorphic maps "inherit" many properties of analytic maps. Let $f : X \to Y$ be a nonconstant holomorphic map between between Riemann surfaces. Then, as with an analytic map from $\mathbb{C}$ to $\mathbb{C}$,

1. f is an **open mapping** (taking open sets to open sets);

2. If $g : X \to Y$ is another holomorphic map, and f and g agree on a subset $S \subseteq X$ with a limit point in X, then $f = g$;

3. $f^{-1}(y)$, $y \in Y$, is a **discrete** subset of X.

Theorem. If X is a compact Riemann surface and $f : X \to Y$ is a nonconstant holomorphic map, then f **is onto,** Y **is compact, and** $f^{-1}(y) \subseteq X$**,** $y \in Y$**, is a non-empty finite set**.

Proof. $f(X)$ is open by the openness of the mapping; $f(X)$ is a compact subset of Hausdorff space and hence also closed. Since Y is connected, $f(X)$ must be all of Y. A discrete subset of a compact space is finite. **Q.e.d.**

Theorem (Local normal form of a holomorphic map). If $f : X \to Y$ is a nonconstant holomorphic map, and $p \in X$, there exists a unique positive integer $m = \mathbf{mult_p(f)}$ (the multiplicity of f at p) such that, for every coordinate chart $\phi : U \subseteq X \to \mathbb{C}$ "centered at p" (i.e., $\phi(p) = 0$) and every coordinate chart $\psi : V \subseteq Y \to \mathbb{C}$ centered at $f(p)$, $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ has the local normal form $z \mapsto z^m$.

Proof. Let w be the complex coordinate on $\phi(U) \subseteq \mathbb{C}$. Let $T(w) = \sum_{i=m}^{\infty} c_i w^i$ be the Taylor series of $\psi \circ f \circ \phi^{-1}(w)$. Since $T(0) = 0$, $m \geq 1$, and $T(w) = w^m S(w)$, with $S(w)$ analytic at 0 and $S(0) \neq 0$. It follows that $S(w)$ has a local "mth root," $R(w)$. Let $z = z(w) = wR(w)$. We have $z(0) = 0$ and $z'(0) = R(0) \neq 0$, so on an open subset of U, $z(w)$ is a new complex coordinate for a new chart, ϕ_1, centered at p. Reading through this new chart, f has the form $z \mapsto z^m$. (Uniqueness of m is left to the reader.) **Q.e.d.**

Defintions. A point $q \in X$ for which $\text{mult}_q(f) > 1$ is called a **ramification point**; the image of a ramification point (in Y) is called a **branch point**.

The branch set and its preimage, which contains the ramification points, are **discrete** subsets of Y, resp. X: in local coordinates $w = h(z)$, ramification occurs at the isolated points z_0, where $h'(z_0) = 0$.

Here is the crucial **global** property of a holomorphic map between **compact** surfaces:

Theorem. If $f : X \to Y$ is a nonconstant holomorphic map between compact surfaces, there exists a unique positive integer d such that, **for every** $y \in Y$,

$$\sum_{p \in f^{-1}(y)} \text{mult}_p(f) = d.$$

Remark. d is called the "degree" of f. The theorem explains why f is also called a **branched covering map**: the branch locus $B \subset Y$ and its preimage $f^{-1}(B)$ are discrete and hence finite (by compactness of Y). Thus, away from finitely many points, f is a covering map of degree d (every point in $Y \setminus B$ is contained in an open set U whose pre-image is a disjoint union of d open sets, each homeomorphic to U).

Sketch of the proof of the theorem. The open unit disk $D \subseteq \mathbb{C}$ is a Riemann surface, and for the holomorphic map $f : D \to D$, defined by $z \to z^m$, the theorem is clearly true: 0 is the unique point in $f^{-1}(0)$, and the multiplicity at 0 is m; if $a \in D$, $a \neq 0$, $f^{-1}(a)$ consists of m distinct points (the m mth roots of a), at which the multiplicity of f is 1. Thus the total multiplicity over every point in D is m. A general nonconstant holomorphic map is a "disjoint union" of such power maps, which shows that $d_y : Y \to \mathbb{N}$, which a priori depends on $y \in Y$, is a locally constant map. But d_y is also continuous; the connectedness of Y implies that d_y is constant. **Q.e.d.**

1.4 THE RIEMANN-HURWITZ RELATION

Topologically, compact oriented surfaces are completely classified by the **genus** $g \geq 0$. All such surfaces admit triangulations; for any triangulation,

$$\#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\} = 2 - 2g \quad \text{(Euler characteristic).}$$

If $f : X_g \to Y_h$ is a covering map of degree d between compact oriented surfaces of genera g, h, resp., then $2g - 2 = d(2h - 2)$. For branched coverings (in particular, for holomorphic maps) we have:

Theorem (the Riemann-Hurwitz relation). If $f : X_g \to Y_h$ is a nonconstant holomorphic map of degree d between compact Riemann surfaces of genera g,h, resp., then

$$2g - 2 = d(2h - 2) + \sum_{p \in X} (\text{mult}_p(f) - 1).$$

Proof. Let Y be triangulated so that the branch locus $B \subset Y$ is contained in the vertex set. Let v,e, f be the number of vertices, edges and faces respectively. The triangulation lifts through the covering of degree d to a triangulation of X which has de edges and df faces, but only

$$dv - \sum_{b \in B} (d - |f^{-1}(b)|)$$

vertices. Hence

$$2 - 2g = dv - de + df - \sum_{b \in B} (d - |f^{-1}(b)|).$$

Since $dv - de + df = d(2 - 2h)$, it suffices to show that

$$\sum_{b \in B} (d - |f^{-1}(b)|) = \sum_{p \in X} (\text{mult}_p(f) - 1).$$

Proof, cont. Let $B = \{b_1, b_2, \ldots, b_n\}$. We make use of the trivial fact that $|f^{-1}(b_i)| = \sum_{x \in f^{-1}(b_i)} 1$, together with the constancy of the degree $\sum_{x \in f^{-1}(b_i)} \text{mult}_x(f) = d$, to rewrite the sum

$$\begin{aligned}\sum_{b \in B}(d - |f^{-1}(b)|) &= \sum_{i=1}^{n}(d - |f^{-1}(b_i)|) \\ &= \sum_{i=1}^{n} \sum_{p \in f^{-1}(b_i)} (\text{mult}_p(f) - 1) \\ &= \sum_{p \in X} (\text{mult}_p(f) - 1).\end{aligned}$$

At the final step, we use the fact that $\text{mult}_p(f) = 1$ whenever $p \notin f^{-1}(B)$. **Q.e.d.**

Applications. 1. The genus of the Fermat curve. Let X be the smooth projective plane curve which is the zero locus of the polynomial $F(x, y, z) = x^d + y^d + z^d$. Consider the holomorphic map

$$\pi : X \to \mathbb{P}^1, \quad [x : y : z] \mapsto [x : y].$$

It has degree d, since $\pi^{-1}([x : y])$ is in bijection with the set of dth roots of $-x^d - y^d$. If $x^d = -y^d$, $|\pi^{-1}([x : y])| = 1$ and the multiplicity of π is d. There are d such points, given (in homogeneous coordinates) by $[1 : \omega : 0]$, where ω is a dth root of -1. At all other points, the multiplicity of π is 1. The genus of $\mathbb{P}^1$ is 0 so the RH relation $2g_X - 2 = d(-2) + d(d-1)$ yields

$$g_X = \frac{(d-1)(d-2)}{2}.$$

[In fact, this **degree-genus formula** holds for ANY smooth projective curve of degree d.]

2. Cyclic covers of the line. Let $h(x)$ be a polynomial of degree k, and consider the affine plane curve $C = \{(x, y) \in \mathbb{C}^2 \mid \ \mathbf{y^d} = \mathbf{h(x)}\}$, where $d \geq 2$. If h has distinct roots (first non-trivial assumption), the projection $\pi_x : X \to \mathbb{C}$, $(x, y) \mapsto x$ ramifies with multiplicity d over the roots of h, and is a d-fold covering over all other points in $\mathbb{C}$. We compactify C to $\overline{C}$ by projectivization. Then π_x extends to a map $\overline{\pi}_x : \overline{C} \to \mathbb{P}^1$. What happens "at infinity" (i.e., as $x \to \infty$)? Suppose $k = dt$, $t \geq 1$ (second non-trivial assumption). For $x \neq 0$ (i.e., in a neighborhood of ∞), the map $(x, y) \leftrightarrow (1/x, y/x^t)$ is bianalytic and defines new coordinates $z = 1/x$, $w = y/x^t$. The defining equation of C transforms to

$$\begin{aligned}\mathbf{w^d} &= y^d/x^k = y^d z^k = h(x) z^k = h(1/z) z^k \\ &= (1 - za_1)(1 - za_2) \cdot \cdots \cdot (1 - za_k) = \ \mathbf{g(z)}\end{aligned}$$

where $a_1, \ldots, a_k$ are the roots of $h(x)$. The dth roots of $g(0) \neq 0$ correspond to d points at ∞.

Thus $\overline{\pi}_x : \overline{C} \to \mathbb{P}^1$ is a holomorphic map of degree d between compact Riemann surfaces (in fact, a meromorphic function) which ramifies at k points (over the k distinct zeroes of $h(x)$, but not over ∞) with multiplicity d. The RH relation determines the genus of $\overline{C}$

$$2g_{\overline{C}} - 2 = d(-2) + k(d-1)$$
$$g_{\overline{C}} = (d-1)(k-2)/2.$$

Remarks. **1.** $\overline{C}$ admits an **automorphism** of order d, defined by

$$\alpha : (x, y) \mapsto (x, \omega y),$$

where ω is a primitive dth root of unity. α fixes the k ramification points, and permutes all other points in orbits of length d.

2. If $d = 2$, $\overline{C}$ is **hyperelliptic** and α is the **hyperelliptic involution**, with $k = 2g + 2$ fixed points.

1.5 SINGULARITIES

Consider, again, the affine plane curve $C = \{(x, y) \in \mathbb{C}^2 \mid y^d = h(x)\}$, where $d \geq 2$ and $h(x)$ is a polynomial of degree k. We wish to treat the most general case: we do NOT assume h has distinct roots, and we do NOT assume k is a multiple of d. Thus:

- $h(x) = (x - a_1)^{e_1}(x - a_2)^{e_2} \dots (x - a_r)^{e_r}$, $a_i \in \mathbb{C}$, $e_i \geq 0$, and $\sum_{i=1}^{r} e_i = k$;
- $k = dt - \epsilon$, $t \geq 1$, $0 \leq \epsilon \leq d - 1$.

With these assumptions, C and its compactification $\overline{C} \subset \mathbb{P}^2$ may contain **singular points** which must be "resolved."

Definitions. A point $p = (x_0, y_0)$ on an affine plane curve $f(x, y) = 0$ is singular if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. A singularity is **monomial** if there are local coordinates (z, w) centered at p in which the defining equation has the form $z^n = w^m$, $n, m > 1$.

A monomial singularity may occur **at** ∞**.** For C, a cyclic covering of the line, defined by $y^d = h(x)$, the projection $\pi_x : (x, y) \mapsto x$ is a coordinate chart on the affine portion. For the points at ∞, we change to the coordinates $z = 1/x$, $w = y/x^t$. In the new coordinates, the defining equation

$$y^d = (x - a_1)^{e_1}(x - a_2)^{e_2} \dots (x - a_r)^{e_r} = h(x)$$

transforms to

$$\mathbf{w}^{\mathbf{d}} = y^d / x^{k+\epsilon} = y^d z^{k+\epsilon} = h(x) z^{k+\epsilon} = h(1/z) z^{k+\epsilon}$$
$$= z^{\epsilon}(1 - z a_1)^{e_1}(1 - z a_2)^{e_2} \dots (1 - z a_r)^{e_r} = \mathbf{z}^{\epsilon}\mathbf{g}(\mathbf{z}).$$

Since $g(0) \neq 0$, the equation "looks like" the singular curve $w^d = z^{\epsilon}$ near ∞ (i.e., near $z = 0$) .

We also get a monomial singularity on C at any root of $h(x)$ with **multiplicity** > 1, e.g., if $e_1 > 1$, then, near $x = a_1$, the equation is

$$\begin{aligned} y^d &= a_1^{\epsilon}(x-a_1)^{e_1}(a_1-a_2)^{e_2}\dots(a_1-a_r)^{e_r} \\ &\approx \text{constant}\cdot X^{e_1} \qquad (\text{where } X = x - a_1). \end{aligned}$$

Theorem. On an affine plane curve, a monomial singularity of type $z^n = w^m$ is resolved by removing the singular point and adjoining $(n, m) = \gcd(n, m)$ points.

Proof. **Case 1.** If $n = m$, $z^n - w^n$ factors as

$$z^n - w^n = \prod_{i=0}^{n-1}(z - \zeta^i w),$$

where ζ is a primitive n th root of unity. Each factor defines a smooth curve. The singularity is resolved by removing the common point $(0, 0)$ and replacing it with n distinct points.

Case 2. If $(n, m) = 1$ (relatively prime), there exist $a, b \in \mathbb{Z}$ such that $an + bm = 1$. The map $\phi : (z, w) \mapsto z^b w^a$ defines a "hole chart" whose domain is the curve minus the singular point $\{(0, 0)\}$ and whose co-domain is the "punctured" plane $\mathbb{C} \setminus \{0\}$. The inverse chart is $\phi^{-1} : s \mapsto (s^m, s^n)$. By continuity, ϕ extends uniquely to the closure of the domain ("restoring" the singular point).

Case 3. If $(n, m) = c$, there exist $a, b \in \mathbb{Z}$ such that $n = ac$ and $m = bc$, and $(a, b) = 1$. Then

$$z^n - w^m = (z^a)^c - (w^b)^c = \prod_{i=1}^{c}(z^a - \zeta^i w^b),$$

where ζ is a primitive c th root of unity. Case 2 applies to each of the c factors; thus c points are adjoined to fill c holes. **Q.e.d.**

Corollary (genus of a cyclic cover of the line). Let $y^d = h(x)$, $d \geq 2$, define the cyclic covering $\pi_x : \overline{C} \to \mathbb{P}^1$. Let the polynomial $h(x)$ have r roots of multiplicities $e_1, \dots, e_r$. Assume $\sum_i^r e_i \equiv 0 \pmod{d}$ (to avoid branching at ∞). The genus of $\overline{C}$ is

$$g = 1 + \frac{(r-2)d - \sum_{i=1}^{r}(d, e_i)}{2}.$$

Proof. $\pi_x : \overline{C} \to \mathbb{P}^1$ is a d-sheeted branched covering; over a zero of multiplicity e, there are (d, e) points, each of multiplicity $d/(d, e)$. Apply the RH relation. **Q.e.d.**

[**Remark.** For connectivity of $\overline{C}$, $f(x, y) = y^d - h(x)$ must be irreducible. Exercise: this is the case iff the gcd of the set $\{d, e_1, \dots, e_k\}$ is 1.]

2 Lecture II

1. Galois coverings
2. Automorphisms with fixed points
3. Monodromy
4. Permutation groups
5. Galois coverings of the line
6. The Galois extension problem

2.1 GALOIS COVERINGS

A **Galois covering** is a branched covering $f : X \to Y$, where f is the quotient map of a group action on X. The group $G(f) = G(f, X, Y)$ is called the **Galois group** of the covering. If X and Y are compact Riemann surfaces and f is holomorphic, then $G(f) \leq \mathrm{Aut}(X)$. $G(f)$ acts as a **transitive** permutation group on the sheets of the covering (in particular, on the fibre over each $y \in Y$). At a ramification point $p \in X$, two or more sheets come together, so p must have a non-trivial **isotropy subgroup** $G_p \leq G(f)$. If q is in the $G(f)$-orbit of p, i.e., if there exists $g \in G(f)$ such that $q = g \cdot p$, then $G_q = g \cdot G_p \cdot g^{-1}$.

Example. Let $y^d = h(x)$, $d \geq 2$, define the cyclic covering $\pi_x : \overline{C} \to \mathbb{P}^1$. The automorphism $\alpha : \overline{C} \to \overline{C}$, defined by

$$\alpha : (x, y) \mapsto (x, \omega y),$$

where ω is a primitive dth root of unity, generates the (cyclic) Galois group $G(\pi_x) = \langle \alpha \rangle \simeq \mathbb{Z}_d$.

Example, cont. Let $h(x)$ have r roots of multiplicities $e_1, \ldots, e_r$, and assume $\sum_i^r e_i \equiv 0 \pmod d$ (so there is no branching at ∞). We know that over the root of multiplicity e_i, the sheets come together at (d, e_i) points, in sets of $d/(d, e_i)$ sheets. If $e_i > 1$, these are ramification points. At each of these points, the isotropy subgroup can only be $\langle \alpha^{(d,e)} \rangle \simeq \mathbb{Z}_{d/(d,e)}$. Conversely, a point in $\overline{C}$ a with non-trivial isotropy subgroup is a ramification point lying over a zero of multiplicity > 1.

The Galois group of a holomorphic map between compact Riemann surfaces need not be cyclic, but the isotropy groups always are.

2.2 AUTOMORPHISMS WITH FIXED POINTS

Lemma. Let G be a **finite** group of automorphisms of a compact Riemann surface X. Let $G_p \leq G$ be the isotropy subgroup of a point $p \in X$. Then G_p is (finite) cyclic.

Proof sketch. Let $g \in G_p$. In a coordinate (z) centered at p, g is represented by a Taylor series of the form $\sum_{n=1}^{\infty} a_n(g)z^n$ with zero constant term (since p is fixed by g). $a_1(g) \neq 0$ (since g is an automorphism, $\text{mult}_q(g) = 1$ for all $q \in X$). Claim: the map $\theta : G_p \to \mathbb{C}^*$, defined by $\theta : g \mapsto a_1(g)$, is a group **homomorphism**. Proof of claim: If $h \in G_p$, the Taylor series for h, in the same coordinate system, has coefficient $a_1(h) \neq 0$. Multiplication of power series shows $a_1(g \circ h) = a_1(g) \cdot a_1(h)$. (Q.e.d. claim.) Thus $\theta(G_p)$ is a finite subgroup of $\mathbb{C}^*$. The only finite subgroups of $\mathbb{C}^*$ are cyclic. **Exercise:** Complete the proof by showing that ker(θ) is trivial. **Q.e.d.**

2.3 MONODROMY

There is a finite permutation group associated to ANY branched covering $f : X \to Y$ between compact surfaces (Galois or not). It is known as the **monodromy group** of the covering. This group, together with a canonical set of generators, determines the branched covering, up to up to homeomorphism (biholomorphism, in the category of Riemann surfaces).

Let f have degree d, and let $Y^* = Y - B$, where $B = \{b_1, b_2, \ldots, b_n\} \subset Y$ is the (finite) branch set and $X^* = X - f^{-1}(B)$. Note that $f^{-1}(B)$ is a finite set containing the ramification points, and possibly some other points. The restricted map

$$f^* : X^* \to Y^*$$

is an **ordinary** (unramified) d-sheeted **covering map**.

Choose a basepoint $y_0 \in Y^*$, and let

$$F = (f^*)^{-1}(y_0) = \{x_1, x_2, \ldots x_d\} \subset X^*,$$

the **fibre** over the basepoint. A loop γ_j, based at y_0 and winding once counterclockwise around the puncture created by the removal of b_j (and not winding around any other puncture), has a **unique lift** to a path $\widetilde{\gamma_{j,i}}$ starting at x_i, $i = 1, 2, \ldots d$, with a well-defined endpoint belonging to F.

Lemma. For each $j \in \{1, 2, \ldots, n\}$, the "endpoint of lift" map

$$\rho_j : i \mapsto \text{endpoint of } \widetilde{\gamma}_{j,i} \in F, \quad i \in \{1, 2, \ldots, d\}$$

is a bijection (hence, an element of S_d).

Proof. Suppose the endpoint of $\widetilde{\gamma}_{j,i}$, say, x_l, coincides with the endpoint of $\widetilde{\gamma}_{j,k}$. Then there is a path in X from x_i to x_k, namely, $(\widetilde{\gamma}_{j,k})^{-1} \circ \widetilde{\gamma}_{j,i}$, which is a lift of the trivial path $(\gamma_j)^{-1} \circ \gamma_j = \{y_0\} \in Y$. This implies $x_i = x_k$. **Q.e.d.**

In other words, after choosing a basepoint $y_0 \in Y^*$, and choosing n loops based at y_0 winding once counterclockwise around one of the n punctures , we obtain a set $\{\rho_1, \ldots, \rho_n\} \subseteq S_d$ of n **permutations of the fibre** F over the basepoint y_0. Definition The **monodromy group** of the branched covering $f : X \to Y$ is the subgroup $M(f) = M(f, X, Y) \leq S_d$ generated by $\{\rho_1, \ldots, \rho_n\}$.

The **cycle structure** of $\rho_j \in M(f)$ encodes the ramification data of the original **branched** covering $f : X \to Y$ above the branch point $b_j \in Y$.

Proposition. The number of points in the fibre $f^{-1}(b_j)$ is the number of cycles in the monodromy generator ρ_j (1-cycles are counted). The multiplicity of f at a point in the fibre $f^{-1}(b_j)$ is the length of the corresponding cycle in ρ_j.

Proof sketch. Number the sheets of $f^* : X^* \to Y^*$ so that x_i belongs to the i th sheet. Suppose there are at least 3 sheets, and one of the cycles of ρ_j is (123). Let γ_j be a loop in Y^* based at y_0 and winding once counterclockwise around the jth puncture (and not around any other puncture). By assumption, the lift $\widetilde{\gamma}_{j,1}$, which starts at x_1 ends at x_2; the lift $\widetilde{\gamma}_{j,2}$, which starts at x_2 ends at x_3; and the lift $\widetilde{\gamma}_{j,3}$ which starts at x_3 ends at x_1. It follows that $(\gamma_j)^3$ lifts to a loop in X^* based at x_1, crossing sheets $1, 2$ and 3, but no other sheets.

Proof sketch, cont. In a sufficiently small punctured neighborhood $V_j \subset Y^*$ (punctured at b_j), every point has three distinct pre-images on sheets 1, 2 and 3 (and possibly other pre-images on other sheets). There exists a "hole chart" whose domain is a union of open subsets of sheets 1, 2 and 3, and whose codomain is contained in V_j. In the local coordinates, $f : z \mapsto z^3$. To fill the hole in the domain we must adjoin a point of multiplicity 3. Analogous constructions are made for each cycle (including 1-cycles) of ρ_j. **Q.e.d.**

EXAMPLE. Let $f : X \to Y$ be a 6-sheeted branched covering, branched over $\{b_1, b_2, \dots, b_n\} \subset Y$, and suppose $\rho_2 = (135)(46)(2) \in S_6$. Then $f^{-1}(b_2) \subset X$ consists of three points: one of multiplicity 3 (where the sheets 1, 3 and 5 come together); one of multiplicity 2 (where sheets 4 and 6 come together); and one other point (on sheet 2) of multiplicity 1.

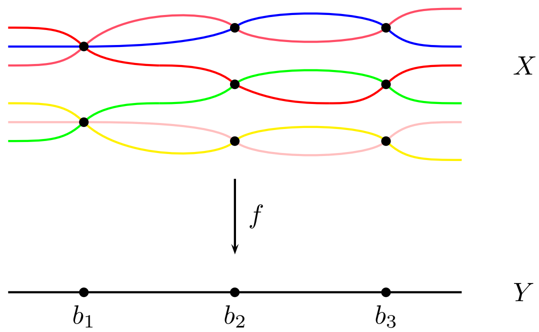
If $f : X \to Y$ is a **Galois** covering, each generator $\rho_j \in M(f)$ is **uniform**, i.e., a product of cycles of the same length, where every symbol $1, 2, \dots d$ appears in exactly one cycle. This is because each cycle of ρ_j is the (cyclic!) isotropy subgroup of the corresponding ramification point; these points comprise an orbit of $G(f)$; hence the cycles are conjugate, and therefore, have the same length. Every symbol appears because $G(f)$ acts transitively on the sheets.

For a Galois covering, the ramification data is encoded by assigning a **branch index** to each point in the branch set $\{b_1, b_2, \dots, b_n\} \subset Y$. The index $r_j > 1$ assigned to b_j means that ρ_j is a product of d/r_j cycles of length r_j, where d is the degree of the covering.

Example. For a 6-sheeted Galois covering, the possible branch indices are 2, 3, and 6. ρ_j is either (a) a product of 2 3-cycles; (b) a product of 3 2-cycles; or (c) a single 6-cycle.

Here is a picture of a 6-fold Galois covering: The branch indices are $(r_1, r_2, r_3) = (3, 2, 2)$. Note: this is NOT a cyclic covering of the line; we shall see that the Galois group is $S_3 \simeq D_6$.

The Riemann-Hurwitz relation for a Galois covering. Branching indices



$\{r_1, r_2, \ldots, r_n\}$ yield a convenient form of the ramification term in the Riemann-Hurwitz relation:

$$\sum_{p \in X} \mathrm{mult}_p(f) = \sum_{i=1}^{n} \frac{|G|}{r_i}(r_i - 1).$$

If g is the genus of X, and h the genus of Y, we have

$$\begin{aligned} 2g - 2 &= |G|(2h-2) + \sum_{i=1}^{n} \frac{|G|}{r_i}(r_i - 1) \\ &= |G|\left(2h - 2 + \sum_{i=1}^{n}(1 - 1/r_i)\right). \end{aligned}$$

Example, cont. For the 6-fold Galois covering pictured previously, with branching indices $(3, 2, 2)$, the RH relation yields $g = h = 0$ (exercise). Thus the Galois group $G(f)$ has order 6 and acts on $\mathbb{P}^1$. It cannot be $\mathbb{Z}_6$, because, as a group of rotations of the sphere, it would have two fixed points where the rotation axis meets the sphere, implying branching indices $(6, 6)$. The only other group of order 6 is $S_3 \simeq D_6$. A **dihedron** with two 3-sided faces is realized on the sphere by inscribing an equilateral triangle on the equator, and taking the two face-centers to be the North and South poles. The dihedron is preserved by a 3-fold **rotation** about the polar axis and any of three conjugate **half-turns** about a line joining a vertex with the opposite edge midpoint. There is one orbit of length 2 (the poles), fixed by the 3-fold rotation (yielding branch index 3); and two orbits of length 3 (vertices and edge-midpoints) fixed by a half-turn (yielding branch indices $(2, 2)$). **Exercise.** Generalize to D_{2n}, $n \geq 2$, acting on $\mathbb{P}^1$, with branch indices $(2, 2, n)$.

The Galois group can be defined for a non-Galois (arbitrary branched) covering. Definition The Galois group $G(f, X, Y)$ of the branched covering map $f : X \to Y$ is the group of **covering transformations** of $f^* : X^* \to Y^*$, that is, the homeomorphisms $h : X^* \to X^*$ (automorphisms in the category of Riemann surfaces) such that $f^* = f^* \circ h$.

We shall show that $G(f, X, Y)$ is isomorphic to the group of permutations of the fibre F which commute with the elements of $M(f,X,Y)$. We first extract the purely algebraic part.

2.4 TWO PERMUTATION GROUPS

Let $K \leq H$ be a subgroup-group pair. Let

$$K^* = \cap_{h \in H} h^{-1} K h,$$

the **core of** K **in** H, and let $N_H(K) = \{h \in H \mid hK = Kh\}$, the **normalizer of** K **in** H. Assume that the index $[H : K^*]$ (hence also $[H : K]$, and $[N_H(K) : K]$) is finite. There are two natural finite permutation groups defined on the set $R = \{Kh \mid h \in H\}$ of **right cosets of** K **in** H:

- $(H/K^*, R)$ (monodromy type - action on the right);
- $(N_H(K)/K, R)$ (Galois type - action on the left).
- the right (monodromy) action $H/K^* \times R \to R$ is

$$K^*h_2 \cdot \mathbf{Kh} = KhK^*h_2 = \mathbf{Khh_2}$$

- the left (Galois) action $N_H(K)/K \times R \to R$ is

$$Kh_1 \cdot \mathbf{Kh} = Kh_1Kh = \mathbf{Kh_1h} \quad (h_1 \in N_H(K)).$$

Exercise. Show:

1. The actions are **well-defined** and **faithful**, i.e. a group element that fixes every coset in R is the identity.

2. The actions **commute**: $(Kh_1)(Khh_2) = (Kh_1h)(K^*h_2)$.

3. The monodromy action is **transitive**.

4. The Galois action is **regular**: if $h_1 \in N_H(K)$, and $Kh_1h = Kh$, then $h_1 \in K$ (i.e., all isotropy subgroups are trivial).

5. If K **is normal in** H (i.e, $K^* = K$, $N_H(K) = H$), the two groups are isomorphic ($\simeq H/K$) and the actions reduce to the **left and right regular representations** of H/K on itself.

Our goal is to show that $M(f,X,Y)$, and $G(f,X,Y)$ fit into this algebraic scheme.

Recall we have a branched covering map $f : X \to Y$, of degree d, with branch set $B = \{b_1, b_2, \ldots, b_n\} \subset Y$, a basepoint $y_0 \in Y^* = Y - B$, and fibre

$$F = (f^*)^{-1}(y_0) = \{x_1, x_2, \ldots x_d\} \subset X^* = X - f^{-1}(B),$$

over the basepoint. By standard covering space theory, f induces an **imbedding of the fundamental groups** $\{\pi_1(X^*, x_i), i = 1, \ldots, d\}$, (all isomorphic), as a conjugacy class of subgroups $D_i \leq \Gamma$, of index d, where $\Gamma = \pi_1(Y^*, y_0)$.

$\Gamma = \pi_1(Y^*, y_0)$ acts on the fibre $F = f^{-1}(y_0) \subset X^*$, by the "endpoint of lift" map. The isotropy subgroup of x_i is D_i; the kernel of the action is $D^* = \cap_{i=1}^d D_i$. The action is **transitive**, by the **connectivity** of X^*: there exists a path $l_j \subset X^*$ from x_1 to x_j, for each $j \in \{1, \ldots, d\}$. This path projects to a loop $f(l_j)$, based at y_0, defining an element of Γ which takes x_1 to x_j. $f(l_j)$ may wind several times around one or more punctures. If it winds, say, twice around b_1, and three times around b_5, it is homotopic to $\gamma_1^2\gamma_5^3$, where γ_j is defined as before: a loop based at y_0 which winds once counterclockwise about the puncture at b_j (and not around any other puncture).

It follows that there is a surjective homomorphism

$$\theta : \Gamma \to M(f, X, Y)$$

with kernel D^*. Equivalently, $M(f, X, Y) \simeq \Gamma/D^*$.

Let $|\Gamma/D_1|_r$ be the set of **right cosets** of D_1 in Γ. There is a **natural bijection** between this set and F: let $\epsilon_i \in \Gamma$, $i = 1, \ldots, n$, lift to a path in X from $x_1 \in F$ to $x_i \in F$. (These exist by the transitivity of Γ on F; we may assume $\epsilon_1 = \text{id}$.) $\cup_i^n D_1\epsilon_i = \Gamma$ since every $\gamma \in \Gamma$ has a lift starting at x_1 (and ending at $x_j \in F$, for some j). $D_1\epsilon_i \cap D_1\epsilon_j = \emptyset$ if $i \neq j$ by uniqueness of path lifting.

Definition Permutation groups (G, A), (G', A') are **isomorphic** if there exist a group isomorphism $i : G \to G'$ and a bijection $b : A \to A'$ such that the diagram

$$\begin{array}{ccccc} G & \times & A & \to & A \\ i\downarrow & & b\downarrow & & b\downarrow \\ G' & \times & A' & \to & A' \end{array}$$

is commutative.

Theorem. The permutation groups $(\Gamma/D^*, |\Gamma/D_1|_r)$ and $(M(f, X, Y),\ F)$ are isomorphic via the group isomorphism $\rho_j \leftrightarrow D^*\gamma_j$, and the natural bijection $F \leftrightarrow |\Gamma/D_1|_r$.

Remark. This is an instance of a more general theorem which will be useful later on: **Theorem.** If (G, A) is a **transitive** permutation group (G a group, A a set) with isotropy subgroup $G_1 \leq G$, and $\theta : \Gamma \to G$ a **surjective** homomorphism, with $\theta^{-1}(G_1) = D$, then (G, A) is isomorphic (as a permutation group) to $(\Gamma/D^*, |\Gamma/D|_r)$.

There is a natural way to define a permutation group **automorphism**. For the permutation group (G, A), an automorphism is a bijection $\sigma : A \to A$ (equivalently, $\sigma \in S_A$) such that

$$\begin{array}{ccccc} G & \times & A & \to & A \\ \text{id}\downarrow & & \sigma\downarrow & & \sigma\downarrow \\ G & \times & A & \to & A \end{array}$$

commutes. Hence we define

$$\begin{aligned} \text{Aut}(G, A) &\stackrel{\text{def}}{=} \{\sigma \in S_A \mid \sigma\mathbf{g} = \mathbf{g}\sigma \text{ for all } g \in G\} \\ &= \text{Cent}_{S_A}(G). \end{aligned}$$

Theorem. $G(f,X,Y) \simeq N_\Gamma(D_1)/D_1$.

Proof. $G(f,X,Y)$ is the automorphism group of the permutation group $(M(f,X,Y),F)$. Identifying the latter with $(\Gamma/D^*, |\Gamma/D_1|_r)$, we seek to **characterize the bijections** τ which make the diagram

$$\begin{array}{ccccc} \Gamma/D^* & \times & |\Gamma/D_1|_r & \to & |\Gamma/D_1|_r \\ \text{id}\downarrow & & \tau\downarrow & & \tau\downarrow \\ \Gamma/D^* & \times & |\Gamma/D_1|_r & \to & |\Gamma/D_1|_r \end{array}$$

commute. Commutativity means for all $\gamma \in \Gamma$, $\tau(D_1\gamma) = \tau(D_1)\gamma$. In particular, if $\gamma \in D_1$, $\tau(D_1) = \tau(D_1)\gamma$. **Suppose** $\tau(D_1) = D_1\varepsilon$, for some $\varepsilon \in \Gamma$. Then $D_1\varepsilon = D_1\varepsilon\gamma$. Since $\gamma \in D_1$ was arbitrary,

$$\begin{aligned} D_1\varepsilon &= D_1\varepsilon D_1 \quad \text{(as cosets)} \\ \varepsilon^{-1}D_1\varepsilon &= \varepsilon^{-1}D_1\varepsilon D_1, \end{aligned}$$

which implies $D_1 \subseteq \varepsilon^{-1}D_1\varepsilon$ and hence $\varepsilon \in N_\Gamma(D_1)$. **Q.e.d.**

If the covering $f: X \to Y$ is **Galois**, $D_1 = D^* = D$. Then $G(f) \simeq M(f)$, and both are isomorphic to the finite group $G = \Gamma/D \leq$ **Aut**(X), of order d. The surjective homomorphism $\theta : \Gamma \to M(f)$ yields a **presentation of** G, as follows. $\Gamma = \pi_1(Y^*, y_0)$, has presentation

$$\langle a_1, b_1, \ldots, a_h, b_h, \gamma_1, \ldots, \gamma_n \mid \prod_{i=1}^{h}[a_i, b_i] \prod_{j=1}^{n} \gamma_j = \text{id}\rangle,$$

where h is the genus of Y and $[a,b] = a^{-1}b^{-1}ab$. The order of $\theta(\gamma_j)$ is $r_j > 1$ (in $M(f) \leq S_d$, $\theta(\gamma_j) = \rho_j$, a product of d/r_i cycles of length r_i). G is **generated** by $g_1, k_1, \ldots, g_h, k_h$ (images of a_i, b_i, resp., under θ), and $\rho_1, \ldots, \rho_n$ (images of the γ_j's). In addition to the **order relations** $\rho_i^{r_i} = \text{id}$, there is the **"long" relation**

$$\prod_{i=1}^{h}[g_i, k_i] \prod_{j=1}^{n} \rho_j = \text{id}.$$

2.5 GALOIS COVERINGS OF THE LINE

In the case $h = 0$ ($f : X \to \mathbb{P}^1$), the long relation is simply $\prod_{j=1}^{n} \rho_j = \mathbf{id}$, and it has the following topological explanation: a loop winding once around **all** the punctures can be shrunk to a point "around the back" of the sphere.

Theorem. Let $f : X \to \mathbb{P}^1$ be a Galois covering with Galois group G of order $d < \infty$, and branching indices $\{r_1, \ldots, r_n\}$, all $r_i > 1$. Then G is generated by n elements of orders r_i, $i = 1, \ldots, n$, whose product is the identity. Conversely, any finite group G with an n-element generating set is the Galois group of a Galois covering of the line, branched over $n+1$ points.

Corollary. Every finite group is a group of automorphisms of a compact Riemann surface.

Theorem (Harvey '66). Let $A = \{a_1, a_2, \dots, a_n\}$, $n \geq 2$, be a multi-set of integers with $a_i > 1$. Then A is the set of branching indices of a d-fold **cyclic** covering of the line if and only if $d = \text{lcm}(A) = \text{lcm}(A - \{a_i\})$, $i = 1, \dots, n$.

Proof. If $\text{lcm}(A) < d$, A cannot generate a cyclic group of order d. Since the product of the elements of A is trivial, any one of them is redundant. Hence the removal of any one of the generators must not reduce the lcm. **Q.e.d.**

2.6 THE GALOIS EXTENSION PROBLEM

Given Galois coverings $f : X \to Y$ and $g : Y \to Z$, under what conditions is $g \circ f : X \to Z$ also a Galois covering? Equivalently,

- Does $G(f) \leq \text{Aut}(X)$ **extend** to a larger group of automorphisms?
- Does $G(g) \leq \text{Aut}(Y)$ **lift** to a subgroup of $\text{Aut}(X)$?

These questions can be answered in terms of monodromy, but a more fruitful approach is via **uniformization**. We'll take this up in the next lecture.

3 Lecture III

1. Uniformization
2. The Dirichlet region
3. Surface groups
4. Triangle groups
5. Uniformization of automorphisms

3.1 UNIFORMIZATION

Uniformization Theorem (Klein, Poincarè, Koebe). A simply connected Riemann surface, up to conformal equivalence, is one of the following:

1. the complex plane $\mathbb{C}$;
2. the Riemann sphere $\mathbb{P}^1$;
3. the upper half plane $\mathbb{U} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

Each simply connected surface has a canonical **complete metric of constant curvature.** On $\mathbb{U}$ the metric is $|dz|/\text{Im}(z)$, with curvature $\equiv -1$.

The uniformization theorem implies that every Riemann surface has one of these three as its universal cover. More generally, every Riemann surface is conformally equivalent to a quotient $\tilde{X}/\Gamma$, where $\tilde{X}$ is one of the simply connected surfaces, and Γ is a **discrete** group of $\text{Isom}^+(\tilde{X})$ (orientation-preserving isometries), acting **properly discontinuously**.

Definitions (1) A subgroup Γ of a topological group such as $\text{Isom}^+(\tilde{X})$ is **discrete** iff any infinite sequence $\{\gamma_n \in \Gamma\}$ converging (in the subspace topology) to id, is eventually constant ($\gamma_n = \text{id}$ for all $n \geq N$, for some $N < \infty$). (2) Γ acts **properly discontinuously** on a topological space such as $\tilde{X}$ if every $x \in \tilde{X}$ is contained in an open set U_x such that $\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset\}$ is finite.

If Γ acts properly discontinuously on $\tilde{X}$, the set D of points with non-trivial stabilizer is **discrete** (possibly empty). Deleting D yields a **covering map**

$$\tilde{X} - D \to (\tilde{X} - D)/\Gamma$$

(possibly infinite-sheeted) which transfers the metric on $\tilde{X}$ to the quotient, making it a Riemann surface (punctured at a discrete set of points). The metric extends canonically to the closure (with filled-in punctures).

We focus on the case where $\tilde{X}/\Gamma$ is **compact**. This is automatic if $\tilde{X} = \mathbb{P}^1$ (itself compact). If $\tilde{X} = \mathbb{C}$, we get compact quotients (tori) when Γ is a lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$, with $\omega_1/\omega_2 \notin \mathbb{R}$.

When $\tilde{X} = \mathbb{U}$, $\text{Isom}^+(\tilde{X})$ is the **real Möbius group**

$$\left\{ z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad-bc=1 \right\} \simeq PSL(2,\mathbb{R}).$$

Discrete subgroups of $\text{PSL}(2,\mathbb{R})$ are **Fuchsian groups**. The **limit set** of a Fuchsian group Γ is the set of accumulation points of its orbits; by discreteness, these lie on the ideal boundary $\partial\mathbb{U} = \mathbb{R} \cup \{\infty\}$. There are three types of elements in $\text{PSL}(2,\mathbb{R})$: **elliptic** elements, with trace $= 2$ and a single fixed point in $\mathbb{U}$; **parabolic** elements, with trace < 2 and a single fixed point in $\partial\mathbb{U}$; and **hyperbolic** elements, with trace > 2 and two fixed points in $\partial\mathbb{U}$. Hyperbolic and parabolic elements have infinite order; an elliptic element can have infinite order, but **an elliptic element in a Fuchsian group has finite order** (by discreteness).

Lemma. Two elements of $\text{PSL}(2,\mathbb{R})$ commute if and only if they have the same fixed point set.

Remarks on the proof. In a general transformation group, commuting elements merely preserve each other's fixed point set (exercise). In a general group, conjugate elements have conjugate centralizers (exercise). From the second statement, it suffices to examine centralizers of representatives of conjugacy classes. In $\text{PSL}(2,\mathbb{R})$, parabolic elements are conjugate to $z \mapsto z \pm 1$, whose fixed point set is $\{\infty\} \subset \partial\mathbb{U}$. Hyperbolic elements are conjugate to $z \mapsto \lambda z$, for some real $\lambda > 1$, having fixed point set $\{0, \infty\} \subset \partial\mathbb{U}$. Elliptic elements are conjugate to rotations fixing the origin (after a standard conformal transformation mapping $\mathbb{U}$ to the interior of the unit disk $\mathbb{D}$). It can be shown that the centralizer of any one of these element types is the largest cyclic subgroup of $\text{PSL}(2,\mathbb{R})$ containing the element; this can be characterized as the set of elements having the same fixed point set.

A Fuchsian group with a finite limit set is called **elementary**. These are either infinite cyclic, finite cyclic, or "infinite dihedral," and they are generated, respectively, by a single parabolic element, a single elliptic element, or a pair consisting of a hyperbolic and an elliptic element of order 2. In the latter case, the elliptic element interchanges the two ideal fixed points of the hyperbolic element while fixing (setwise) the geodesic joining them.

We consider mostly **cocompact** Fuchsian groups, i.e, with compact quotient space. These cannot contain parabolic elements, and must be non-elementary. It suffices for our purposes to further restrict to **cofinite area** groups, whose quotient has finite (hyperbolic) area. For such groups, one always has a fundamental domain, known as a **Dirichlet region**, which is a convex geodesic polygon with finitely many sides, entirely contained in $\mathbb{U}$. Recall that a closed subset $D \subset \mathbb{U}$ is a **fundamental domain** for Γ if: (i) $\cup_{\gamma \in \Gamma}(\gamma D) = \mathbb{U}$; and (ii) Int$(D) \cap$ Int$(\gamma D) = \emptyset$ unless $\gamma =$ id.

3.2 THE DIRICHLET REGION

Let Γ be a cocompact, cofinite area Fuchsian group (henceforth, we will just say "Fuchsian group").

Choose $p \in \mathbb{U}$ which is not fixed by any nontrivial element of Γ. The **Dirichlet region** for Γ, based at p, is the set

$$D_p = \{z \in \mathbb{U} \mid d(z,p) \leq d(\gamma z, p), \forall \gamma \in \Gamma\},$$

where d denotes hyperbolic distance. It is not hard to see that D_p is a finite intersection of half-planes bounded by geodesics. A bounding geodesic segment is called a **side** of the region. A point where two distinct sides intersect is called a **vertex**. The set $\{\gamma D_p \mid \gamma \in \Gamma\}$ is called a **Dirichlet tesselation** of $\mathbb{U}$. A particular γD_p is called a **face** of the tesselation. **Neighboring** faces share a common side.

If an element of Γ has a fixed point $q \in \mathbb{U}$, its orbit Γq intersects the Dirichlet region D on its **boundary** ∂D. The isotropy subgroups of the points in Γq are (finite) cyclic groups generated by elliptic elements (all conjugate in Γ). Suppose $u \in \partial D$ is fixed by an elliptic element $\gamma \in \Gamma$. If γ has order $k \geq 3$, u must be a **vertex** of D, at which three or more sides meet at angles $\leq 2\pi/k < \pi$. If $k = 2$, u might be the **midpoint of a side**; in this case, it is convenient to adjoin u to the vertex set, where two "half-sides" meet at the angle $= \pi$.

The set of vertices of D is partitioned into subsets (**vertex cycles**) whose elements belong to the same Γ orbit. If two vertices are in the same cycle, their isotropy subgroups are conjugate. Hence there is a **period** associated with each vertex cycle; it is the common order of the elliptic generator of the isotropy subgroup. **The vertex cycles with period > 1 are in bijection with conjugacy classes of elliptic elements in Γ.**

Lemma. The internal angles at the vertices of a vertex cycle of period k in a Dirichlet region sum to $2\pi/k$.

Proof. Let $v_1, \ldots, v_t$ be the vertices in a cycle, and let θ_i be the internal angle at v_i, $i = 1, \ldots, t$. Let $H \leq \Gamma$ be the (finite, cyclic) isotropy subgroup of

v_1. Then there are $|H| = k$ faces containing vertex v_1 and having internal angle θ_1 at v_1; similarly, there are k faces containing v_j and having internal angle θ_j at v_j. There exists $\gamma_j \in \Gamma$ such that $\gamma_j v_j = v_1$. Thus γ_j adds k more faces to the total set of faces surrounding v_1. Of course, the total angle around v_1 is 2π. Summing over all j, we have

$$k(\theta_1 + \theta_2 + \cdots + \theta_t) \leq 2\pi.$$

Exercise: complete the proof by showing that every face containing v_1 has been counted in this procedure, hence the inequality is actually equality. **Q.e.d.**

Sides s_1, s_2 of a Dirichlet region D for Γ are **congruent** if there is a **side-pairing** $\gamma \in \Gamma$ such that $s_2 = \gamma s_1$. In this case, D and γD are neighboring faces. A side may be congruent to itself (if its midpoint is fixed by an elliptic element of order 2). Since no side belongs to more than two faces of the Dirichlet tesselation, *no more than two sides can be congruent.* For if a side s were congruent with $\gamma_1 s$ and $\gamma_2 s$, then it would belong to three faces: D, $\gamma^{-1}D$, and $\gamma_1^{-1}D$ (unless $\gamma = \gamma_1$). Hence, if we count a side whose midpoint is fixed by an elliptic element of order 2 as a pair of (congruent) sides, then **the number of sides of D is even.**

Lemma. The k side-pairing elements of a $2k$-sided Dirichlet region for Γ are a finite **generating set** for Γ.

Proof. Let $\Lambda \leq \Gamma$ be the subgroup generated by the side-pairing elements of a Dirichlet region D for Γ. The strategy is to show that the connected set $\mathbb{U}$ is the **disjoint** union of two closed sets,

$$X = \cup_{\lambda \in \Lambda} \lambda D \quad \text{and} \quad Y = \cup_{\gamma \in \Gamma - \Lambda} \gamma D.$$

Any union of faces is closed (exercise). Clearly $X \neq \emptyset$. Thus if we show that $X \cap Y = \emptyset$, it will follow that $Y = \emptyset$, i.e., $\Lambda = \Gamma$. Let $\lambda \in \Lambda$ be arbitrary, and suppose γD, $\gamma \in \Gamma$, is a neighboring face of λD. Then D is a neighboring face of $\gamma^{-1}\lambda D$. Hence $\gamma^{-1}\lambda \in \Lambda$, which forces $\gamma \in \Lambda$. This is true for each of the finitely many neighbors of λD. There are possibly finitely many other faces which share only a vertex with λD. Let $\gamma_1 D$ be one of them. Since $\gamma_1 D$ is a "a neighbor of a neighbor of $\ldots$ a neighbor of" λD (finitely many!), the previous argument, applied finitely many times, shows that $\gamma_1 \in \Lambda$. Thus all the faces surrounding any vertex of λD are Λ-translates of D, and none is a $(\Gamma - \Lambda)$-translate. This shows that $X \cap Y = \emptyset$. **Q.e.d.**

Let Γ have a Dirichlet region D with $2k \geq 4$ sides, $r \geq 0$ vertex cycles with periods $m_i > 1$, $i = 1, 2, \ldots r$, and $s \geq 0$ other vertex cycles (with period 1). By the **Gauss-Bonnet** theorem, the hyperbolic area of D is

$$\begin{aligned}\mu(D) &= \pi(2k-2) - \sum \text{internal angles} \\ &= \pi(2k-2) - \sum_{i=1}^{r} \frac{2\pi}{m_i} - 2\pi s \\ &= 2\pi\left[k - 1 - s - \sum_{i=1}^{r} \frac{1}{m_i}\right] \\ &= 2\pi\left[\mathbf{k} - \mathbf{1} - \mathbf{s} - \mathbf{r} + \sum_{i=1}^{r} 1 - \frac{1}{m_i}\right].\end{aligned}$$

Claim: $k-1-s-r = 2h-2$, where h is the **genus** of $\mathbb{U}/\Gamma$.

To verify the claim, we need the following

Theorem. Let Γ be a Fuchsian group with Dirichlet region D. The (compact) quotient Riemann surface $\mathbb{U}/\Gamma$ is homeomorphic to the **orbifold** D/Γ.

(**Comments on the proof.** An orbifold is a compact surface with finitely many **cone points**, where the total angle surrounding a point is $2\pi/k$ for some $k > 1$. There is a cone point in D/Γ for every vertex cycle with period $k > 1$. To define an **open**, continuous, bijective mapping between the two spaces, one uses the **local finiteness** of D: every point has an open neighborhood which meets only finitely many of its Γ-translates.)

Proof of claim. The orbifold D/Γ has $s + r$ **vertices,** k **edges and** 1 **face**. Thus, if D/Γ has genus h, the Euler characteristic is $2 - 2h = s + r - k + 1$, and the claim follows.

The area $\mu(D)$ **of a Dirichlet region is a numerical invariant of** Γ. This is a consequence of the fact that any two sufficiently "nice" fundamental regions for Γ have the same area. (**Exercise**: prove this for finite polygonal fundamental regions.) From the formula

$$\mu(D) = 2\pi\left[2h - 2 + \sum_{i=1}^{r} 1 - \frac{1}{m_i}\right]$$

we get, for every Fuchsian group Γ, a well-defined **signature**

$$(\mathbf{h}; \mathbf{m_1}, \ldots, \mathbf{m_r}) \qquad h, r \geq 0; \quad \text{if } r > 0, m_i > 1.$$

Theorem (Poincaré, Maskit). There exists a Fuchsian group with signature $(h; m_1, \ldots, m_r)$ if and only if

$$\left[2h - 2 + \sum_{i=1}^{r} 1 - \frac{1}{m_i}\right] > 0.$$

Proof sketch. Construct a $4h + r$-sided regular polygon (it is convenient to make a conformal tansformation and work in the interior of the unit disk $\mathbb{D}$). In counterclockwise order, label the first $4h$ sides $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots,$

$\alpha_h, \beta_h, \alpha_h^{-1}, \beta_h^{-1}$. On the last r sides, erect external isosceles triangles with apex angles $2\pi/m_i$. Delete the bases and label the equal sides of the isosceles triangles ξ_i, ξ_i^{-1}. Expand or contract the resulting polygonal region (which has $4h+2r$ sides) until it has the required area. Let $a_i, b_i \in \mathrm{PSL}(2,\mathbb{R})$ pair α_i with α_i^{-1} and β_i with β_i^{-1}, respectively. Let $e_i \in \mathrm{PSL}(2,\mathbb{R})$ pair ξ_i with ξ_i^{-1}. Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be the group generated by these elements. Claim: Γ is Fuchsian, and the polygonal region is a fundamental polygon for Γ, with r singleton vertex cycles of periods m_i (the apices of the isosceles triangles) and one other vertex cycle (the $4h+r$ vertices of the original regular polygon) with period 1. **Q.e.d.**

Corollary. A Fuchsian group Γ with signature $(h; m_1, \ldots, m_r)$ has **presentation**

$$\Gamma = \langle a_1, b_1, \ldots, a_h, b_h, e_1, \ldots, e_r \mid e_1^{m_1} = e_2^{m_2} = \cdots = e_r^{m_r} = \mathrm{id}, \prod_{i=1}^{h}[a_i, b_i]\prod_{j=1}^{r} e_j = \mathrm{id}\rangle.$$

Proof. We already know that Γ is *generated* by the given (side-pairing) elements. It remains to verify that the relations hold and that no other relations are needed to define the group.

Proof, cont. Let $\phi : \mathbb{U} \to \mathbb{U}/\Gamma$ be the quotient map. Remove from $\mathbb{U}$ all the fixed points of elliptic elements of Γ, and remove from $S = \mathbb{U}/\Gamma$ the images of those points, obtaining S_0. The restricted map $\phi' : \mathbb{U}_0 \to S_0$ is an unbranched Galois covering map (infinite sheeted), with Galois group Γ. From the theory of covering spaces,

$$\Gamma \simeq \pi_1(S_0)/\phi'_*(\pi_1(\mathbb{U}_0)),$$

where ϕ'_* is the imbedding of fundamental groups induced by ϕ' (basepoints suppressed). Since S_0 is a surface of genus g punctured at $r \geq 0$ points, its fundamental group has presentation

$$\langle a_1, b_1, \ldots, a_h, b_h, e_1, \ldots, e_r \mid \prod_{i=1}^{h}[a_i, b_i]\prod_{j=1}^{r} e_j = \mathrm{id}\rangle.$$

If $r = 0$, we are done. Hence assume $r > 0$.

Proof, cont. We need to show that $\phi'_*(\pi_1(\mathbb{U}_0))$ is the **smallest** normal subgroup of $\pi_1(S_0)$ containing $e_1^{m_1}, \ldots, e_r^{m_r}$. It certainly must contain these elements: a loop winding once counterclockwise around the jth puncture of S_0 (and not around any other punctures) must be traversed exactly m_j times until it lifts to a loop in $\mathbb{U}_0$. Assuming e_j represents such a loop, $e_j^{m_j} \in \phi'_*(\pi_1(\mathbb{U}_0))$. On the other hand, $\pi_1(\mathbb{U}_0)$ is generated by infinitely many loops $\lambda_1, \lambda_2, \ldots$ winding around each of infinitely many punctures. Each λ_i winds around a puncture lying over the jth puncture in S_0, for some $j \in \{1, \ldots, r\}$. Thus $\phi'_*(\lambda_i) = e_j^{m_j}$. if $u \in \phi'_*(\pi_1(\mathbb{U}_0))$, and $\tilde{u}$ is a preimage in $\pi_1(\mathbb{U}_0)$, then $\tilde{u} =$

$\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\ldots$, and hence u is a product of elements from the set $\{e_1^{m_1},\ldots,e_r^{m_r}\}$. **Q.e.d.** **Lemma.** Fuchsian groups are isomorphic if and only if they have the same signature (up to re-ordering of the periods.)

Proof. Exercise.

3.3 SURFACE GROUPS

A **torsion-free** Fuchsian group has signature $(g;-)$, $g>1$, and presentation

$$\Lambda_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i] = \mathrm{id}\rangle.$$

It is called a "surface" group, since it is isomorphic to $\pi_1(X_g)$. **Theorem.** Any compact Riemann surface X_g of genus $g>0$ is **conformally equivalent** to the orbit space $\mathbb{U}/\Lambda_g$, where Λ_g is a surface group of genus g.

Proof. The orbifold D/Λ_g is a manifold (since there are no "cone" points). It inherits a **conformal** structure from $\mathbb{U}$. **Q.e.d.**

Theorem. Let $\Lambda, \Lambda' \leq \mathrm{PSL}(2,\mathbb{R})$ be two surface groups of fixed genus $g>1$. The compact surfaces $\mathbb{U}/\Lambda$ and $\mathbb{U}/\Lambda'$ are **conformally equivalent** if and only if Λ and Λ' are **conjugate** in $\mathrm{PSL}(2,\mathbb{R})$.

Proof. Let $g : \mathbb{U}/\Lambda \to \mathbb{U}/\Lambda'$ be a conformal homeomorphism between the two compact surfaces. Any homeomorphism, in particular, g, lifts to the universal cover, i.e., there exists $T \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$g[z]_\Lambda = [T(z)]_{\Lambda'}.$$

For any $S \in \Lambda$, $g[S(z)]_\Lambda = [TS(z)]_{\Lambda'} = [T(z)]_{\Lambda'}$. Hence $TS(z) = VT(z)$ for some $V \in \Lambda'$. This is true for all $z \in \mathbb{U}$, hence, $TST^{-1} = V$. Thus $T\Lambda T^{-1} \leq \Lambda'$. In fact, equality holds, since Λ and Λ' are isomorphic. Thus Λ and Λ' are conjugate in $\mathrm{PSL}(2,\mathbb{R})$. Conversely, if $T\Lambda T^{-1} = \Lambda'$, the map $[z]_\Lambda \mapsto [T(z)]_{\Lambda'}$ is a conformal homeomorphism. **Q.e.d.**

3.4 TRIANGLE GROUPS

Let $\Delta \in \mathbb{U}$ be a geodesic triangle with vertices a, b, c, at which the interior angles are π/n, π/m, π/r respectively. Reflections in the sides of Δ generate a discrete group of isometries of $\mathbb{U}$ having Δ as fundamental domain. **The orientation-preserving subgroup (of index 2) is a Fuchsian group with signature** $(0; n, m, r)$. To see why, let e_1 be the product of the two reflections in the sides incident with vertex a; geometrically, this is a rotation (orientation-preserving) about vertex a through an angle $2\pi/n$. Define e_2 and e_3 similarly as rotations about b and c through angles $2\pi/m$, $2\pi/r$, respectively. The product $e_1e_2e_3$ is easily seen to be trivial (write it as the product of six side reflections). Let D be the four-sided region formed by the union of Δ with its reflection across the side ab. e_1 and e_3 pair the sides of D, so D is a Dirichlet region for the group $\Gamma_\Delta = \langle e_1, e_3\rangle$ with presentation $\langle e_1, e_2, e_3 \mid e_1^n = e_2^m = e_3^r = e_1e_2e_3 = \mathrm{id}\rangle$.

The construction of Γ_Δ works as just well if the defining geodesic triangle is in $\mathbb{C}$ or $\mathbb{P}^1$. In these cases, we get **euclidean** and **spherical** triangle groups. The hyperbolic area of a Fuchsian triangle group with signature (n, m, r) (short for $(0; n, m, r)$) is

$$\mu(D) = 2\pi\left[1 - (\frac{1}{n} + \frac{1}{m} + \frac{1}{r})\right].$$

Hence there is a Fuchsian triangle group with signature (n, m, r) if and only if the bracketed quantity is positive. If it is 0 or negative we get

$$(2,4,4),\ (3,3,3),\ (2,3,6) \quad (euclidean);$$
$$(2,2,n), n \geq 3,\ (2,3,3),\ (2,3,4),\ (2,3,5) \quad (spherical).$$

The spherical groups are finite: dihedral, tetrahedral, octahedral, icosahedral, respectively.

Lemma (Hurwitz). The Fuchsian group with Dirichlet region of smallest hyperbolic area is the triangle group with signature $(2, 3, 7)$.

Proof. An entertaining exercise. Minimize from the general signature $(h; m_1, \ldots, m_r)$, showing, successively, that $h = 0$; $3 \leq r \leq 4$; $r = 3$ and $m_1 = 2$, etc.

Since 1-(1/2+1/3+1/7)=1/42, **the hyperbolic area** A **of a Dirichlet region for a Fuchsian group satisfies** $A \geq \pi/21$.

3.5 UNIFORMIZATION OF AUTOMORPHISMS

Let Γ be Fuchsian, and $\Gamma_1 \leq \Gamma$ a subgroup of finite index d. If D_1 and D are (respective) Dirichlet regions, a simple geometric argument yields $\mu(D_1) = d\mu(D)$. This is none other than the **Riemann-Hurwitz relation** for the holomorphic map

$$\rho : \mathbb{U}/\Gamma_1 \to \mathbb{U}/\Gamma, \qquad \rho : [z]_{\Gamma_1} \mapsto [z]_\Gamma,$$

which sends the Γ_1-orbit of $z \in \mathbb{U}$ to the Γ-orbit which contains it.

If we start with a surface group Λ_g, $g > 1$, and take $\Gamma = N(\Lambda_g)$, its **normalizer** in PSL$(2, \mathbb{R})$, then, **provided** Γ **is Fuchsian**, we have a Galois covering with Galois group Γ/Γ_1, which must be the **full automorphism group** of the compact Riemann surface $\mathbb{U}/\Lambda_g$.

Theorem. The normalizer of a (noncyclic) Fuchsian group is Fuchsian.

Proof. Let Γ be a (noncyclic) Fuchsian group, and $N(\Gamma) = N_{\text{PSL}(2,\mathbb{R})}(\Gamma)$ its normalizer. If $N(\Gamma)$ is not Fuchsian, there is an infinite sequence of distinct elements n_i tending to id. For $\gamma \in \Gamma$, $\gamma \neq$ id, $n_i^{-1}\gamma n_i$ is an infinite sequence in Γ tending to γ, which must be eventually constant, since Γ is Fuchsian. Thus for all sufficiently large i, n_i and γ **commute**. An abelian Fuchsian group is cyclic, for otherwise there are commuting elements of PSL$(2, \mathbb{R})$ with different fixed point sets. Since we are assuming Γ is not cyclic, it is not abelian, so there is an element $\gamma' \in \Gamma$ which does not commute with γ. On the other hand, imitating the first part of the proof, for sufficiently large i, n_i commutes with γ' as well.

Hence both γ and γ' have the same fixed point set, which implies that they commute, a contradiction. **Q.e.d.**

Corollary (Hurwitz). The automorphism group of a compact Riemann surface of genus $g > 1$ is **finite**, with order $\leq 84(g-1)$.

Proof. The surface group Λ_g is non-cyclic, hence its normalizer $N(\Lambda_g)$ is Fuchsian. The area of a Dirichlet region for Λ_g is $2\pi(2g-2)$. Let $A \geq \pi/21$ be the area of a Dirichlet region for $N(\Lambda_g)$. Then

$$|\mathrm{Aut}(\mathbb{U}/\Lambda_g))| = [N(\Lambda_g) : \Lambda_g] \leq \frac{2\pi(2g-2)}{A} \leq 84(g-1). \quad \textbf{Q.e.d.}$$

Remark. A group of $84(g-1)$ automorphisms of a compact surface of genus $g > 1$ is called a **Hurwitz group**. The smallest Hurwitz group is PSL$(2,7)$ (order 168) acting in genus $g = 3$. There are infinitely many Hurwitz genera (and also infinitely many non-Hurwitz genera) (A.M. Macbeath.)

We have shown that every **Riemann surface transformation group**, $G \times X_g \to X_g$, where X_g is a compact Riemann surface of genus $g > 1$, and $G \leq \mathrm{Aut}(X_g)$, can be represented entirely in terms of Fuchsian groups acting on the universal covering space $\mathbb{U}$:

$$\frac{\Gamma}{\Lambda_g} \times \frac{\mathbb{U}}{\Lambda_g} \to \frac{\mathbb{U}}{\Lambda_g}, \qquad \overline{\gamma} : [z] \mapsto [\gamma z],$$

where Λ_g is a surface group, and Γ is a subgroup of $N(\Lambda_g)$, such that $\Gamma/\Lambda_g \simeq G$. $\overline{\gamma}$ denotes the element $\gamma\Lambda_g$ of the factor group; $[z]$, $[\gamma z]$ denote the Λ_g-orbits of $z, \gamma z \in \mathbb{U}$.

But Λ_g could imbed as a normal subgroup of Γ in more than one way, so it is more precise to associate the action with a short exact sequence

$$\{\mathrm{id}\} \to \Lambda_g \to \mathbf{\Gamma} \xrightarrow{\rho} \mathbf{G} \to \{\mathrm{id}\}.$$

The epimorphism ρ, which determines the G-action, is called a "smooth" or "surface-kernel" epimorphism (**SKEP**).

In studying actions on X_g, we are free to vary Λ_g in its conjugacy class (within PSL$(2,\mathbb{R})$). Since conjugate subgroups have conjugate normalizers, we may assume Γ is a fixed representative of its conjugacy class. Suppose two SKEPS, $\rho, \rho' : \Gamma \to G$ differ by pre- and post composition by automorphisms α, β of Γ, G, respectively. That is, the diagram

$$\begin{array}{ccccccccc} \{\mathrm{id}\} & \to & \Lambda_g & \xrightarrow{i} & \mathbf{\Gamma} & \xrightarrow{\rho} & \mathbf{G} & \to & \{\mathrm{id}\} \\ & & \| & & \alpha\downarrow & & \beta\downarrow & & \\ \{\mathrm{id}\} & \to & \Lambda_g & \xrightarrow{j} & \mathbf{\Gamma} & \xrightarrow{\rho'} & \mathbf{G} & \to & \{\mathrm{id}\} \end{array}$$

commutes. Then the actions are **topologically conjugate** (E.K. Lloyd '72), that is, there exists an orientation-preserving **homeomorphism** H such that

$$\begin{array}{ccccc} G & \times_\rho & \mathbb{U}/i(\Lambda_g) & \to & \mathbb{U}/i(\Lambda_g) \\ \beta\downarrow & & H\downarrow & & H\downarrow \\ G & \times_{\rho'} & \mathbb{U}/j(\Lambda_g) & \to & \mathbb{U}/j(\Lambda_g) \end{array}$$

commutes.

Lloyd's result is a consequence of a deep result going back to Nielsen ('27) and later generalized by Zieschang ('66): **every automorphism of a Fuchsian group is geometrically realized**, that is, given $\alpha \in \mathrm{Aut}(\Gamma)$, there exists a homeomorphism h such that the diagram

$$\begin{array}{ccccc} \Gamma & \times & \mathbb{U} & \rightarrow & \mathbb{U} \\ \alpha\downarrow & & h\downarrow & & h\downarrow \\ \Gamma & \times & \mathbb{U} & \rightarrow & \mathbb{U} \end{array}$$

commutes.

Topological conjugacy is an **equivalence relation** on Riemann surface transformation groups. It is, of course, weaker than conformal conjugacy. In the latter case, two G-actions are merely conjugate within the full automorphism group of a single (conformal equivalence class of) surface. In contrast, topologically conjugate G-actions may occur on distinct surfaces. Topological conjugacy is best studied in the context of Teichmüller spaces.

4 Lecture IV

Lecture IV:

1. Teichmüller spaces of Fuchsian groups
2. The extension problem
3. Generalized Lefschetz curves
4. Accola-Maclachlan and Kulkarni curves
5. Appendix: *Dessins d'enfant*

4.1 TEICHMÜLLER SPACES

As before, Γ denotes a co-compact Fuchsian group with signature $(h; m_1, m_2, \ldots m_r)$. Let $\mathcal{L} = \mathrm{PSL}(2, \mathbb{R})$, and let $R(\Gamma)$ be the **representation space** of all injective homomorphisms $r : \Gamma \rightarrow \mathcal{L}$ such that the image $r(\Gamma)$ is Fuchsian. $R(\Gamma)$ is topologized as a subspace of the product of $2h + r$ copies of $\mathcal{L}$, by assigning to $r \in R(\Gamma)$ the point

$$(r(a_1), r(b_1), \ldots, r(a_h), r(b_h), r(e_1), \ldots r(e_r)) \in \mathcal{L}^{2h+r}.$$

$r_1, r_2 \in R(\Gamma)$ are *equivalent* if their images are **conjugate** in $\mathcal{L}$. Definition The **Teichmüller space of** Γ, $T(\Gamma)$, is the set of equivalence classes $[r : \Gamma \rightarrow \mathcal{L}]$, endowed with the quotient topology from $R(\Gamma)$.

Example: surface groups. $T(\Lambda_g)$, where Λ_g is a surface group, is the ordinary Teichmüller space $\mathcal{T}_g$ of "marked" Riemann surfaces,

$$\mathcal{T}_g = \{\mathbb{U}/[r : \Lambda_g \rightarrow \mathcal{L}] \mid [r] \in T(\Lambda_g)\}.$$

Let $\mathrm{Aut}^+(\Gamma)$ be the group of automorphisms of Γ which are both *type-* and *orientation-preserving*. [Type-preserving automorphisms preserve elliptic, parabolic, hyperbolic types. Orientation-preserving automorphisms carry the "long" relator to a conjugate of itself but not of its inverse.] $\alpha \in \mathrm{Aut}^+(\Gamma)$ **induces a homeomorphism of** $T(\Gamma)$ defined by $[\alpha] : [r] \mapsto [r \circ \alpha]$. The subgroup $\mathrm{Inn}(\Gamma) \leq \mathrm{Aut}^+(\Gamma)$ of inner automorphisms acts trivially by the definition of $T(\Gamma)$. Hence the **Teichmüller modular group for** Γ,

$$\mathrm{Mod}(\Gamma) = \frac{\mathrm{Aut}^+(\Gamma)}{\mathrm{Inn}(\Gamma)} = \mathrm{Out}^+(\Gamma),$$

acts (possibly not faithfully) on $T(\Gamma)$.

Theorem (Kravetz, Maclachlan, Harvey). The *Teichmüller modular group* acts **properly discontinuously** on $T(\Gamma)$. The stabilizer of a point $[r] \in T(\Gamma)$ is isomorphic to the finite subgroup $N_{\mathcal{L}}(r(\Gamma))/r(\Gamma)$.

Proof of the second statement. If $[\alpha] \in \mathrm{Mod}(\Gamma)$ fixes $[r]$, then $[r \circ \alpha] = [r]$ and there exists $t \in \mathcal{L}$ such that, for all $\gamma \in \Gamma$, $r \circ \alpha(\gamma) = tr(\gamma)t^{-1}$. It follows that $t \in N_{\mathcal{L}}(r(\Gamma))$. If $t \in r(\Gamma)$, $\alpha \in \mathrm{Inn}(\Gamma)$ and hence $[\alpha]$ is the identity in $\mathrm{Mod}(\Gamma)$. Thus the stabilizer of $[r]$ is isomorphic to a subgroup of $N_{\mathcal{L}}(r(\Gamma))/r(\Gamma)$. On the other hand, if $t \in N_{\mathcal{L}}(r(\Gamma))$, the map $\beta_t : r(\gamma) \mapsto tr(\gamma)t^{-1}$ is a type- and orientation-preserving automorphism of $r(\Gamma)$, whence $\alpha_t = r^{-1} \circ \beta_t \circ r$ is a type- and orientation-preserving automorphism of Γ. α_t is inner if and only if $t \in r(\Gamma)$. This establishes the isomorphism. The normalizer of a cocompact Fuchsian group Γ has a fundamental domain of finite area $A = \mu(N_{\mathcal{L}}(\Gamma))$ which satisfies $\pi/21 \leq A \leq \mu(\Gamma)$. Hence the index $[N_{\mathcal{L}}(\Gamma) : \Gamma]$ is finite. **Q.e.d.**

Surface groups, cont. $\mathrm{Mod}(\Lambda_g)$ is the "mapping class group" of outer automorphisms of the fundamental group of a surface of genus g. The isotropy subgroup of $[r] \in T(\Lambda_g)$,

$$N_{\mathcal{L}}(r(\Lambda_g))/r(\Lambda_g),$$

is (the attentive listener will spot) none other than $\mathrm{Aut}(\mathbb{U}/r(\Lambda_g))$.

In most cases (with important exceptions!), the action of $\mathrm{Mod}(\Gamma)$ on $T(\Gamma)$ is **faithful**. This is true, for example, if $\Gamma = \Lambda_g$, $g > 2$. Then the orbit spaces

$$T(\Lambda_g)/\mathrm{Mod}(\Lambda_g),$$

are the so-called **moduli spaces**. Higher dimensional generalizations of orbifolds, they are "almost manifolds," depending on $3g-3$ complex numbers which parametrize conformal equivalence classes of surfaces. The **singular set** (where the manifold structure breaks down) consists of precisely those surfaces with a non-trivial automorphism group (Nielsen-Kerckhoff).

Note: the proof of the following theorem, and some others which follow, is beyond the scope of these lectures. **Theorem (Teichmüller, Ahlfors, Bers, Macbeath, Singerman, ...).** The Teichmüller space of a Fuchsian group Γ with signature $(h; m_1, \ldots, m_r)$ is homeomorphic to an **open ball** in the Euclidean space $\mathbb{C}^{3h-3+r}$.

$3h-3+r$ is the (complex) **Teichmüller dimension** of Γ.

Theorem (Greenberg '63). An inclusion $i : \Gamma \to \Gamma_1$ of Fuchsian groups induces a imbedding of Teichmüller spaces,

$$\bar{i} : T(\Gamma_1) \to T(\Gamma), \quad \bar{i} : [r] \mapsto [r \circ i],$$

whose image is a closed subspace.

The imbedding $\bar{i} : T(\Gamma_1) \to T(\Gamma)$ may be a **surjection** even though $i(\Gamma)$ is a proper subgroup of Γ_1. For this to occur, the dimensions of $T(\Gamma)$ and $T(\Gamma_1)$ must be equal. The list of subgroup pairs $\Gamma < \Gamma_1$ for which the Teichmüller dimensions are equal, was partially determined Greenberg ('63) and completed by D. Singerman ('72). For such a pair, we have the following possibility: if Γ covers a group action on the Riemann surfaces in $T(\Gamma)$, the **action may extend on all the surfaces** to larger group action covered by Γ_1.

Example 1. $\Gamma(2;-)$ is a subgroup of index 2 in $\Gamma_1(0;2,2,2,2,2,2)$. One can check that the Teichmüller dimensions are both $=3$. Now $\Gamma(2;-)=\Lambda_2$ covers the trivial action on every surface of genus 2. But all surfaces of genus 2 are hyperelliptic (2-fold cyclic coverings of $\mathbb{P}^1$); the trivial action always extends to a $\mathbb{Z}_2$-action with $2g+2=6$ branch points.

Example 2. All triangle groups have Teichmüller dimension 0. It follows that

- all triangle groups of a given signature are conjugate in $\mathcal{L}$;
- a Fuchsian group which contains a triangle group is itself a triangle group.

It is a subtle problem to recognize, **given a signature** $\sigma_1=(h;m_1,\ldots,m_r)$ for a Fuchsian group Γ_1, **the possible signatures** $\sigma=(h';n_1,\ldots,n_s)$ for a **subgroup** Γ. By the Riemann-Hurwitz relation for the holomorphic map $\mathbb{U}/\Gamma \to \mathbb{U}/\Gamma_1$, the ratio of the hyperbolic areas $\mu(\sigma)/\mu(\sigma_1)$ must be a positive integer N (equal to the index $[\Gamma_1:\Gamma]$). It is also clearly necessary that the periods of Γ be divisors of the periods of Γ_1. Necessary and sufficient conditions were given by Singerman ('70): **there exists a finite permutation group** G**, transitive on** N **points, and an epimorphism** $\theta : \Gamma \to G$, with precise conditions on the cycle structures of $\theta(x_i)$, where the x_i, $i=1,\ldots r$, are the elliptic elements of Γ_1. (The required permutation group is the natural (monodromy type) action of Γ_1 on the left cosets of Γ (cf. Lecture II).)

Imposing the **extra condition** that the Teichmüller **dimensions** of $\Gamma < \Gamma_1$ be **equal** severely restricts the possibilities for the signature pairs σ, σ_1. The complete list of pairs (Greenberg '63, Singerman '72) comprises seven infinite families and one sporadic pair (for normal inclusions $\Gamma \triangleleft \Gamma$), plus four infinite families and seven sporadic pairs (for non-normal inclusions $\Gamma < \Gamma$). We give a selection (involving triangle groups) on the next slide. **Exercise:** Verify the Riemann-Hurwitz relation for each pair.

Notes: Cases N6 and N8 are **normal** inclusions; the rest are non-normal. "Cyclic admissible" means that the sub-signatures (σ) are **potential signatures**

Case	σ	σ_1	$[\Gamma(\sigma_1) : \Gamma(\sigma)]$	Conditions
N6	$(0; k, k, k)$	$(0; 3, 3, k)$	3	$k \geq 4$
N8	$(0; k, k, u)$	$(0; 2, k, 2u)$	2	$u\|k$, $k \geq 3$
T1	$(0; 7, 7, 7)$	$(0; 2, 3, 7)$	24	-
T4	$(0; 8, 8, 4)$	$(0; 2, 3, 8)$	12	-
T8	$(0; 4k, 4k, k)$	$(0; 2, 3, 4k)$	6	$k \geq 2$
T9	$(0; 2k, 2k, k)$	$(0; 2, 4, 2k)$	4	$k \geq 3$
T10	$(0; 3k, k, 3)$	$(0; 2, 3, 3k)$	4	$k \geq 3$

Table 1: Cyclic-admissible signatures (σ) and possible extensions (σ_1)

for a cyclic group action (equivalently, potential branching indices for a cyclic (Galois) covering of $\mathbb{P}^1$.)

Some geometric intuition: Consider the "T9" inclusion $\Gamma(2k, 2k, k) < \Gamma_1(2, 4, 2k)$**, index** 4. First notice that T9 is equivalent to two successive extensions of the "N8" type $(a, a, b) < (2, a, 2b)$ (normal, index 2):

1. $(2k, 2k, k) \triangleleft (2, 2k, 2k)$
2. $(2, 2k, 2k) \triangleleft (2, 4, 2k)$

Construct a hyperbolic isosceles triangle (in $\mathbb{U}$) with apex angle $2\pi/k$ and base angles $2\pi/2k$. ($k \geq 3$.)

1. drop a perpendicular from the apex to the midpoint m of the base, creating two congruent right triangles (with angles $2\pi/2k$ at the apex and $\pi/2$ at m).
2. draw a perpendicular from m to each of the two opposite sides.

The original isosceles triangle with angles $2\pi\{1/2k, 1/2k, 1/k\}$ has been **subdivided into four congruent triangles** with angles $2\pi\{1/2, 1/4, 1/2k\}$.

4.2 THE EXTENSION PROBLEM

Given the existence of a genus g surface with a known group G of automorphisms, it is natural to ask if the (topological type of the) G- action can be **extended to a larger group** of automorphisms of some (possibly different) surface of genus g, or if it is always the **full** group of automorphisms. Let $X = \mathbb{U}/\Lambda_g$, and let the G-action be uniformized by a Fuchsian group Γ, i.e., $G \simeq \Gamma/\Lambda_g$. Then we have a short exact sequence

$$\{\mathrm{id}\} \rightarrow \Lambda_g \hookrightarrow \Gamma \xrightarrow{\rho} G \rightarrow \{\mathrm{id}\}.$$

If $\sigma(\Gamma)$ does not appear in the Greenberg-Singerman list, then G is always the full group of automorphisms. If $\sigma(\Gamma)$ appears as the first member of a Greenberg-Singerman pair σ, σ_1, then the G action **might extend** – that is ρ

might extend to ρ_1, a SKEP with the **same kernel**, onto a larger group G_1, uniformized by Γ_1 with signature σ_1.

In that case, we have a commuting diagram of short exact sequences,

$$\begin{array}{ccccccccc} \{\text{id}\} & \to & \Lambda_g & \hookrightarrow & \mathbf{\Gamma} & \stackrel{\rho}{\to} & \mathbf{G} & \to & \{\text{id}\} \\ & & \| & & \mu\downarrow & & \nu\downarrow & & \\ \{\text{id}\} & \to & \Lambda_g & \hookrightarrow & \mathbf{\Gamma_1} & \stackrel{\rho'}{\to} & \mathbf{G_1} & \to & \{\text{id}\} \end{array}$$

where μ, ν are inclusion maps. The inclusion μ can be given explicitly, since the signatures and hence presentations of Γ, Γ_1 are given. The problem then is to determine **conditions on** G which permit an extension to G_1 so that the diagram commutes. This has been done recently for all Greenberg-Singerman pairs (Bujalance, Cirre, Conder, 2002). Unfortunately, there is no general algorithm; the problem must be handled on a case-by-case method. Here, we will consider extendability of certain cyclic covers of $\mathbb{P}^1$.

4.3 GENERALIZED LEFSCHETZ CURVES

The generalized Lefschetz curves have (irreducible) equations of the form $y^n = x(x-1)^b(x+1)^c$, where $1+b+c \equiv 0 \pmod n$, and we assume $1 \le b, c \le n-1$. These curves are treated comprehensively by Kallel and Sjerve (2003); Lefschetz considered the case where n is a prime.

We have shown (cf. Lecture I) that the genus of such a curve is $g = [(n+1-(n,b)-(n,c)]/2$. We also know (cf. Lecture II) that it is a cyclic Galois covering of $\mathbb{P}^1$, with Galois group $\mathbb{Z}_n$, generated by three elements of orders n, (n,b), (n,c) whose product is the identity. By Harvey's theorem, $\mathbf{lcm}((n,b),(n,c)) = n$. In terms of Fuchsian groups, we have

$$\{\text{id}\} \;\to\; \Lambda_g \;\hookrightarrow\; \Delta \;\stackrel{\rho}{\to}\; \mathbb{Z}_n \;\to\; \{\text{id}\},$$

where Δ is a **triangle group** with signature $(n,(n,b),(n,c))$.

For definiteness, we focus on a **specific case**: $n = 2k \ge 6$**,** $b = 1$**,** $c = n-2$. This yields the signature $(2k, 2k, k)$ for Δ, and there is a potential extension (type T9) of the $\mathbb{Z}_{2k}$-action to a G_{8k}-action with covering group Δ_1,

$$\Delta = \langle x_1, x_2, x_3\rangle \quad (2k, 2k, k)$$
$$\Delta_1 = \langle y_1, y_2, y_3\rangle \quad (2, 4, 2k)$$

An explicit imbedding of $\mu : \Delta \to \Delta_1$ is given by

$$\mu : x_1 \to y_2^2 y_3 y_2^2, \quad x_2 \mapsto y_3, \quad x_3 \mapsto y_2 y_3^2 y_2^{-1}.$$

We seek a group G_{8k} and an inclusion $\nu : \mathbb{Z}_{2k} \to G_{8k}$ such that

$$\begin{array}{ccccccccc} \{\text{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta & \stackrel{\rho}{\to} & \mathbb{Z}_{2k} & \to & \{\text{id}\} \\ & & \| & & \mu\downarrow & & \nu\downarrow & & \\ \{\text{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta_1 & \stackrel{\rho'}{\to} & \mathbf{G_{8k}} & \to & \{\text{id}\} \end{array}$$

commutes.

Let $\mathbb{Z}_{2k} = \langle a \mid a^{2k} = \text{id}\rangle$. The skep $\rho : \Delta \to \mathbb{Z}_{2k}$ determines a **generating vector**

$$\langle \rho(x_1), \rho(x_2), \rho(x_3)\rangle \in \mathbb{Z}_{2k}^3$$

for the $\mathbb{Z}_{2k}$-action. We may assume, up to an automorphism of $\mathbb{Z}_{2k}$, that $\rho(x_1) = a$. If $\rho(x_2) = a^i$ and $\rho(x_3) = a^j$, then, since ρ is a skep, $\rho(x_1)\rho(x_2)\rho(x_3) = a^{1+i+j} = \text{id}$. Equivalently, $1 + i + j \equiv 0 \pmod{2k}$. Example: $\langle a, a, a^{2k-2}\rangle$ is a possible generating vector for $\mathbb{Z}_{2k}$. The existence of $\nu : \mathbb{Z}_{2k} \to G_{8k}$, we shall see, depends on the form of the generating vector.

Before proceeding, we note that the Riemann-Hurwitz relation yields $k = g + 1$, so we are looking to extend a $\mathbb{Z}_{2g+2}$ action to a G_{8g+8}-action on the genus g Lefschetz curve

$$y^{2g+2} = x(x-1)(x+1)^{2g}.$$

Recall that the "T9" extension (index 4) is equivalent to two successive normal (index 2) extensions of type "N8."

- $\mathbf{\Delta(2k, 2k, k)} \lhd \Delta_0(2, 2k, 2k)$ (index 2);
- $\Delta_0(2, 2k, 2k) \lhd \mathbf{\Delta_1(2, 4, 2k)}$ (index 2)

The first of these must cover an extension of $\mathbb{Z}_{2k}$ to a group $G_{4k} \rhd \mathbb{Z}_{2k}$ which can be constructed as follows: let $\alpha \in \text{Aut}(\mathbb{Z}_{2k})$ have order ≤ 2. Let t be a new generator of order 2 such that conjugation by t acts on $\mathbb{Z}_{2k} = \langle a\rangle$ as α does. Then

$$G_{4k} = \langle a, t \mid a^{2k} = t^2 = 1, tat^{-1} = \alpha(a)\rangle.$$

If $\alpha(a) = a^{-1}$, then $G_{4k} \simeq D_{4k}$, the **dihedral** group of order $4k$; if $\alpha(a) = a$, then $G_{4k} \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. If $k \neq p^s$ (p an odd prime) there exists an involutory automorphism α, $\alpha(a) \neq a, a^{-1}$. In this case G_{4k} is a (non-dihedral, non-abelian) **semi-direct product** $\mathbb{Z}_2 \ltimes_\alpha \mathbb{Z}_{2k}$.

For the initial N8 extension, we have

$$\Delta = \langle x_1, x_2, x_3\rangle \quad (2k, 2k, k)$$
$$\Delta_0 = \langle z_1, z_2, z_3\rangle \quad (2, 2k, 2k)$$

and an explicit imbedding $\mu_0 : \Delta \to \Delta_0$ is given by

$$\mu_0 : x_1 \to z_3^{-1} z_2 z_3, \quad x_2 \mapsto z_2, \quad x_3 \mapsto z_3^2.$$

We seek a skep $\rho_0 : \Delta_0 \to \langle a, t\rangle = G_{4k}$ such that

$$\begin{array}{ccccccccc}
\{\text{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta & \xrightarrow{\rho} & \langle a\rangle & \to & \{\text{id}\} \\
 & & \| & & \mu_0 \downarrow & & \downarrow & & \\
\{\text{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta_0 & \xrightarrow{\rho_0} & \langle a, t\rangle & \to & \{\text{id}\}
\end{array}$$

commutes. It is not difficult to verify that

$$\rho_0 : z_1 \mapsto t, \quad z_2 \mapsto ta, \quad z_3 \mapsto a^{-1}$$

will do. That is, $\langle t, ta, a^{-1}\rangle$ is a Δ_0-generating vector for the G_{4k}-action.

For a second N8 extension (of the G_{4k} action), we need $\beta \in \mathrm{Aut}(G_{4k})$, of order 2, which interchanges ta and a^{-1} (the last two elements of the G_{4k} generating vector). Hence let s be a new generator such that conjugation by s acts as β does, i.e.,

$$sas^{-1} = a^{-1}t.$$

Equivalently, $(sa)^2 = ts^2$. Since $s^2 \in \langle a, t\rangle$ (for an index 2 extension), and $s^2 \notin \langle a\rangle$, either $s^2 = t$, or $s^2 = \mathrm{id}$. **Claim:** $s^2 = t$**.** It follows that $(sa)^2 = \mathrm{id}$, and hence we have an extended group

$$G_{8k} = \langle s, a \mid s^4 = a^{2k} = (sa)^2 = \mathrm{id},\ s^2as^2 = \alpha(a)\rangle,$$

containing $G_{4k} = \langle s^2, a\rangle$, acting with Δ_1- generating vector

$$\langle sa, s, a\rangle \quad (2, 4, 2k).$$

4.4 ACCOLA-MACLACHLAN AND KULKARNI CURVES

These curves arise from certain choices of $\alpha \in \mathrm{Aut}(\mathbb{Z}_{2k})$ as considered in the previous section.

Case 1. $\alpha(a) = a$, i.e., α is trivial and $G_{4k} = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. With $k = g+1$, we have the **Accola-Maclachlan surface** of genus g, with equation $y^{2g+2} = x(x-1)(x+1)^{2g}$, and full automorphism group

$$G_{8g+8} = \langle s, a \mid s^4 = a^{2g+2} = (sa)^2 = [s^2, a] = \mathrm{id}\rangle$$

The curve was identified by Accola and Maclachlan (independently) in 1968. It is hyperelliptic with **hyperelliptic involution** s^2. Note that $G_{8g+8}/\langle s^2\rangle \simeq D_{4g+4}$, the dihedral group of order $4g+4$.

Case 2. $g \equiv -1 \pmod 4$, and $\alpha(a) = a^{g+2}$. (Exercise: verify that α is an automorphism under the stated condition.) Here we have

$$G_{8g+8} = \langle s, a \mid s^4 = a^{2g+2} = (sa)^2 = \mathrm{id},\ s^2as^2 = \mathbf{a^{g+2}}\rangle.$$

The (non-hyperelliptic) curve was identified by R.S. Kulkarni in 1991, and is known as the **Kulkarni curve**. An equation of the curve is

$$y^{2g+2} = x(x-1)^{g+2}(x+1)^{g-1}.$$

Corollary. Let $m(g)$ be the order of the largest group of automorphisms of a compact Riemann surface of genus $g > 1$. Then $8g+8 \leq |\mathrm{Aut}(X)| \leq 84(g-1)$.

Remark. There exist genera g for which $m(g) = 8g+8$, i.e., the lower bound is sharp.

4.5 APPENDIX: DESSINS D'ENFANT

Definition. A Belyi curve is a compact Riemann surface admitting a meromorphic function with at most three critical values.

Hence Lefschetz curves are Belyi.

Theorem (Belyi, Wolfart, Koeck). X is a Belyi curve if and only if it is defined over a **number field** (finite field extension of $\mathbb{Q}$.)

Let (X, f) be a Belyi pair, i.e., X is a curve and $f : X \to \mathbb{P}^1$ a Belyi function having three or fewer critical values. Up to a Möbius transformation, the three critical values can be assumed to lie in the set $\{0, 1, \infty\}$. The preimage $f^{-1}([0, 1])$ of the unit interval is a **bipartite graph imbedded on** X, with, say, black vertices at $f^{-1}(0)$ and white vertices at $f^{-1}(1)$.

Grothendieck realized that any finite bipartite graph ("child's drawing" or *dessin d'enfant*) defines a canonical compact Riemann surface on which the graph imbeds geometrically, that is, with geodesic edges.

If all the white vertices have valence 2, they can be considered as edge midpoints, and erased, resulting in an ordinary graph. When imbedded on a surface, these special dessins are called **maps**. There is an intimate connection between maps and $(2, b, c)$-triangle groups. They key object, once again, is a pair of permutation groups arising from the triangle group and a certain finite index subgroup.

A **map** is an embedding of a finite connected graph $\mathcal{G}$ on a compact oriented surface X such that the complement $X \setminus \mathcal{G}$ is a union of 2-cells, called **faces**.

- $\mathcal{G}$ may have loops and multiple edges;
- a directed edge is called a **dart**;
- edges mostly carry two darts, although "free edges" are possible.

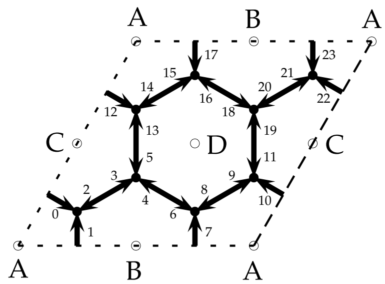
If there are n darts, label them with the symbols $0, 1, 2, 3, \ldots, n-1$ (in some convenient order) and define the monodromy group to be the subgroup of S_n generated by:

- $x \equiv$ product of dart-label cycles at the **vertices**;
- $y \equiv$ product of dart-label pairs on the **edges**,

where the cyclic ordering of dart-labels at a vertex is determined by the orientation of the ambient surface. It is easy to show that the cycles of $(xy)^{-1}$ correspond to dart-label cycles bounding the **faces**.

In the figure, which shows a map on a torus, the monodromy group ($\leq S_{24}$) has generators

- $x = (0\ 1\ 2)\ (3\ 4\ 5)\ (6\ 7\ 8)\ \ldots$ (Eight 3-cycles $\leftrightarrow$ **vertices**)
- $y = (2\ 3)\ (4\ 6)\ \ldots$ (Twelve 2-cycles $\leftrightarrow$ **edges**)



Note: $(xy)^{-1} = (5\ 6\ 9\ 19\ 16\ 14)\ \ldots$ (Four 6-cycles $\leftrightarrow$ **faces**)

Definition. Two maps $\mathcal{M}_1$, $\mathcal{M}_2$ with n darts are *equivalent* if their monodromy groups G_1, G_2 are **strongly conjugate** in S_n. This means there exists a single permutation in S_n which **simultaneously** conjugates the generators $x_1, y_1 \in G_1$ to the corresponding generators $x_2, y_2 \in G_2$.

Definition. The *automorphism group* of a map $\mathcal{M}$ with n darts is the **centralizer** of its monodromy group in S_n. (Rationale: $\mathcal{M}$ is non-trivially equivalent *to itself* if there is a permutation in S_n which commutes with both monodromy generators.) Aut($\mathcal{M}$) is **well-defined** on equivalence classes of one-vertex maps: conjugate subgroups have conjugate centralizers.

Definitions. The *type* of a map $\mathcal{M}$ is (m, r) where m is the lcm of the vertex valencies and r is the lcm of the face valencies. $\mathcal{M}$ is:

- **uniform** if *all* vertices have valence m and *all* faces have valence r;
- **regular** if Aut($\mathcal{M}$) is *transitive* on the darts.

Regular $\implies$ Uniform: every dart has the same local incidence relations.

Let $\Gamma = \Gamma(m, r)$ be the group with presentation

$$\Gamma(m, r) = \langle \xi_1, \xi_2, \xi_3 \mid \xi_1^m = \xi_2^2 = \xi_3^r = \xi_1\xi_2\xi_3 = 1 \rangle.$$

There is an obvious surjective homomorphism

$$\theta : \xi_1 \mapsto x, \quad \xi_2 \mapsto y, \quad \xi_3 \mapsto (xy)^{-1}.$$

onto the monodromy group $G = \langle x, y \rangle$ of a map $\mathcal{M}$ of type (m, r). Let $G_\delta \leq G$ be the isotropy subgroup of a dart δ.

Definition. The **canonical map subgroup** for $\mathcal{M}$ is

$$M \equiv \theta^{-1}(G_\delta) \leq \Gamma(m, r).$$

Note: M is (up to conjugacy) independent of the choice of δ (Reason: the underlying graph $\mathcal{G}$ is connected $\leftrightarrow$ G is *transitive* on the darts $\leftrightarrow$ all G_δ are conjugate.)

Let $M^* \equiv \bigcap_{\gamma \in \Gamma} \gamma^{-1} M \gamma$ (= *the core of* M *in* Γ). Let $|\Gamma/M|$ = the set of cosets M in Γ. Γ/M^* acts faithfully and transitively on $|\Gamma/M|$. Let $D \equiv$ the set of darts of $\mathcal{M}$.

Lemma (Jones, Singerman, '78). The permutation groups (G, D) and $(\Gamma/M^*, |\Gamma/M|)$ are isomorphic.

- $\mathcal{M}$ is uniform $\iff$ M is torsion-free;
- $\mathcal{M}$ is regular $\iff$ M is torsion-free and normal ($M = M^*$).

Via M we obtain, for a map $\mathcal{M}$,

$$X \equiv \text{the } \textbf{canonical Riemann surface} \text{ of } \mathcal{M} = \frac{\mathcal{U}}{M},$$

where $\mathcal{U} \equiv \mathbb{C}, \mathbb{P}^1$, or H^2.

- X contains $\mathcal{M}$ **geometrically**: edges are geodesics; face-centers and edge midpoints are well-defined;
- Aut($\mathcal{M}$) is a group of **conformal automorphisms** of X.

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